On Permutiple Numbers and the Role of Symmetry in Finding New Examples from Old

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Abstract

A permutiple is a natural number whose representation in some base, $b > 1$, is an integer multiple of a number whose base-b representation has the same collection of digits. In this article, we summarize efforts in finding such numbers. Recent work utilizes graph-theoretical and finite-state-machine constructions. We detail how the inherent symmetry of these constructions filters down to techniques for finding new permutiples from old, enabling us to see previous work through a new lens.

1 Introduction

A *permutiple* is a number which is an integer multiple of some permutation of its digits in a natural-number base [11]. We may also describe permutiples as simply the result of a digit-preserving multiplication. The reader may find other descriptions of equivalent and similar notions in Sloane's Online Encyclopedia of Integer Sequences [20], for instance, "numbers whose digits can be permuted to get a proper divisor," or "numbers p such that p and np are anagrams," where $2 < n < b$.

A well-known base-10 example is $p = 87912 = 4 \cdot 21978$. The techniques outlined in this article will enable the reader to obtain new examples from p provided above. For example, the permutiple having the same digits as p, $79128 = 4 \cdot 19782$, may be viewed as the result of a "rotation" of an object, to be defined later, which is associated with p. Another permutiple, also obtained from p, is $71208 = 4 \cdot 17802$, which may be viewed as a "rotation" and a "reflection" of this same object mentioned above.

A specific case of permutiple numbers which has received much attention are those which are multiples of their digit reversals. The most widely known base-10 examples include the first example in the previous paragraph, as well as $98901 = 9 \cdot 10989$. These numbers are known by several names: *palintiple* numbers [8, 9, 10], *reversal numbers* [20], as well as similarly-defined, but equivalent, cases of this phenomenon known as *reverse multiples* [16, 19, 23, 24], and *reverse divisors* [3, 21]. For the present effort, we retain the use of the term "palintiple" to be consistent with the terminology used by Hoey [8] and other work [9, 10, 11, 12, 13, 14]. The large variety of palintiple (reverse-multiple) types can be organized using a graph-theoretical construction by Sloane [19] called *Young graphs*, which are a modification of the work of Young [23, 24].

On a historical note, one of the earliest mentions of palintiple numbers may be found in W. W. R. Ball's *Mathematical Recreations and Essays* [1], where readers learn that $8712 = 4 \cdot 2178$ and $9801 = 9 \cdot 1089$ are the only 4-digit numbers which are multiples of their reversals. This "odd fact," which we will reference as such, is one of two singled out by G. H. Hardy in his well-known work, *A Mathematician's Apology* [7], in order to illustrate his idea of a quintessentially "non-serious" theorem:

> *"These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals to a mathematician. The proofs are neither difficult nor interesting - merely tiresome. The theorems are not serious; and it is plain that one reason (though perhaps not the most important) is the extreme speciality of both the enunciations and proofs, which are* **not capable of any significant generalization***."*

While no one has argued that the "odd fact" is a mathematically "serious" statement, several authors [9, 16, 17, 19] have questioned Hardy's assertion stated in bold above. One of these references, Pudwell's *Digit Reversal without Apology* [17], challenges Hardy's statements directly both in its very title, and by its own assertion that "Hardy's comment may have been short-sighted." The most prolific and well-known mathematician to weigh in on the matter is N. J. A. Sloane [19], who, with regard to Hardy's comment in bold, states that "it seems fair to say that Hardy was wrong." In fairness to Hardy, we mention an article in which Weisgerber [22] contends that these authors[1] misinterpreted what Hardy had actually meant by the words "general," "serious," and "significant." To his credit, Hardy makes a substantial effort to make clear what he had in mind when using these words. That said, the extent to which the above qualities apply to mathematical statements, even if it is in the exact sense which Hardy had intended, is highly subjective. We leave it for readers to decide for themselves whether the generalizations of the "odd fact" presented both here and in other works are "significant."

Another specific, well-studied case of permutiple numbers involves those which are multiples of cyclic permutations of their digits. A base-10 example of such a number is $714285 = 5 \cdot 142857$. Several varieties on this theme exist: *cyclic numbers*, *transposable numbers* [6, 15], and *parasitic numbers* [20].

Efforts beyond cyclic and reversal permutations include an article by Qu and Curran [18] who examine the case of permutiples which are multiples of $(b^{b-1} - 1)/(b - 1)^2$. In base 10, these are multiples of 123456789, and two base-10 examples include $493827156 = 4 \cdot 123456789$ and $987654312 = 8 \cdot 123456789$. Other work [11, 12] details elementary methods for finding new permutiple examples from the digits of known examples, such as the case presented in the first paragraph.

Hoey [8] uses methods from formal language theory, namely, a finite-state-

[1] The reader will notice that these include the author of this article.

machine construction, to find and characterize all base-10 palintiples. In this same work, Hoey also constructs machines which recognize palintiples in other bases, while leaving their general properties as open questions. The state diagrams of these machines bear strong resemblance to Sloane's [19] construction, and Sloane acknowledges the connection between Young graphs and formal language theory. Faber and Grantham [4] also use finite-state-machine techniques to find pairs of integers whose sum is the reverse of their product in an arbitrary base. A base-10 example of such a pair is 3 and 24 since $3 + 24 = 27$ and $3 \cdot 24 = 72$. Recent work by the author [13, 14] also uses a finite-state-machine construction and its state graph, called the *Hoey-Sloane graph*, to find permutiples. This graph describes single-digit multiplication in a chosen base and multiplier, where each carry is less than the multiplier. The states of the machine are the possible carries of a digit-preserving multiplication, and the input alphabet consists of ordered pairs from the so-called *mother graph*, which describes how digits may be permuted in a single-digit multiplication. Digit-preserving multiplications are represented by input strings consisting of certain multiset combinations of mother-graph cycles which enable walks on the Hoey-Sloane graph beginning and ending with the zero state.

In this article, we draw particular attention to the reflective symmetry of the mother graph, and we detail how the Hoey-Sloane graph inherits this symmetry. From these investigations, we see how notions of symmetry arise when finding new permutiples from known examples, and how previous efforts [11, 12] with similar objectives fit into a broader framework.

2 Summary of previous work

2.1 Elementary definitions and results

We use the notation $(d_k, d_{k-1}, \ldots, d_0)_b$ to denote the natural number $\sum_{j=0}^{k} d_j b^j$, where $0 \leq d_j < b$ for all $0 \leq j \leq k$. With this notation, we state the definition of a permutiple number [11, Definition 1].

Definition 1. Let n and b be natural numbers where $1 < n < b$, and let σ be a permutation on $\{0, 1, 2, \ldots, k\}$. We say that $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-*permutiple* provided

$$(d_k, d_{k-1}, \ldots, d_1, d_0)_b = n \cdot (d_{\sigma(k)}, d_{\sigma(k-1)}, \ldots, d_{\sigma(1)}, d_{\sigma(0)})_b.$$

In the case that the digit permutation, σ, is not relevant, we may refer to $(d_k, d_{k-1}, \ldots, d_0)_b$ as simply an (n, b)-permutiple. We refer to the collection of base-b permutiples having n as their multiplier as (n, b)-*permutiples.*

An algorithm for carrying out single-digit multiplication gives us the next result, which is stated and proved in previous work [11, Theorem 1].

Theorem 1. *Let $(d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b, σ)-permutiple, and let c_j be the j^{th} carry. Then, $bc_{j+1} - c_j = nd_{\sigma(j)} - d_j$ for all $0 \leq j \leq k$.*

Every carry in a digit-preserving multiplication is less than the multiplier [11, Theorem 2]. This fact is of particular importance in what follows, and we state the full result below.

Theorem 2. *Let* $(d_k, d_{k-1}, \ldots, d_0)_b$ *be an* (n, b, σ)*-permutiple, and let* c_j *be the* j^{th} *carry. Then,* $c_j \leq n-1$ *for all* $0 \leq j \leq k$*.*

For an (n, b, σ)-permutiple, $(d_k, d_{k-1}, \ldots, d_0)_b$, [11, 12] develop elementary, matrix-based techniques for finding permutations, π, which yield another permutiple, $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$. These methods rely on the notion of permutiple *conjugacy* to sort new examples into *conjugacy classes* [11, 12, Definition 2].

Definition 2. Suppose $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b)-permutiple. Then, an (n, b, τ_1)-permutiple, $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \ldots, d_{\pi_1(0)})_b$, and an (n, b, τ_2)-permutiple, $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \ldots, d_{\pi_2(0)})_b$, are said to be *conjugate* if $\pi_1\tau_1\pi_1^{-1} = \pi_2\tau_2\pi_2^{-1}$.

In the case of repeated digits, this definition requires that we assume the collection $\{d_k, d_{k-1}, \ldots, d_0\}$ is a multiset.

To illustrate the above ideas, we may apply the techniques featured in previous work [11, 12] to a known example, $p = (d_4, d_3, d_2, d_1, d_0)_{10} = (8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$, to compute elements, $(d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_{10}$, of the conjugacy class containing p, all of which are shown in Table 1. Note that ψ is the 5-cycle $(0, 1, 2, 3, 4)$, ρ is the reversal permutation, and ε is the identity permutation.

$(4, 10, \tau)$-Example	π	τ
$(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$	ε	ρ
$(8, 7, 1, 9, 2)_{10} = 4 \cdot (2, 1, 7, 9, 8)_{10}$	$(1, 2)$	$(1, 2)\rho(1, 2)$
$(7, 9, 1, 2, 8)_{10} = 4 \cdot (1, 9, 7, 8, 2)_{10}$	ψ^4	$\psi^{-4}\rho\psi^4$
$(7, 1, 9, 2, 8)_{10} = 4 \cdot (1, 7, 9, 8, 2)_{10}$	$(1, 2)\psi^4$	$\psi^{-4}(1, 2)\rho(1, 2)\psi^4$

Table 1: The conjugacy class of $p = (8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$.

All of the examples in Table 1 are conjugate since $\pi\tau\pi^{-1} = \rho$ for every π and τ listed. An example with the same digits which is not a member of the above conjugacy class is $(7, 8, 9, 1, 2)_{10} = 4 \cdot (1, 9, 7, 2, 8)_{10}$, which may also be found using previous work [12].

2.2 Permutiple graphs and the mother graph

We now state a definition of a permutiple's graph [13, Definition 3], which is a description of the digit permutation in terms of the digits themselves rather than their indexing set.

Definition 3. Let $p = (d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b, σ)-permutiple. We define a directed graph, called the *graph of* p, denoted by G_p, to consist of the collection of base-b digits as vertices, and the collection of directed edges $E_p = \{(d_j, d_{\sigma(j)}) \mid 0 \leq j \leq k\}$. A graph, G, for which there is a permutiple, p, such that $G = G_p$ is called a *permutiple graph.*

We note that for the remainder of this effort, all graphs are directed graphs, and we may refer to a "directed graph" as simply a "graph," or a "directed edge" as an "edge."

A permutiple's conjugacy class gives us additional information about its graph; when two (n, b)-permutiples, p_1 and p_2, are conjugate, their graphs are the same [13, Theorem 4].

Theorem 3. *Suppose* $(d_k, d_{k-1}, \ldots, d_0)_b$ *is an* (n, b, σ)*-permutiple. Also, suppose* $p_1 = (d_{\pi_1(k)}, d_{\pi_1(k-1)}, \ldots, d_{\pi_1(0)})_b$ *is an* (n, b, τ_1)*-permutiple, and* $p_2 = (d_{\pi_2(k)}, d_{\pi_2(k-1)}, \ldots, d_{\pi_2(0)})_b$ *is an* (n, b, τ_2)*-permutiple. Then, if* p_1 *and* p_2 *are members of the same conjugacy class, then* $G_{p_1} = G_{p_2}$.

Each conjugate permutiple in Table 1 has the same graph displayed in Figure 1.

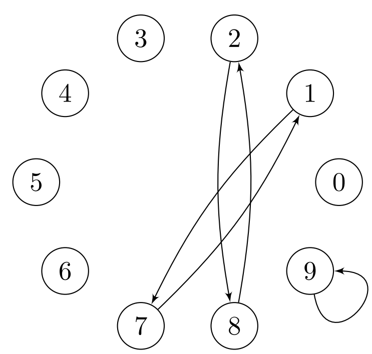


Figure 1: The graph with edges $\{(d_j, d_{\tau(j)}) \mid 0 \le j \le 4\}$ for each example in Table 1.

Permutiple graphs enable us to classify examples according to their graph [13, Definition 6]. Although the example $p = (7, 8, 9, 1, 2)_{10} = 4 \cdot (1, 9, 7, 2, 8)_{10}$ has the same digits as those in Table 1, the graph of p immediately distinguishes it from the others as it belongs to an entirely different conjugacy class.

Definition 4. Let p be an (n, b)-permutiple with graph G_p. We define the *class of* p to be the collection, C, of all (n, b)-permutiples, q, such that G_q is a subgraph of G_p. We also define the graph of the class to be G_p, which we denote by G_C and call the *graph of* C.

Theorem 3, coupled with the above definition, tells us that conjugate permutiples are always members of the same permutiple class.

The next result narrows down the possible collection of edges of a permutiple graph [13, Theorem 3].

Theorem 4. *Let* $p = (d_k, d_{k-1}, \ldots, d_0)_b$ *be an* (n, b, σ)*-permutiple with graph* G_p*. Then, for every edge,* $(d_j, d_{\sigma(j)})$*, of* G_p*, it must be that* $\lambda\big(d_j + (b-n)d_{\sigma(j)}\big) \le n - 1$ *for all* $0 \le j \le k$*, where* λ *gives the least non-negative residue modulo* b*.*

The above allows us to gather all possible edges of a permutiple graph into a single graph [13, Definition 4].

Definition 5. The (n, b)-*mother graph*, denoted by M, is the graph having all base-b digits as its vertices and the collection of all edges, (d_1, d_2), which satisfy the inequality $\lambda\big(d_1 + (b-n)d_2\big) \leq n-1$.

The next result is also fundamental to the methods presented both here and in other work [13, Theorem 6].

Theorem 5. *Let C be an (n, b)-permutiple class. Then, G_C is a union of cycles of M.*

Example 1. The $(4, 10)$-mother graph is displayed in Figure 2. Letting $p = (8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$ from Table 1, and C be the $(4, 10)$-permutiple class with graph $G_C = G_p$, Figure 2 highlights the edges of G_C in bold red.

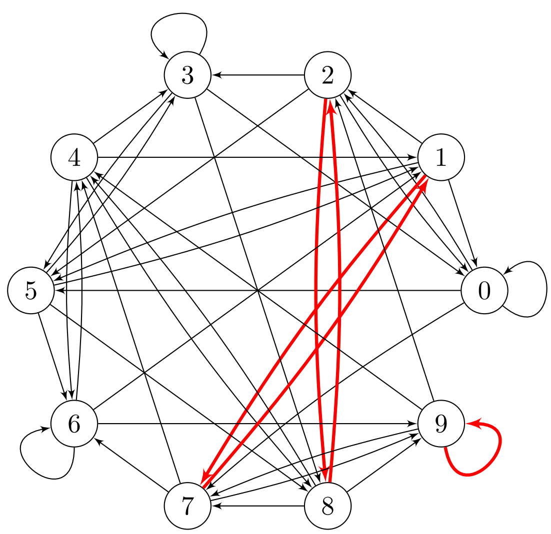


Figure 2: The $(4, 10)$-mother graph with the edges of of G_C displayed in bold red.

2.3 A finite-state-machine description of permutiple numbers and the Hoey-Sloane graph

To create a framework for describing digit-preserving multiplication, we construct a finite-state machine which describes single-digit multiplication by a number, n, less than the base, b. Here, the carries take on a central role in the discussion. By Theorem 2, all of the carries in a digit-preserving multiplication are less than the multiplier, n. Thus, the possible states of the machine are non-negative integers less than n. The input alphabet is the collection of edges of the mother graph, M, and the statement of Theorem 1 motivates the definition of the state-transition function,

$$c_2 = (nd_2 - d_1 + c_1) \div b, \tag{1}$$

where the input (d_1, d_2) enables a transition from state c_1 to state c_2. This transition defines a labeled edge on the state diagram as seen in Figure 3.

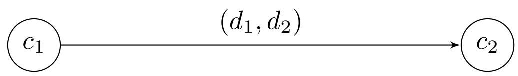


Figure 3: A labeled edge on the state diagram.

The first carry, c_0, of a single-digit multiplication is zero by definition [11]. In the present context, this is to say that the only possible initial state is zero. Also, assuming the product of a single-digit multiplication is a $(k+1)$-digit number, $(d_k, d_{k-1}, \ldots, d_0)$, the state c_{k+1} must also be zero, otherwise, the result would be a $(k+2)$-digit number. Thus, the zero state is the only possible accepting state. We call the above machine the (n, b)-*Hoey-Sloane machine*, and its state diagram is called the (n, b)-*Hoey-Sloane graph*, which we denote by Γ.

We note here that an input, (d_1, d_2), which allows a transition from state c_1 to state c_2 is generally not unique. In this way, the edge label on the state diagram from c_1 to c_2 may not be a single input, but a list of inputs. That is, we assign a collection of suitable inputs to each edge on Γ by the mapping $(c_1, c_2) \mapsto \{(d_1, d_2) \in E_M \mid c_2 = (nd_2 - d_1 + c_1) \div b\}$, where E_M is the collection of edges of M. We also note that we could alternatively define a multigraph where a unique multi-edge is assigned to each input in E_M, as done in other work [14]. We give some details of this construction later.

We let L denote the language of input strings accepted by the (n, b)-Hoey-Sloane machine. We may describe L as finite sequences of edge-label inputs which define walks on Γ whose initial and final states are the zero state. Such walks we call L-*walks*. Members of L which produce permutiple numbers are called (n, b)-*permutiple strings.*

In this new setting, Theorem 5 gives us the following result [13, Corollary 1].

Corollary 6. *Let* $s = (d_0, \widehat{d_0})(d_1, \widehat{d_1}) \cdots (d_k, \widehat{d_k})$ *be a member of* L*. If* s *is a permutiple string, then the collection of ordered-pair inputs of* s *is a union of cycles of* M*.*

The converse of Corollary 6 is not true in general; it is easy to find members of L whose ordered pairs make up cycles on M, but are not permutiple strings [13]. In order for an input string, $s = (d_0, \widehat{d_0})(d_1, \widehat{d_1}) \cdots (d_k, \widehat{d_k})$, in L to qualify as a permutiple string, it must be that the two collections of base-b digits formed by the left and right components of the inputs of s must be the same. In other words, $\{d_k, \ldots, d_1, d_0\}$ and $\{\widehat{d_k}, \ldots \widehat{d_1}, \widehat{d_0}\}$ must form the same multiset. Since every member of L describes a valid single-digit, base-b multiplication by n, multiset unions of cycles of M which can be ordered into an element, s, of L, allow us to form permutiple strings. Since s may be visualized as an L-walk on the (n, b)-Hoey-Sloane graph, Γ, the edges of Γ associated with the inputs of s must define a strongly-connected subgraph of Γ containing the zero state.

It is now advantageous to reestablish some additional terminology and notation used in previous efforts [13]. In this article, a union of multisets is denoted by $\uplus$. For instance, $\{1,2,2,3\} \uplus \{2,3,4\} = \{1,2,2,2,3,3,4\}$. Let $C_0, C_1, \ldots, C_m$ be the cycles of M. For each C_j, construct a labeled subgraph of Γ which we denote by Γ_j. The edges of Γ_j are pairs, (c_1, c_2), for which the collection $\mathscr{E}_j = \{(d_1, d_2) \in C_j \mid c_2 = (nd_2 - d_1 + c_1) \div b\}$ is nonempty. The edge label is then the list of elements of $\mathscr{E}_j$. With the collection $\mathscr{E}_j$ in hand, we exclude any state from the collection of vertices where the indegree and outdegree are both zero. We refer to Γ_j as the *image of* C_j*,* or simply as a *cycle image.* We now suppose I is a multiset whose support is a subset, J, of $\{0, 1, \ldots, m\}$. The key observation in previous efforts [13] is that L-walks can only occur on subgraphs of Γ which are strongly connected and contain the zero state. Thus, our search for permutiple strings is reduced to considering cycle image unions, $\Gamma_J = \bigcup_{j \in J} \Gamma_j$ (edge labels included), which are strongly-connected and contain the zero state. If the corresponding multiset union of mother-graph cycles, $C_I = \biguplus_{j \in I} C_j$, can be ordered into a string, s, belonging to L, then s is a permutiple string. This is to say that a strongly-connected graph, Γ_J, containing the zero state is a necessary condition for being able to order the elements of C_I into a permutiple string. Later, in Example 3, we provide a counterexample which shows that these conditions are not sufficient for this same purpose.

Example 2. The cycles of G_C from Example 1 are $C_0 = \{(9,9)\}$, $C_1 = \{(2,8),(8,2)\}$, and $C_2 = \{(1,7),(7,1)\}$. Table 2 depicts the cycle images corresponding to the cycles of G_C.

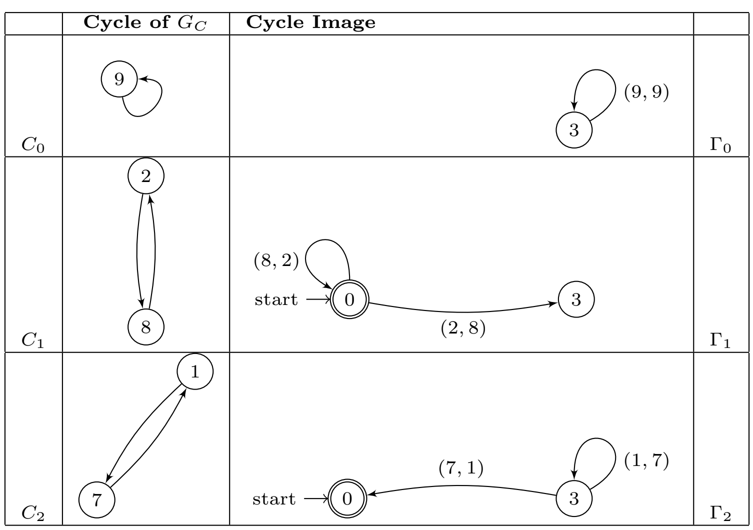


Table 2: Cycles of G_C and their corresponding cycle images.

The $(4, 10)$-Hoey-Sloane graph is shown in Figure 4. The graph of the union of the cycle images, $\Gamma_J = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, corresponding to the cycles of G_C is highlighted in bold red.

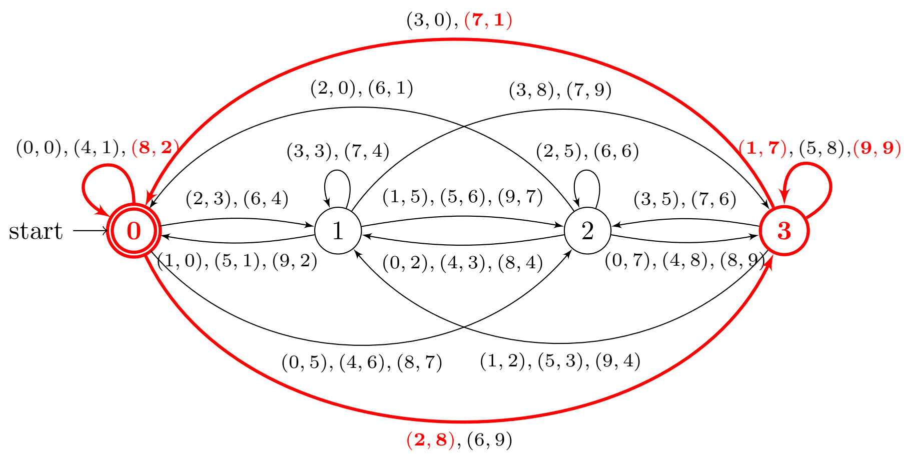


Figure 4: The $(4, 10)$-Hoey-Sloane graph with cycle images of G_C in bold red.

As a concrete example of the above ideas, consider the multiset union of mother-graph cycles

$$\begin{aligned} C_I &= C_0 \uplus C_1 \uplus C_1 \uplus C_2 \uplus C_2 \\ &= \{(9, 9), (8, 2), (8, 2), (2, 8), (2, 8), (7, 1), (7, 1), (1, 7), (1, 7)\}. \end{aligned}$$

With the help of Γ_J, there are multiple ways to order C_I into a member of L. For instance, $(8, 2)(8, 2)(2, 8)(9, 9)(1, 7)(1, 7)(7, 1)(2, 8)(7, 1)$ is an input string in L which yields a new example, $(7, 2, 7, 1, 1, 9, 2, 8, 8)_{10} = 4 \cdot (1, 8, 1, 7, 7, 9, 8, 2, 2)_{10}$. Another example, $(8, 2)(2, 8)(1, 7)(7, 1)(2, 8)(9, 9)(1, 7)(7, 1)(8, 2)$, corresponds to $(8, 7, 1, 9, 2, 7, 1, 2, 8)_{10} = 4 \cdot (2, 1, 7, 9, 8, 1, 7, 8, 2)_{10}$.

2.4 A multigraph representation of the Hoey-Sloane machine

The question remaining from the above is which multiset unions of mother-graph cycles yield permutiple strings. The author used a multigraph construction to answer this question [14]. Instead of assigning a collection of mother-graph inputs, $\mathscr{E}_j$, to a single edge, (c_1, c_2), as described above, we may define a multiset of labeled multi-edges where each element of $\mathscr{E}_j$ is individually assigned an edge, (c_1, c_2). That is, distinct inputs, (d_1, d_2) and $(\widehat{d}_1, \widehat{d}_2)$, determine two distinct multi-edges connecting state c_1 to state c_2, as depicted in Figure 5.

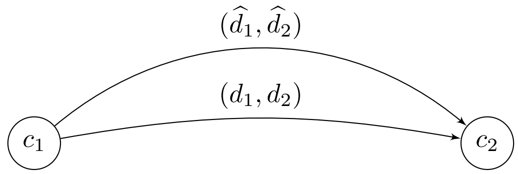


Figure 5: Two multi-edges representing two possible transitions from state c_1 to state c_2.

From previous work, we know that an edge label, (d_1, d_2), cannot appear on distinct edges of Γ [14, Theorem 5], and that every input, (d_1, d_2), enables some transition, (c_1, c_2) [14, Theorem 6]. Thus, for each input edge, (d_1, d_2), of M, we may find a unique pair, (c_1, c_2), which solves Equation (1). That is, letting N be the collection of non-negative integers less than the multiplier, n, the above determines a well-defined mapping, $\mu : E_M \to N \times N \times E_M$, given by $(d_1, d_2) \mapsto (c_1, c_2, (d_1, d_2))$. From there, we construct the *multi-image of* C_j, denoted by Δ_j, in similar fashion to the cycle images. To define the edges of Δ_j, apply μ to each element of the mother-graph cycle C_j to obtain its corresponding labeled edge, $(c_1, c_2, (d_1, d_2))$. As with the cycle images, we exclude any state from the collection of vertices where the indegree and outdegree are both zero. We may now state conditions which are both necessary and sufficient for the existence of permutiple strings [14, Corollary 2].

Corollary 7. *Let* $\{C_0, C_1, \ldots, C_m\}$ *be the collection of cycles of* M*, and let* Δ_j *be the corresponding multi-image of* C_j*, Also, let* I *be a multiset whose support is a subset of* $\{0, 1, \ldots, m\}$*. Then, a multiset union of mother-graph cycles,* $C_I = \biguplus_{j \in I} C_j$*, may be ordered into a permutiple string if and only if the corresponding multigraph union of the cycle multi-images,* $\Delta_I = \biguplus_{j \in I} \Delta_j$*, contains the zero state, is strongly connected, and the indegree is equal to the outdegree at each vertex.*

If we apply μ to all the edges of M, we obtain the (n, b)*-Hoey-Sloane multigraph* [14], which we denote by Δ to distinguish it from Γ. Although Γ and Δ are different objects, they are both equivalent representations of the Hoey-Sloane machine, as mentioned earlier.

Example 3. Picking up where Example 2 leaves off, we note that the multigraph representation of each cycle multi-image, Δ_j, looks the same as the graph of its cycle-image counterpart, Γ_j, shown in Table 2. The multi-image union, $\Delta_I = \Delta_0 \uplus \Delta_1 \uplus \Delta_1 \uplus \Delta_2 \uplus \Delta_2$, corresponding to C_I from Example 2, is shown in Figure 6. Since Δ_I contains the zero state, is strongly connected, and the indegree is equal to the outdegree at each vertex, Corollary 7 guarantees that we may find orderings like the one given in Example 2. In fact, we may use Figure 6 to determine all orderings by traversing all Eulerian circuits beginning and ending with the zero state.

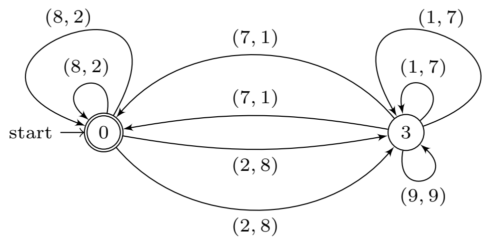


Figure 6: The multi-image union $\Delta_I = \Delta_0 \uplus \Delta_1 \uplus \Delta_1 \uplus \Delta_2 \uplus \Delta_2$ corresponding to the multiset union $C_I = C_0 \uplus C_1 \uplus C_1 \uplus C_2 \uplus C_2$ of mother-graph cycles.

We also mentioned in Example 2 that not every multiset union of mother-graph cycles, C_I, can be ordered into a permutiple string. As an example, consider $C_I = C_1 \uplus C_1 \uplus C_2$. Inspecting the corresponding multi-image union, $\Delta_I = \Delta_1 \uplus \Delta_1 \uplus \Delta_2$, which is depicted in Figure 7, we see that ordering C_I into a member of L is not possible. We may arrive at this same conclusion by using Corollary 7; the indegrees and outdegrees are not equal for both vertices.

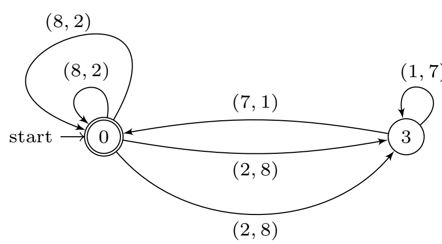


Figure 7: The multi-image union $\Delta_I = \Delta_1 \uplus \Delta_1 \uplus \Delta_2$ corresponding to the multiset union $C_I = C_1 \uplus C_1 \uplus C_2$ of mother-graph cycles.

3 Reflective symmetry of the mother graph and Hoey-Sloane graph

We begin with a definition.

Definition 6. Let G be a directed graph whose vertices are base-b digits, $B = \{0, 1, \ldots, b-1\}$, with a collection of directed edges, E. Also, let $\overline{d}_1 = b - 1 - d_1$ and $\overline{d}_2 = b-1-d_2$, where (d_1, d_2) is an edge of G. Finally, let $\overline{E}$ be the collection of directed edges $\{(\overline{d}_1, \overline{d}_2) \mid (d_1, d_2) \in E\}$. Then, the directed graph having B as vertices and $\overline{E}$ as edges is called the *reflection* of G and is denoted by $\overline{G}$. If G is its own reflection, that is, if $G = \overline{G}$, then we say that G is *symmetric*.

Remark 8. The reader will notice that to obtain the reflection of a graph whose vertices are the collection of base-b digits, we simply apply the reversal permutation, ρ, to the vertices. In this case, symmetric graphs are simply those for which ρ is a graph automorphism.

The following is a basic consequence of the above definition.

Corollary 9. *Let G_1 and G_2 be subgraphs of a directed graph whose vertices are base-b digits. Then, $\overline{G_1 \cup G_2} = \overline{G}_1 \cup \overline{G}_2$.*

Supposing that (d_1, d_2) is an edge of M, we notice that

$$\overline{d}_1 + (b-n)\overline{d}_2 \equiv n - 1 - (d_1 + (b-n)d_2) \pmod{b}.$$

Since $\lambda(d_1 + (b-n)d_2) \leq n-1$ by definition, we have $\lambda(\overline{d}_1 + (b-n)\overline{d}_2) \leq n-1$. This observation yields our first new theorem.

Theorem 10. *The (n, b)-mother graph is symmetric.*

From the above, we may say that reflections of mother-graph cycles are again mother-graph cycles. We also draw the reader's attention to the reflective symmetry of the $(4, 10)$-mother graph in Figure 2.

Now, for an edge, (c_1, c_2), on Γ with edge label (d_1, d_2), we notice that if we let $\overline{c}_1 = n-1-c_1$ and $\overline{c}_2 = n-1-c_2$ (treating c_1 and c_2 as base-n digits), then

$$\begin{aligned}(n\overline{d}_2 - \overline{d}_1 + \overline{c}_1) \div b &= \big(n(b-1-d_2) - (b-1-d_1) + n-1-c_1\big) \div b \\ &= n-1-c_2 \\ &= \overline{c}_2.\end{aligned}$$

Since $c_2 \leq n-1$, we have $\overline{c}_2 \leq n-1$. These facts give us our next result.

Theorem 11. *If (c_1, c_2) is an edge on the (n, b)-Hoey-Sloane graph with edge label (d_1, d_2), then $(\overline{c}_1, \overline{c}_2)$ is also an edge on the (n, b)-Hoey-Sloane graph with edge label $(\overline{d}_1, \overline{d}_2)$.*

Theorem 11 tells us, in a sense to be made precise below, that Γ inherits the reflective symmetry of M.

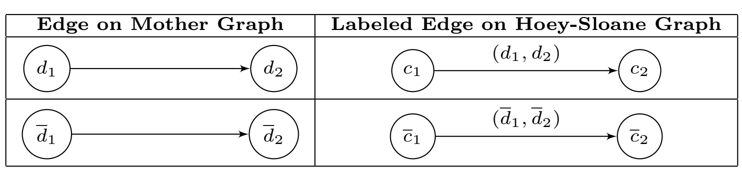


Table 3: Reflective symmetry of the (n, b)-mother graph and the (n, b)-Hoey-Sloane graph.

From the above, we may say that reflections of cycle images are cycle images of reflections of mother-graph cycles. We state the above more precisely as a corollary, but first we need to take care of some details since the edge labels of the Hoey-Sloane graph make it a bit more complex than the mother graph. In particular, we define what we mean by the reflection of a subgraph of Γ.

Definition 7. Let Γ_0 be a subgraph of Γ. We define the *reflection* of Γ_0, denoted by $\overline{\Gamma}_0$, to be the edge-labeled graph with the following properties:

1. $\bar{c}$ is a vertex of $\overline{\Gamma}_0$ whenever c is a vertex of Γ_0.
2. $(\bar{c}_1, \bar{c}_2)$ is an edge of $\overline{\Gamma}_0$ whenever (c_1, c_2) is an edge of Γ_0.
3. $(\overline{d}_1, \overline{d}_2)$ is an edge label of $\overline{\Gamma}_0$ whenever (d_1, d_2) is an edge label of Γ_0.

If Γ_0 is its own reflection, that is, if $\Gamma_0 = \overline{\Gamma}_0$, then we say Γ_0 is *symmetric*.

With the above, we may say more precisely what we mean by the "reflective symmetry" of Γ. In particular, with Definition 7, Theorem 11 gives us the following corollaries.

Corollary 12. *The (n, b)-Hoey-Sloane graph is symmetric.*

The reader may observe the reflective symmetry of the $(4, 10)$-Hoey-Sloane graph in Figure 4.

Corollary 13. *Let C_0 be a cycle of M. Then, Γ_0 is the cycle image of C_0 if and only if $\overline{\Gamma}_0$ is the cycle image of $\overline{C}_0$.*

$\overline{\Gamma}_0$ also inherits the topology of Γ_0, which gives us the following.

Corollary 14. *Let Γ_0 be a strongly-connected subgraph of Γ. Then, $\overline{\Gamma}_0$ is a strongly-connected subgraph of Γ.*

The following defines some useful notation.

Definition 8. Suppose that $\mathscr{C} = \{C_0, \ldots, C_m\}$ is the collection of cycles of G_C, and suppose that $\{\Gamma_0, \ldots, \Gamma_m\}$ is the collection of corresponding cycle images of $\mathscr{C}$. We denote the union of cycle images, $\bigcup_{j=0}^{m} \Gamma_j$, by Γ_C.

Letting C be a permutiple class, we may now say when the reflection, $\overline{G}_C$, of G_C, is also the graph of a permutiple class.

Theorem 15. *Let C be an (n, b)-permutiple class, and let G_C be its graph. Then, the following statements are equivalent.*

1. *$\overline{G}_C$ is the graph of a permutiple class.*
2. *The zero state is a vertex of $\overline{\Gamma}_C$.*
3. *The state $n - 1$ is a vertex of Γ_C.*

Proof. (*1* $\implies$ *2*). If $\overline{G}_C$ is the graph of a permutiple class, then the zero state is trivially a vertex of $\overline{\Gamma}_C$.

(*2* $\implies$ *3*). Now suppose that the zero state is a vertex of $\overline{\Gamma}_C$. Then, by Theorem 11 and Definition 7, the state $\overline{0} = n - 1$ must be a vertex of Γ_C.

(*3* $\implies$ *1*). Suppose that the state $c_j = n - 1$ is a vertex of Γ_C. Since G_C is the graph of the permutiple class C, we may choose an (n, b, σ)-permutiple,

$p = (d_k, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, \ldots, d_{\sigma(0)})_b$, with carries $c_k, \ldots, c_j = n-1, \ldots, c_1$, $c_0 = 0$, such that $G_C = G_p$. Let $s = (d_0, d_{\sigma(0)}) \cdots (d_k, d_{\sigma(k)})$ be the permutiple string of p, and let

$$S = \{(0, c_1), (c_1, c_2), \ldots, (c_{j-1}, n-1), (n-1, c_{j+1}) \ldots, (c_{k-1}, c_k), (c_k, 0)\}$$

be the associated sequence of state transitions induced by the inputs of s on the L-walk of p, which is visualized in Figure 8.

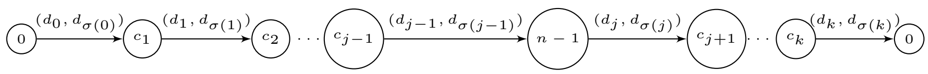


Figure 8: The L-walk on Γ_C corresponding to p.

We examine the reflections of the input sequence, $\overline{s} = (\overline{d}_0, \overline{d}_{\sigma(0)}) \cdots (\overline{d}_k, \overline{d}_{\sigma(k)})$, and the state-transition sequence,

$$\overline{S} = \{(n-1, \overline{c}_1), (\overline{c}_1, \overline{c}_2), \ldots, (\overline{c}_{j-1}, 0), (0, \overline{c}_{j+1}), \ldots, (\overline{c}_{k-1}, \overline{c}_k), (\overline{c}_k, n-1)\},$$

of p. Now, both of these describe a closed walk on Γ, pictured in Figure 9, but since the initial and final states are not zero, $\overline{s}$ is not a permutiple string.

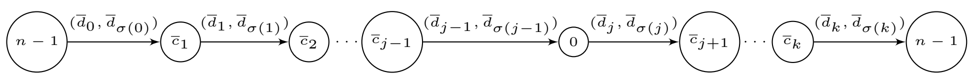


Figure 9: The reflection of the L-walk of p on $\overline{\Gamma}_C$ (not an L-walk).

However, a cyclic permutation of the sequence $\overline{S}$ does yield an L-walk. That is, $\overline{s}_{\psi^j} = (\overline{d}_{\psi^j(0)}, \overline{d}_{\sigma\psi^j(0)}) \cdots (\overline{d}_{\psi^j(k)}, \overline{d}_{\sigma\psi^j(k)})$, is a permutiple string with the state-transition sequence

$$\overline{S}_{\psi^j} = \{(0, \overline{c}_{j+1}), \ldots, (\overline{c}_{k-1}, \overline{c}_k), (\overline{c}_k, n-1), (n-1, \overline{c}_1), (\overline{c}_1, \overline{c}_2), \ldots, (\overline{c}_{j-1}, 0)\}.$$

We write $\overline{p}_{\psi^j}$ to denote the associated permutiple, $(\overline{d}_{\psi^j(k)}, \ldots, \overline{d}_{\psi^j(0)})_b = n \cdot (\overline{d}_{\sigma\psi^j(k)}, \ldots, \overline{d}_{\sigma\psi^j(0)})_b$. Now, since the inputs in the string s make up the edges of G_C, we know that the collection of inputs of $\overline{s}_{\psi^j}$ make up the edges of $\overline{G}_C$. This is to say that $\overline{G}_C$ is the graph of a permutiple class since $\overline{G}_C = G_{\overline{p}_{\psi^j}}$. $\square$

Example 4. Consider the $(4, 10)$-permutiple example $p = (8, 6, 7, 1, 2)_{10} = 4 \cdot (2, 1, 6, 7, 8)_{10}$ and the class, C, with graph $G_C = G_p$. Figure 10 shows G_C and its reflection, $\overline{G}_C$.

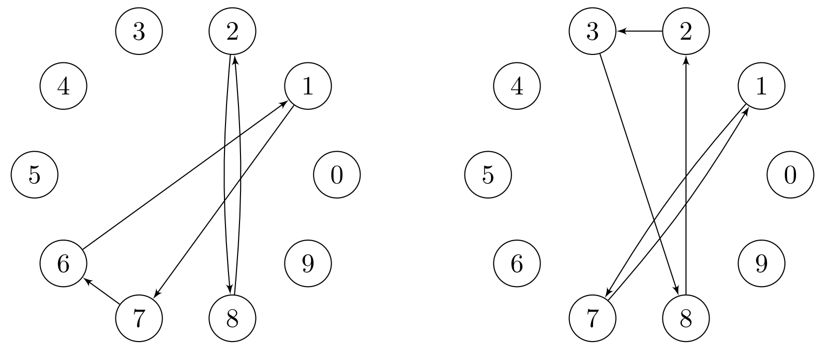


Figure 10: The permutiple graph G_C (left) and its reflection $\overline{G}_C$ (right).

We examine the correspondence between the cycles of G_C and their cycle images as seen in Table 4. Since Γ_C contains the vertex $n-1=3$, we may apply Theorem 15 and conclude that $\overline{G}_C$ is the graph of some permutiple class. We may confirm this by examining the cycles of $\overline{G}_C$ and their corresponding cycle images shown in Table 5.

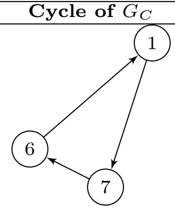
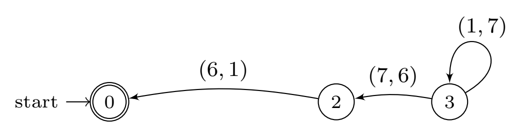
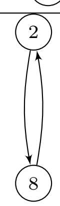
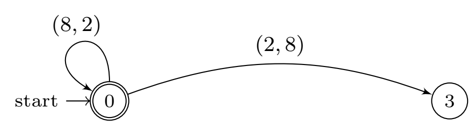
	Cycle of G_C	**Cycle Image**	
C_0			Γ_0
C_1			Γ_1

Table 4: The cycles of G_C and their corresponding cycle images.

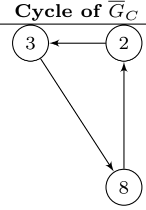
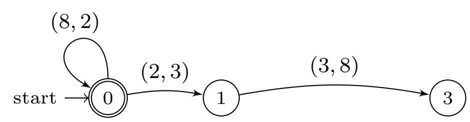
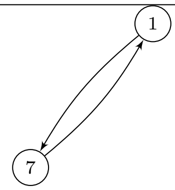
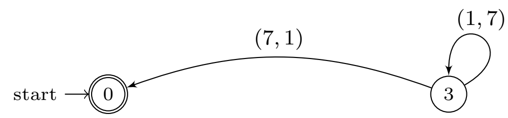
	Cycle of $\overline{G}_C$	**Cycle Image**	
$\overline{C}_0$			$\overline{\Gamma}_0$
$\overline{C}_1$			$\overline{\Gamma}_1$

Table 5: The cycles of $\overline{G}_C$ and their corresponding cycle images.

The resulting cycle-image union, $\overline{\Gamma}_C = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, forms a strongly-connected graph (as guaranteed by Corollary 14) containing the zero state. From this graph, we may construct permutiple strings, such as $(2,3)(3,8)(1,7)(7,1)(8,2)$, which corresponds to the $(4,10)$-permutiple $(8,7,1,3,2)_{10} = 4 \cdot (2,1,7,8,3)_{10}$.

4 Symmetric classes

Definition 9. Let C be an (n,b)-permutiple class, and let G_C be its graph. If the state $n-1$ is a vertex of Γ_C, we call the (n,b)-permutiple class with graph $\overline{G}_C$, guaranteed by Theorem 15, the *reflection* of C, and denote it by $\overline{C}$. If $C = \overline{C}$, then we say that C is a *symmetric class*. The *symmetric closure* of C, denoted by $\widehat{C}$, is the permutiple class with graph $G_C \cup \overline{G}_C$.

Lemma 16. *Let Γ_0 and Γ_1 be subgraphs of Γ. Then, $\overline{\Gamma_0 \cup \Gamma_1} = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$.*

Proof. Let c be a vertex of $\overline{\Gamma_0 \cup \Gamma_1}$. Then, $\bar{c}$ is a vertex of $\Gamma_0 \cup \Gamma_1$. Thus, $\bar{c}$ is a vertex of Γ_0 or Γ_1, from which it follows that c is a vertex of $\overline{\Gamma}_0$ or $\overline{\Gamma}_1$. The other containment is proved similarly. Moreover, the above argument is identical in form for edges and edge labels. $\square$

Corollary 17. *Let $\Gamma_0, \ldots, \Gamma_m$ be subgraphs of Γ. Then,* $\overline{\bigcup_{j=0}^{m} \Gamma_j} = \bigcup_{j=0}^{m} \overline{\Gamma}_j$.

Theorem 18. *Let C be an (n,b)-permutiple class, and let G_C be its graph. Also, suppose that the state $n-1$ is a vertex of Γ_C. Then, the symmetric closure of C is a symmetric class. Also, all of the following are equivalent:*

1. *C is a symmetric class.*
2. *C is the symmetric closure of itself.*
3. *Γ_C is a symmetric subgraph of the (n,b)-Hoey-Sloane graph.*

4. G_C is a symmetric graph.

Proof. By Corollary 9, the reflection of $G_{\widehat{C}} = G_C \cup \overline{G}_C$ is itself. Thus, $\widehat{C}$ is its own reflection. By definition, we conclude that $\widehat{C}$ is symmetric. To demonstrate the equivalence, we show that items *1*, *2*, and *3* are all equivalent to item *4*.

(*1* $\iff$ *4*). The permutiple class C has graph G_C, and, by definition, the permutiple class $\overline{C}$ has graph $\overline{G}_C$. From this, the equivalence between the statements $C = \overline{C}$ and $G_C = \overline{G}_C$ is clear.

(*2* $\iff$ *4*). If $C = \widehat{C}$, then $G_C = G_{\widehat{C}} = G_C \cup \overline{G}_C$. By Corollary 9, we then have that $\overline{G}_C = \overline{G_C \cup \overline{G}_C} = \overline{G}_C \cup G_C = G_C$. Conversely, if G_C is a symmetric graph, then $G_C = \overline{G}_C = G_C \cup \overline{G}_C$. Thus, C and its symmetric closure have the same graph, from which it follows, by definition, that C is its own symmetric closure.

(*3* $\iff$ *4*). Suppose G_C is a symmetric graph, and let $\mathscr{C} = \{C_0, \ldots, C_m\}$ be the collection of cycles of G_C, and let $\overline{\mathscr{C}} = \{\overline{C}_0, \ldots, \overline{C}_m\}$ be the collection of cycles of $\overline{G}_C$. Also, let $\mathscr{I} = \{\Gamma_0, \ldots, \Gamma_m\}$ be the collection of cycle images of $\mathscr{C}$, and let $\overline{\mathscr{I}} = \{\overline{\Gamma}_0, \ldots, \overline{\Gamma}_m\}$ be the collection of cycle images of $\overline{\mathscr{C}}$. Since $G_C = \overline{G}_C$, it follows that $\mathscr{C} = \overline{\mathscr{C}}$, from which we have $\mathscr{I} = \overline{\mathscr{I}}$. Therefore, by Corollary 17, the union of cycle images of G_C, that is, $\Gamma_C = \bigcup_{j=0}^{m} \Gamma_j$, is a symmetric subgraph of Γ. Supposing now that Γ_C is symmetric, let (d_1, d_2) be an edge of G_C. Then, (d_1, d_2) is an edge label of Γ_C. By our assumption, we may say that $(\overline{d}_1, \overline{d}_2)$ is also an edge label of Γ_C, which is only possible if $(\overline{d}_1, \overline{d}_2)$ is an edge of G_C. It follows that (d_1, d_2) is also an edge of $\overline{G}_C$. A similar argument shows that the edges of $\overline{G}_C$ are contained in the edges of G_C. Therefore, we may conclude that $G_C = \overline{G}_C$. □

Example 5. We again consider the $(4, 10)$-permutiple $p = (8, 6, 7, 1, 2)_{10} = 4 \cdot (2, 1, 6, 7, 8)_{10}$ and the class C with graph $G_C = G_p$, which is depicted in Figure 10. The graph, $G_{\widehat{C}}$, of the symmetric closure, $\widehat{C}$, is seen in Figure 11 and is a symmetric graph as guaranteed by Theorem 18.

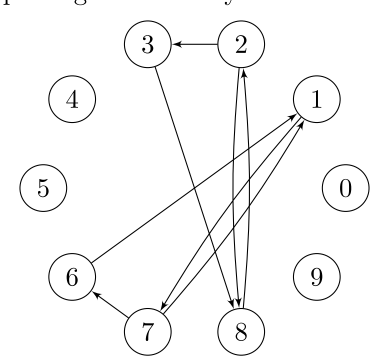


Figure 11: The graph, $G_{\widehat{C}}$, of the symmetric closure, $\widehat{C}$, of C.

The cycle images of $\widehat{C}$ are Γ_0, Γ_1, $\overline{\Gamma}_0$, and $\overline{\Gamma}_1$, given in Example 4. The union of these, $\Gamma_{\widehat{C}}$, is shown in Figure 12. We note that the configuration of the vertices and edges is to make the literal reflective symmetry more obvious.

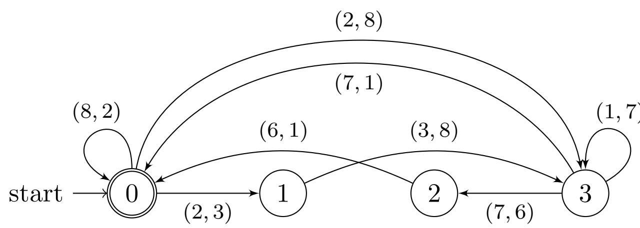


Figure 12: The cycle-image union, $\Gamma_{\widehat{C}} = \Gamma_0 \cup \Gamma_1 \cup \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, of the cycles of $G_{\widehat{C}}$.

We see that both $(8, 6, 7, 1, 2)_{10} = 4 \cdot (2, 1, 6, 7, 8)_{10}$ and $(8, 7, 1, 3, 2)_{10} = 4 \cdot (2, 1, 7, 8, 3)_{10}$, given in Example 4, are members of the symmetric closure of C.

5 New permutiples from old

We begin this section with a corollary to the argument in the proof of Theorem 15.

Corollary 19. *Let* $(d_k, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, \ldots, d_{\sigma(0)})_b$ *be an* (n, b, σ)*-permutiple with carry sequence* $c_k, \ldots, c_1, c_0 = 0$*. Then, for every* $0 < j \leq k$ *such that* $c_j = n - 1$*, we have that* $(\overline{d}_{\psi^j(k)}, \ldots, \overline{d}_{\psi^j(0)})_b = n \cdot (\overline{d}_{\sigma\psi^j(k)}, \ldots, \overline{d}_{\sigma\psi^j(0)})_b$ *is an* $(n, b, \psi^{-j}\sigma\psi^j)$*-permutiple with carry sequence* $\overline{c}_{\psi^j(k)}, \ldots, \overline{c}_{\psi^j(1)}, \overline{c}_{\psi^j(0)} = 0$.

The above prompts us to define some additional terminology.

Definition 10. Suppose $p = (d_k, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, \ldots, d_{\sigma(0)})_b$ is an (n, b, σ)-permutiple with carry sequence $c_k, \ldots, c_1, c_0 = 0$, where $c_j = n - 1$. Then, we call the permutiple $\overline{p}_{\psi^j} = (\overline{d}_{\psi^j(k)}, \ldots, \overline{d}_{\psi^j(0)})_b = n \cdot (\overline{d}_{\sigma\psi^j(k)}, \ldots, \overline{d}_{\sigma\psi^j(0)})_b$ a *reflective sibling* of p.

Remark 20. We draw the reader's attention to the use of the indefinite article in the above definition; for each $0 < j \leq k$ for which $c_j = n - 1$, the permutiple p has a reflective sibling.

Example 6. We consider the $(4, 10)$-permutiple featured in Example 4, namely,

$$p = (d_4, d_3, d_2, d_1, d_0) = (8, 6, 7, 1, 2)_{10} = 4 \cdot (2, 1, 6, 7, 8)_{10}.$$

Since its carry vector, $(c_4, c_3, c_2, c_1, c_0) = (0, 2, 3, 3, 0)$, contains two values of $n - 1 = 3$, Corollary 19 guarantees that p has two reflective siblings. For $c_1 = 3$,

$$\begin{aligned}
\overline{p}_{\psi} &= (\overline{d}_{\psi(4)}, \overline{d}_{\psi(3)}, \overline{d}_{\psi(2)}, \overline{d}_{\psi(1)}, \overline{d}_{\psi(0)})_{10} \\
&= (\overline{d}_0, \overline{d}_4, \overline{d}_3, \overline{d}_2, \overline{d}_1)_{10} \\
&= (7, 1, 3, 2, 8)_{10} \\
&= 4 \cdot (1, 7, 8, 3, 2)_{10} \\
&= 4 \cdot (\overline{d}_4, \overline{d}_0, \overline{d}_1, \overline{d}_3, \overline{d}_2)_{10} \\
&= 4 \cdot (\overline{d}_{\sigma\psi(4)}, \overline{d}_{\sigma\psi(3)}, \overline{d}_{\sigma\psi(2)}, \overline{d}_{\sigma\psi(1)}, \overline{d}_{\sigma\psi(0)})_{10},
\end{aligned}$$

and for $c_2 = 3$,

$$\begin{aligned}
\overline{p}_{\psi^2} &= (\overline{d}_{\psi^2(4)}, \overline{d}_{\psi^2(3)}, \overline{d}_{\psi^2(2)}, \overline{d}_{\psi^2(1)}, \overline{d}_{\psi^2(0)})_{10} \\
&= (\overline{d}_1, \overline{d}_0, \overline{d}_4, \overline{d}_3, \overline{d}_2)_{10} \\
&= (8, 7, 1, 3, 2)_{10} \\
&= 4 \cdot (2, 1, 7, 8, 3)_{10} \\
&= 4 \cdot (\overline{d}_2, \overline{d}_4, \overline{d}_0, \overline{d}_1, \overline{d}_3)_{10} \\
&= 4 \cdot (\overline{d}_{\sigma\psi^2(4)}, \overline{d}_{\sigma\psi^2(3)}, \overline{d}_{\sigma\psi^2(2)}, \overline{d}_{\sigma\psi^2(1)}, \overline{d}_{\sigma\psi^2(0)})_{10},
\end{aligned}$$

the latter of which, was also mentioned in Example 4.

From past work [11, 12], we have also seen that permutiples with zero carries other than $c_0 = 0$ enable us to find new permutiples. Specifically, we restate a result from previous work [11, Corollary 1].

Theorem 21. *Suppose $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-permutiple with carries $c_k, c_{k-1}, \ldots, c_0$. If $c_j = 0$, then $(d_{\psi^j(k)}, d_{\psi^j(k-1)}, \ldots, d_{\psi^j(1)}, d_{\psi^j(0)})_b$ is an $(n, b, \psi^{-j}\sigma\psi^j)$-permutiple with carries $c_{\psi^j(k)}$, $c_{\psi^j(k-1)}$, $\ldots$, $c_{\psi^j(1)}$, $c_{\psi^j(0)} = c_j = 0$.*

The above inspires the next two definitions.

Definition 11. Suppose $p = (d_k, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, \ldots, d_{\sigma(0)})_b$ is an (n, b, σ)-permutiple with carry sequence $c_k, \ldots, c_1, c_0 = 0$, where $c_j = 0$. Then, we call the permutiple $p_{\psi^j} = (d_{\psi^j(k)}, \ldots, d_{\psi^j(0)})_b = n \cdot (d_{\sigma\psi^j(k)}, \ldots, d_{\sigma\psi^j(0)})_b$ a *rotational sibling* of p.

Definition 12. Suppose p is an (n, b)-permutiple. The collection of reflective and rotational siblings of p together are called the *dihedral siblings* of p.

Example 7. Once again, we consider the $(4, 10)$-permutiple

$$p = (d_4, d_3, d_2, d_1, d_0) = (8, 6, 7, 1, 2)_{10} = 4 \cdot (2, 1, 6, 7, 8)_{10}$$

with carry vector $(c_4, c_3, c_2, c_1, c_0) = (0, 2, 3, 3, 0)$. In Example 6 we found the two reflective siblings of p. We now find its rotational siblings. In doing so, we complete the collection of the dihedral siblings of p.

By Theorem 21, every permutiple is trivially its own rotational sibling since c_0 must always be zero. On the other hand, we have a nontrivial rotational sibling for $c_4 = 0$, namely,

$$\begin{aligned}
p_{\psi^4} &= (d_{\psi^4(4)}, d_{\psi^4(3)}, d_{\psi^4(2)}, d_{\psi^4(1)}, d_{\psi^4(0)})_{10} \\
&= (d_3, d_2, d_1, d_0, d_4)_{10} \\
&= (6, 7, 1, 2, 8)_{10} \\
&= 4 \cdot (1, 6, 7, 8, 2)_{10} \\
&= 4 \cdot (d_1, d_3, d_2, d_4, d_0)_{10} \\
&= 4 \cdot (d_{\sigma\psi^4(4)}, d_{\sigma\psi^4(3)}, d_{\sigma\psi^4(2)}, d_{\sigma\psi^4(1)}, d_{\sigma\psi^4(0)})_{10}.
\end{aligned}$$

We see that, altogether, p has four dihedral siblings: p, p_{ψ^4}, $\overline{p}_{\psi}$, and $\overline{p}_{\psi^2}$.

It is here that we say more about our chosen terminology. Let C be an (n, b)-permutiple class for which Γ_C has the state $n-1$ as a vertex. We consider its symmetric closure, $\widehat{C}$. Letting $\mathscr{I}$ be the collection of cycle images of the cycles of $G_{\widehat{C}}$, we have seen, in general, that the union of the elements of $\mathscr{I}$ is a symmetric subgraph, $\Gamma_{\widehat{C}}$, of the (n, b)-Hoey-Sloane graph. We now consider the collection of closed walks, $\mathscr{W}_\ell$, on $\Gamma_{\widehat{C}}$ which have a fixed length, ℓ. To denote an element of $\mathscr{W}_\ell$, we use the notation $(c_0, c_1, \ldots, c_{\ell-1})$ to mean the state-transition sequence

$$\{(c_0, c_1), (c_1, c_2), \ldots, (c_{\ell-2}, c_{\ell-1}), (c_{\ell-1}, c_0)\}$$

on $\Gamma_{\widehat{C}}$ and its corresponding edge-label sequence,

$$\{(d_0, \widehat{d}_0), (d_1, \widehat{d}_1), \ldots, (d_{\ell-2}, \widehat{d}_{\ell-2}), (d_{\ell-1}, \widehat{d}_{\ell-1})\},$$

where $(d_j, \widehat{d}_j)$ induces the transition from c_j to c_{j+1}. We use this notation regardless of whether the collection is a cycle or a circuit on $\Gamma_{\widehat{C}}$.

Under these conditions, the dihedral group, D_ℓ, acts on $\mathscr{W}_\ell$ by cyclically permuting vertices, edges, and labels, as well as reflecting vertices, edges, and labels in the sense established by Definition 7. More specifically, letting α be a $360°/\ell$ rotation and β a reflection, we may define the action by how these generating elements act on $\mathscr{W}_\ell : \alpha w = (c_{\psi(0)}, c_{\psi(1)}, \ldots, c_{\psi(\ell-1)})$, where ψ is the ℓ-cycle $(0, 1, \ldots, \ell-1)$, and $\beta w = (\overline{c}_0, \overline{c}_1, \ldots, \overline{c}_{\ell-1})$.

In this setting, we may now say that two permutiples, p_1 and p_2, are dihedral siblings if there is a composition of dihedral transformations which acts on the L-walk, w_1, of p_1 to produce the L-walk, w_2, of p_2 (that is, if w_1 and w_2 are in the same orbit under the action of D_ℓ).

To illustrate the above, we consider the dihedral siblings listed in Example 7. The L-walks of p and p_{ψ^4} are visualized in the shape of a regular polygon in Figure 13.

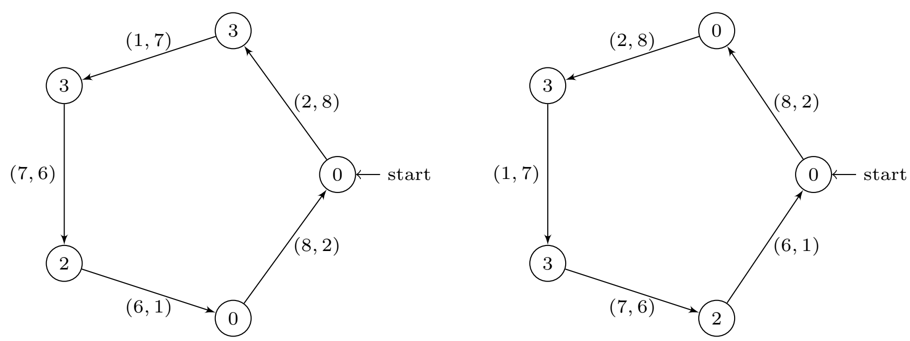


Figure 13: The L-walks corresponding to $p = (8, 6, 7, 1, 2)_{10} = 4 \cdot (2, 1, 6, 7, 8)_{10}$ (left) and its rotational sibling, $p_{\psi^4} = (6, 7, 1, 2, 8)_{10} = 4 \cdot (1, 6, 7, 8, 2)_{10}$ (right).

The reflection of the L-walk corresponding to p is visualized in Figure 14.

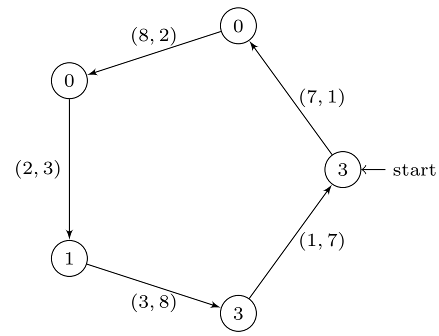


Figure 14: The closed walk resulting from the reflection of the L-walk corresponding to p.

Although the result depicted in Figure 14 is not an L-walk, there are two rotations which do result in L-walks. These are visualized in Figure 15, and correspond to the reflective siblings of p.

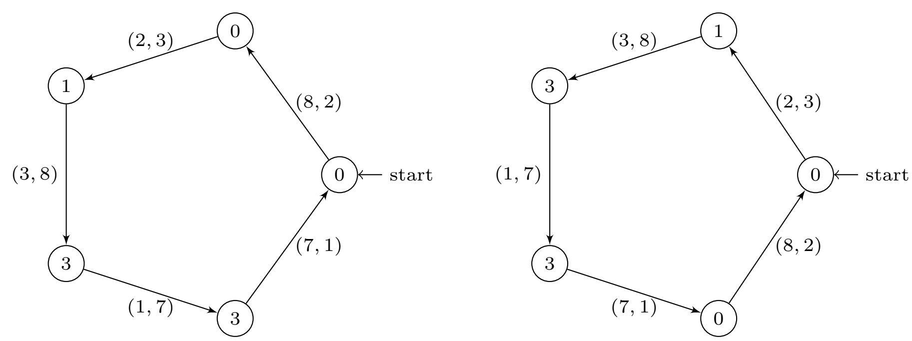


Figure 15: The L-walks corresponding to the reflective siblings of p: $\overline{p}_{\psi} = (7, 1, 3, 2, 8)_{10} = 4 \cdot (1, 7, 8, 3, 2)_{10}$ (left) and $\overline{p}_{\psi^2} = (8, 7, 1, 3, 2)_{10} = 4 \cdot (2, 1, 7, 8, 3)_{10}$ (right).

Here we add that when the configuration of vertices and edges highlights a literal line of symmetry of Γ_C inherited from the mother graph, as is the case in Figure 12, the reflection of an L-walk results in a walk on Γ_C which is a literal reflection over this line of symmetry. Rotations acting on such walks result in L-walks when the zero state occupies the initial position as seen above. These represent the reflective siblings of some initial permutiple. The above helps to explain not only the nomenclature adopted by this effort, but also, perhaps more importantly, the types of permutations which arise when finding new permutiples of a fixed length from old [11, 12] (see Table 1).

The above observations also demonstrate that an (n, b)-permutiple, p, having at least one carry equal to $n-1$ has a reflective sibling consisting of the reflected and cyclically-permuted digits of p. If p belongs to a symmetric class, C, then every reflective sibling of p is still a member of C. More specifically, when the reflected digits form the same multiset as the digits of the original example, we can say more.

Corollary 22. *Suppose that* $p = (d_{\pi(k)}, \ldots, d_{\pi(0)})_b = n \cdot (d_{\pi\sigma(k)}, \ldots, d_{\pi\sigma(0)})_b$ *is an* (n, b, σ)*-permutiple whose* j^{th} *carry is* $n - 1$*. If* $\{d_0, d_1, \ldots, d_k\}$ *and* $\{\overline{d}_0, \overline{d}_1, \ldots, \overline{d}_k\}$ *are the same multiset, where* $d_0 \leq d_1 \leq \cdots \leq d_k$*, then the reflective sibling of* p*,*

$$\overline{p}_{\psi^j} = (\overline{d}_{\pi\psi^j(k)}, \ldots, \overline{d}_{\pi\psi^j(0)})_b = n \cdot (\overline{d}_{\pi\sigma\psi^j(k)}, \ldots, \overline{d}_{\pi\sigma\psi^j(0)})_b,$$

may also be represented as

$$\overline{p}_{\psi^j} = (d_{\rho\pi\psi^j(k)}, \ldots, d_{\rho\pi\psi^j(0)})_b = n \cdot (d_{\rho\pi\sigma\psi^j(k)}, \ldots, d_{\rho\pi\sigma\psi^j(0)})_b,$$

where ρ *is the reversal permutation.*

Proof. Since the reflected digits of p yield the same multiset, we may say that $\overline{d}_0 = d_{\rho(0)} \geq \overline{d}_1 = d_{\rho(1)} \geq \cdots \geq \overline{d}_k = d_{\rho(k)}$. Since $\overline{d}_j = d_{\rho(j)}$ for all $0 \leq j \leq k$, the statement follows. □

Given an (n, b)-permutiple, $p = (d_k, \ldots, d_0)_b$, whose j^{th} carry is $n-1$, it is not difficult to manufacture examples which allow for the application of Corollary 22. Let C be the permutiple class whose graph is $G_C = G_p$. We consider the reflective sibling, $\overline{p}_{\psi^j} = (\overline{d}_{\psi^j(k)}, \ldots, \overline{d}_{\psi^j(0)})_b$, of p, and its class, $\overline{C}$, with graph $\overline{G}_C = G_{\overline{p}_{\psi^j}}$. The multiset union of the digits of p and $\overline{p}_{\psi^j}$ give us a collection of digits whose reflection is itself. Thus, the concatenation of the digit string of p and $\overline{p}_{\psi^j}$ give us an (n, b)-permutiple, $\widehat{p}$, which is a member of the symmetric closure of C, and to which we may apply Corollary 22. Of course, generally speaking, $\widehat{p}$ is only one of many members of $\widehat{C}$ with the same digits. We may easily find other examples by using the multi-image union, Δ_I, corresponding to the multiset union of mother-graph cycles, C_I, which forms the collection of inputs constituting the permutiple string of $\widehat{p}$. Counting such examples becomes a problem of counting multi-Eulerian circuits of a directed graph, which is a problem taken up by Farrell and Levine [5].

The next example illustrates the above points, verifies Corollary 22, and completes our consideration of the class of $(4, 10)$-permutiples featured in Examples 4 through 7.

Example 8. Consider the $(4, 10)$-permutiple example $p = (8, 6, 7, 1, 2)_{10} = 4 \cdot (2, 1, 6, 7, 8)_{10}$, whose second carry is $n - 1 = 3$, and its reflective sibling, $\overline{p}_{\psi^2} = (8, 7, 1, 3, 2)_{10} = 4 \cdot (2, 1, 7, 8, 3)_{10}$, given in Example 6. A concatenation of these two examples gives an element,

$$\widehat{p} = (8, 6, 7, 1, 2, 8, 7, 1, 3, 2)_{10} = 4 \cdot (2, 1, 6, 7, 8, 2, 1, 7, 8, 3)_{10},$$

of $\widehat{C}$ whose corresponding permutiple string is

$$\widehat{s} = (2, 3)(3, 8)(1, 7)(7, 1)(8, 2)(2, 8)(1, 7)(7, 6)(6, 1)(8, 2).$$

The inputs of $\widehat{s}$ make up the multiset union of mother-graph cycles for this example, $C_I = C_0 \uplus C_1 \uplus \overline{C}_0 \uplus \overline{C}_1$, and the constituent cycles are shown in Tables 4 and 5. The corresponding multi-images are Δ_0, Δ_1, $\overline{\Delta}_0$, and $\overline{\Delta}_1$ (where multigraph reflection is defined analogously as in Definition 7), and these multigraphs look identical to Γ_0, Γ_1, $\overline{\Gamma}_0$, and $\overline{\Gamma}_1$ (also shown in Tables 4 and 5). The multi-image union $\Delta_I = \Delta_0 \uplus \Delta_1 \uplus \overline{\Delta}_0 \uplus \overline{\Delta}_1$ corresponding to C_I is shown in Figure 16, and the reader may verify that Δ_I satisfies the conditions of Corollary 7. The reader may also observe that Δ_I differs from the set-theoretic union, Γ_C, shown in Figure 12.

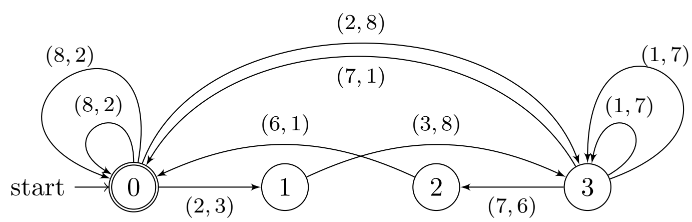


Figure 16: The multi-image union, $\Delta_I = \Delta_0 \uplus \Delta_1 \uplus \overline{\Delta}_0 \uplus \overline{\Delta}_1$, corresponding to the multiset union $C_I = C_0 \uplus C_1 \uplus \overline{C}_0 \uplus \overline{C}_1$ of the cycles of $G_{\widehat{C}}$.

As mentioned more generally, we may use Δ_I in Figure 16 to create new orderings of C_I which produce permutiple strings. For instance, the input string

$$s = (2,3)(3,8)(1,7)(1,7)(7,6)(6,1)(8,2)(8,2)(2,8)(7,1)$$

determines an Eulerian circuit on Δ_I from the zero state back to itself (this is precisely what the conditions of Corollary 7 guarantee [2, 5]). This is to say that s is a permutiple string.

We now verify Corollary 22 by considering the $(4,10)$-permutiple corresponding to s,

$$q = (7,2,8,8,6,7,1,1,3,2)_{10} = 4 \cdot (1,8,2,2,1,6,7,7,8,3)_{10},$$

with carry vector $(c_9, c_8, c_7, c_6, c_5, c_4, c_3, c_2, c_0) = (3,0,0,0,2,3,3,3,1,0)$. The ordered multiset of digits of both $\widehat{p}$ and q is

$$\{d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9\} = \{1,1,2,2,3,6,7,7,8,8\},$$

and clearly $\overline{d}_j = d_{\rho(j)}$ for all $0 \leq j \leq 9$, where ρ is the reversal permutation, $\rho(j) = 9 - j$. Choose $\pi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 0 & 1 & 6 & 5 & 8 & 9 & 3 & 7 \end{pmatrix}$ and

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 6 & 4 & 9 & 5 & 2 & 0 & 8 & 7 & 3 \end{pmatrix} \text{ so that}$$

$$\begin{aligned} q &= (d_{\pi(9)}, d_{\pi(8)}, \ldots, d_{\pi(1)}, d_{\pi(0)})_{10} \\ &= (d_7, d_3, d_9, d_8, d_5, d_6, d_1, d_0, d_4, d_2)_{10} \\ &= (7, 2, 8, 8, 6, 7, 1, 1, 3, 2)_{10} \\ &= 4 \cdot (1, 8, 2, 2, 1, 6, 7, 7, 8, 3)_{10} \\ &= 4 \cdot (d_1, d_9, d_3, d_2, d_0, d_5, d_7, d_6, d_8, d_4)_{10} \\ &= 4 \cdot (d_{\pi\sigma(9)}, d_{\pi\sigma(8)}, \ldots, d_{\pi\sigma(1)}, d_{\pi\sigma(0)})_{10}. \end{aligned}$$

For $c_2 = 3$, we have a reflective sibling,

$$\begin{aligned} \overline{q}_{\psi^2} &= (\overline{d}_{\pi\psi^2(9)}, \overline{d}_{\pi\psi^2(8)}, \ldots, \overline{d}_{\pi\psi^2(1)}, \overline{d}_{\pi\psi^2(0)})_{10} \\ &= (\overline{d}_4, \overline{d}_2, \overline{d}_7, \overline{d}_3, \overline{d}_9, \overline{d}_8, \overline{d}_5, \overline{d}_6, \overline{d}_1, \overline{d}_0)_{10} \\ &= (\overline{3}, \overline{2}, \overline{7}, \overline{2}, \overline{8}, \overline{8}, \overline{6}, \overline{7}, \overline{1}, \overline{1})_{10} \\ &= (6, 7, 2, 7, 1, 1, 3, 2, 8, 8)_{10}. \end{aligned}$$

With all of its hypotheses met, we may now invoke Corollary 22:

$$\begin{aligned} \overline{q}_{\psi^2} &= (d_{\rho\pi\psi^2(9)}, d_{\rho\pi\psi^2(8)}, \ldots, d_{\rho\pi\psi^2(1)}, d_{\rho\pi\psi^2(0)})_{10} \\ &= (d_5, d_7, d_2, d_6, d_0, d_1, d_4, d_3, d_8, d_9)_{10} \\ &= (6, 7, 2, 7, 1, 1, 3, 2, 8, 8)_{10} \\ &= 4 \cdot (1, 6, 8, 1, 7, 7, 8, 3, 2, 2)_{10} \\ &= 4 \cdot (d_1, d_5, d_8, d_0, d_6, d_7, d_9, d_4, d_2, d_3)_{10} \\ &= 4 \cdot (d_{\rho\pi\sigma\psi^2(9)}, d_{\rho\pi\sigma\psi^2(8)}, \ldots, d_{\rho\pi\sigma\psi^2(1)}, d_{\rho\pi\sigma\psi^2(0)})_{10}. \end{aligned}$$

As the above illustrates, Corollary 22 implicitly assumes that a permutiple, p, resides within a symmetric class, C.

Bringing together several of the ideas we have discussed so far, we now consider a base-4 example.

Example 9. Consider the $(3, 4)$-permutiple example $p = (3, 1, 1, 0, 2, 2)_4 = 3 \cdot (1, 0, 1, 2, 3, 2)_4$ with carry vector $(c_5, c_4, c_3, c_2, c_1, c_0) = (0, 1, 2, 2, 1, 0)$. Let C be the permutiple class determined by the graph of p. The $(3, 4)$-mother graph is seen in Figure 17 with the edges of $G_C = G_p$ highlighted in bold red. The cycles and cycle images of G_C are given in Table 6.

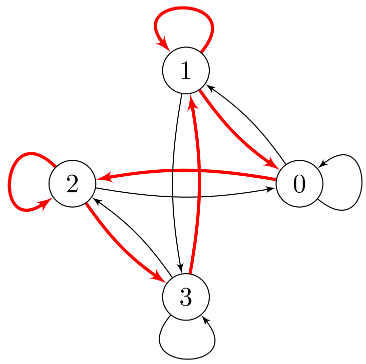


Figure 17: The $(3, 4)$-mother graph with the edges of $G_C = G_p$ featured in bold red.

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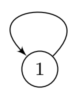
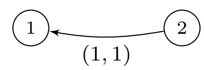
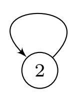
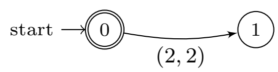
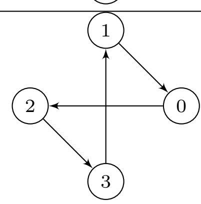
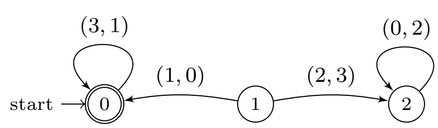
	Cycle of G_C	**Cycle Image**	
C_1			Γ_1
C_2			Γ_2
C_3			Γ_3

Table 6: The cycles of G_C and their corresponding cycle images.

The union of the cycle images, $\Gamma_C = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, is a labeled subgraph of the $(3, 4)$-Hoey-Sloane graph, and is shown in Figure 18 with the labeled edges of Γ_C highlighted in bold red. Also, the configuration of the vertices and edges emphasizes a literal vertical line of symmetry of both Γ and Γ_C inherited from the mother graph. The permutiple string, $s = (2, 2)(2, 3)(0, 2)(1, 1)(1, 0)(3, 1)$, of p determines the L-walk on Γ_C.

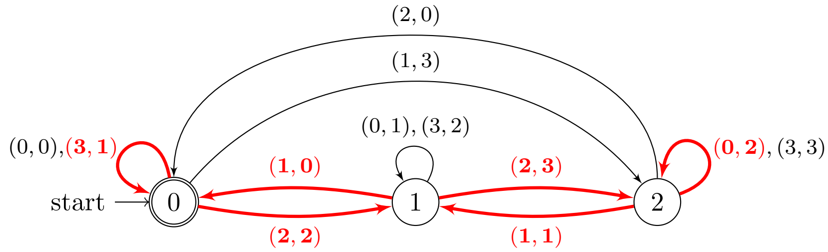


Figure 18: The $(3, 4)$-Hoey-Sloane graph with the labeled edges of Γ_C featured in bold red.

The ordered multiset of the digits of p is

$$\{d_0, d_1, d_2, d_3, d_4, d_5\} = \{0, 1, 1, 2, 2, 3\} = \{\overline{0}, \overline{1}, \overline{1}, \overline{2}, \overline{2}, \overline{3}\},$$

and clearly $\overline{d}_j = d_{\rho(j)}$ for all $0 \leq j \leq 5$, where ρ is the reversal permutation, $\rho(j) = 5 - j$. To apply Corollary 22, we choose $\pi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 0 & 2 & 1 & 5 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 5 & 1 & 3 & 2 & 4 \end{pmatrix}$ so that

$$\begin{aligned} p &= (d_{\pi(5)}, d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_4 \\ &= (d_5, d_1, d_2, d_0, d_3, d_4)_4 \\ &= (3, 1, 1, 0, 2, 2)_4 \\ &= 3 \cdot (1, 0, 1, 2, 3, 2)_4 \\ &= 3 \cdot (d_1, d_0, d_2, d_3, d_5, d_4)_4 \\ &= 3 \cdot (d_{\pi\sigma(5)}, d_{\pi\sigma(4)}, d_{\pi\sigma(3)}, d_{\pi\sigma(2)}, d_{\pi\sigma(1)}, d_{\pi\sigma(0)})_4. \end{aligned}$$

Now, since $c_2 = 2$, Corollary 19 gives us the reflective sibling

$$\begin{aligned} \overline{p}_{\psi^2} &= (\overline{d}_{\pi\psi^2(5)}, \overline{d}_{\pi\psi^2(4)}, \overline{d}_{\pi\psi^2(3)}, \overline{d}_{\pi\psi^2(2)}, \overline{d}_{\pi\psi^2(1)}, \overline{d}_{\pi\psi^2(0)})_4 \\ &= (\overline{d}_3, \overline{d}_4, \overline{d}_5, \overline{d}_1, \overline{d}_2, \overline{d}_0)_4 \\ &= (\overline{2}, \overline{2}, \overline{3}, \overline{1}, \overline{1}, \overline{0})_4 \\ &= (1, 1, 0, 2, 2, 3)_4. \end{aligned}$$

We now have everything we need to apply and verify Corollary 22:

$$\begin{aligned} \overline{p}_{\psi^2} &= (d_{\rho\pi\psi^2(5)}, d_{\rho\pi\psi^2(4)}, d_{\rho\pi\psi^2(3)}, d_{\rho\pi\psi^2(2)}, d_{\rho\pi\psi^2(1)}, d_{\rho\pi\psi^2(0)})_4 \\ &= (d_2, d_1, d_0, d_4, d_3, d_5)_4 \\ &= (1, 1, 0, 2, 2, 3)_4 \\ &= 3 \cdot (0, 1, 2, 3, 2, 1)_4 \\ &= 3 \cdot (d_0, d_1, d_4, d_5, d_3, d_2)_4 \\ &= (d_{\rho\pi\sigma\psi^2(5)}, d_{\rho\pi\sigma\psi^2(4)}, d_{\rho\pi\sigma\psi^2(3)}, d_{\rho\pi\sigma\psi^2(2)}, d_{\rho\pi\sigma\psi^2(1)}, d_{\rho\pi\sigma\psi^2(0)})_4. \end{aligned}$$

Applying the above to $c_3 = 2$, we see that the reflective sibling $\overline{p}_{\psi^3}$ is simply p.

Expressing the above in terms of the action of D_6 on $\mathscr{W}_6$, the representation of the L-walk of p and its reflection (not an L-walk) is shown in Figure 19. A clockwise rotation by two vertices of the graph on the right gives us the representation of the L-walk of the reflective sibling, $\overline{p}_{\psi^2}$, of p.

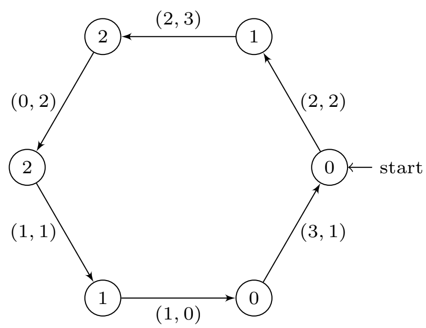


Figure 19: The L-walk of p (left) and its reflection (right).

An interesting feature of the above example is that the reflection of w is also a rotation of w, which we may directly observe in Figure 19.

5.1 Other symmetries involving dihedral siblings

In other work [13], we have observed cases where transposing edge labels is possible by maintaining the same state-transition sequence. We now investigate this question more generally, and begin with a definition.

Definition 13. Let $p = (d_k, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, \ldots, d_{\sigma(0)})_b$, be an (n, b, σ)-permutiple and let $s = (d_0, d_{\sigma(0)})(d_1, d_{\sigma(1)}) \cdots (d_{k-1}, d_{\sigma(k-1)})(d_k, d_{\sigma(k)})$ be its corresponding permutiple string. A permutation, φ, of the inputs of s which results in another permutiple string is called a *symmetry* of p.

Under this definition, all rotational siblings are the result of applying a symmetry of the form ψ^j to the L-walk corresponding to p.

We recall that an edge label, (d_1, d_2), cannot appear on distinct edges of Γ [14, Theorem 5], which makes explicit a basic fact.

Corollary 23. *Suppose that* $(d_k, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, \ldots, d_{\sigma(0)})_b$ *is an* (n, b, σ)*-permutiple, and let* $s = (d_0, d_{\sigma(0)})(d_1, d_{\sigma(1)}) \cdots (d_{k-1}, d_{\sigma(k-1)})(d_k, d_{\sigma(k)})$ *be its corresponding permutiple string. Then, s uniquely defines an* L*-walk on* Γ*.*

In terms of symmetries, we have another result.

Corollary 24. *Suppose that* $p = (d_k, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, \ldots, d_{\sigma(0)})_b$ *is an* (n, b, σ)*-permutiple, let* $S = \{(c_0, c_1), (c_1, c_2), \ldots, (c_{k-1}, c_k), (c_k, c_0)\}$ *be its state-transition sequence, and let* $s = (d_0, d_{\sigma(0)})(d_1, d_{\sigma(1)}) \cdots (d_{k-1}, d_{\sigma(k-1)})(d_k, d_{\sigma(k)})$

be its corresponding permutiple string. If φ is a symmetry of p, then the state-transition sequence of the permutiple string

$$s_\varphi = (d_{\varphi(0)}, d_{\sigma\varphi(0)})(d_{\varphi(1)}, d_{\sigma\varphi(1)}) \cdots (d_{\varphi(k-1)}, d_{\sigma\varphi(k-1)})(d_{\varphi(k)}, d_{\sigma\varphi(k)})$$

is $S_\varphi = \{(c_{\varphi(0)}, c_{\varphi(1)}), (c_{\varphi(1)}, c_{\varphi(2)}), \ldots, (c_{\varphi(k-1)}, c_{\varphi(k)}), (c_{\varphi(k)}, c_{\varphi(0)})\}$.

For a permutiple string, s, we see that transposing inputs, $(d_i, d_{\sigma(i)})$ and $(d_j, d_{\sigma(j)})$, for which their associated state transitions, (c_i, c_{i+1}) and (c_j, c_{j+1}), are equal does not change the path traversed on Γ. That is, we may transpose $(d_i, d_{\sigma(i)})$ and $(d_j, d_{\sigma(j)})$ in s and still have a permutiple string. Furthermore, every symmetry, φ, which leaves S fixed also results in a permutiple string,

$$s_\varphi = (d_{\varphi(0)}, d_{\sigma\varphi(0)})(d_{\varphi(1)}, d_{\sigma\varphi(1)}) \cdots (d_{\varphi(k-1)}, d_{\sigma\varphi(k-1)})(d_k, d_{\sigma\varphi(k)}),$$

yielding an $(n, b, \varphi^{-1}\sigma\varphi)$-permutiple,

$$p_\varphi = (d_{\varphi(k)}, \ldots, d_{\varphi(0)})_b = n \cdot (d_{\sigma\varphi(k)}, \ldots, d_{\sigma\varphi(0)})_b.$$

The next result details the interaction between the symmetries of p which fix S, and the dihedral siblings of p.

Theorem 25. *Let $p = (d_k, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, \ldots, d_{\sigma(0)})_b$ be an (n, b, σ)-permutiple with carries $c_k, \ldots, c_0 = 0$, and let S be the state-transition sequence of the L-walk of p. Also, let φ be a symmetry of p which fixes S, and let ψ be the $(k+1)$-cycle $(0, 1, \ldots, k)$. If p_{ψ^j} is a rotational sibling of p, then $\varphi\psi^j$ is also a symmetry of p. In particular,*

$$p_{\varphi\psi^j} = (d_{\varphi\psi^j(k)}, \ldots, d_{\varphi\psi^j(0)})_b = n \cdot (d_{\sigma\varphi\psi^j(k)}, \ldots, d_{\sigma\varphi\psi^j(0)})_b$$

is an $(n, b, \psi^{-j}\varphi^{-1}\sigma\varphi\psi^j)$-permutiple. If $\overline{p}_{\psi^j}$ is a reflective sibling of p, then

$$\overline{p}_{\varphi\psi^j} = (\overline{d}_{\varphi\psi^j(k)}, \ldots, \overline{d}_{\varphi\psi^j(0)})_b = n \cdot (\overline{d}_{\sigma\varphi\psi^j(k)}, \ldots, \overline{d}_{\sigma\varphi\psi^j(0)})_b$$

is an $(n, b, \psi^{-j}\varphi^{-1}\sigma\varphi\psi^j)$-permutiple.

Proof. Let $S = \{(c_0, c_1), (c_1, c_2), \ldots, (c_{k-1}, c_k), (c_k, c_0)\}$ be the state-transition sequence of the L-walk of p, and let

$$s = (d_0, d_{\sigma(0)})(d_1, d_{\sigma(1)}) \cdots (d_{k-1}, d_{\sigma(k-1)})(d_k, d_{\sigma(k)})$$

be its corresponding permutiple string. Applying φ to s results in the same walk traversed on Γ. Thus, since p_{ψ^j} is a rotational sibling of p, applying ψ^j to s also results in an L-walk. Thus, applying φ and ψ^j to the inputs of s results in another permutiple string,

$$s_{\varphi\psi^j} = (d_{\varphi\psi^j(0)}, d_{\sigma\varphi\psi^j(0)})(d_{\varphi\psi^j(1)}, d_{\sigma\varphi\psi^j(1)}) \cdots (d_{\varphi\psi^j(k)}, d_{\sigma\varphi\psi^j(k)}).$$

Applying a reflection to each digit of p, we obtain a walk on Γ with state-transition sequence $\overline{S} = \{(\overline{c}_0, \overline{c}_1), (\overline{c}_1, \overline{c}_2), \ldots, (\overline{c}_{k-1}, \overline{c}_k), (\overline{c}_k, \overline{c}_0)\}$, which is not an L-walk since $\overline{c}_0 = \overline{0} = n-1$. Since φ fixes the elements of S, it also fixes the elements of $\overline{S}$, so that

$$\overline{S}_\varphi = \{(\overline{c}_{\varphi(0)}, \overline{c}_{\varphi(1)}), (\overline{c}_{\varphi(1)}, \overline{c}_{\varphi(2)}), \ldots, (\overline{c}_{\varphi(k-1)}, \overline{c}_{\varphi(k)}), (\overline{c}_{\varphi(k)}, \overline{c}_{\varphi(0)})\} = \overline{S},$$

which, again, is not an L-walk. However, since $\overline{p}_{\psi^j}$ is a reflective sibling of p, we have $\overline{c}_j = \overline{n-1} = 0$. It follows that

$$\overline{S}_{\varphi\psi^j} = \{(\overline{c}_{\varphi\psi^j(0)}, \overline{c}_{\varphi\psi^j(1)}), (\overline{c}_{\varphi\psi^j(1)}, \overline{c}_{\varphi\psi^j(2)}), \ldots, (\overline{c}_{\varphi\psi^j(k)}, \overline{c}_{\varphi\psi^j(0)})\}$$

is an L-walk, from which we know that

$$\overline{s}_{\varphi\psi^j} = (\overline{d}_{\varphi\psi^j(0)}, \overline{d}_{\sigma\varphi\psi^j(0)})(\overline{d}_{\varphi\psi^j(1)}, \overline{d}_{\sigma\varphi\psi^j(1)}) \cdots (\overline{d}_{\varphi\psi^j(k)}, \overline{d}_{\sigma\varphi\psi^j(k)})$$

is a permutiple string. □

We return to the class, C, of permutiples considered in Example 2.

Example 10. Consider the $(4, 10)$-permutiple

$$p = (7,2,7,1,1,9,2,8,8)_{10} = 4 \cdot (1,8,1,7,7,9,8,2,2)_{10}$$

constructed in Example 2, whose graph is G_C from Example 1. Its permutiple string is $s = (8,2)(8,2)(2,8)(9,9)(1,7)(1,7)(7,1)(2,8)(7,1)$, and, using the $(4, 10)$-Hoey-Sloane graph in Figure 4, the corresponding state-transition sequence is

$$S = \{(0,0),(0,0),(0,3),(3,3),(3,3),(3,3),(3,0),(0,3),(3,0)\}.$$

Using S_j to denote the j^{th} element of S, with indexing beginning at $j = 0$, we see that $S_0 = S_1$, $S_3 = S_4 = S_5$, $S_2 = S_7$, and $S_6 = S_8$. Thus, the composition of any of the transpositions $(0,1)$, $(2,7)$, and $(6,8)$, as well as any permutation which fixes the set $\{3,4,5\}$, is a symmetry which fixes S.

Most of the above symmetries trivially result in the same permutiple string. On the other hand, we may consider only those symmetries which fix S, but permute the inputs of s in a nontrivial way. Those symmetries are precisely the ones which permute elements of S corresponding to distinct edge labels, such as $S_3 = (3,3)$ and $S_4 = (3,3)$, whose corresponding edge labels are $(9,9)$ and $(1,7)$, respectively. Thus, the transposition $\varphi_1 = (3,4)$ is a symmetry yielding a new permutiple string, $s_{\varphi_1} = (8,2)(8,2)(2,8)(1,7)(9,9)(1,7)(7,1)(2,8)(7,1)$. We see that $\varphi_2 = (3,5)$ is another symmetry of p which gives us the permutiple string $s_{\varphi_2} = (8,2)(8,2)(2,8)(1,7)(1,7)(9,9)(7,1)(2,8)(7,1)$. It is clear that φ_1 and φ_2 are the only nontrivial symmetries which fix the state-transition sequence (in the sense that they produce permutiple strings which are distinct from s in the usual set-theoretical context).

We may now use Theorem 25 to find the dihedral siblings of the three examples produced above,

$$\begin{aligned} p &= (7,2,7,1,1,9,2,8,8)_{10} = 4 \cdot (1,8,1,7,7,9,8,2,2)_{10}, \\ p_{\varphi_1} &= (7,2,7,1,9,1,2,8,8)_{10} = 4 \cdot (1,8,1,7,9,7,8,2,2)_{10}, \\ p_{\varphi_2} &= (7,2,7,9,1,1,2,8,8)_{10} = 4 \cdot (1,8,1,9,7,7,8,2,2)_{10}, \end{aligned}$$

by composing φ_1 and φ_2 with the symmetries ψ, ψ^2, and ψ^7 to produce their rotational siblings. The rotational siblings of p are the following:

$$\begin{aligned} p_{\psi} &= (8,7,2,7,1,1,9,2,8)_{10} = 4 \cdot (2,1,8,1,7,7,9,8,2)_{10}, \\ p_{\psi^2} &= (8,8,7,2,7,1,1,9,2)_{10} = 4 \cdot (2,2,1,8,1,7,7,9,8)_{10}, \\ p_{\psi^7} &= (7,1,1,9,2,8,8,7,2)_{10} = 4 \cdot (1,7,7,9,8,2,2,1,8)_{10}. \end{aligned}$$

Applying the rotations to p_{φ_1} and p_{φ_2}, we have the following:

$$\begin{aligned} p_{\varphi_1\psi} &= (8,7,2,7,1,9,1,2,8)_{10} = 4 \cdot (2,1,8,1,7,9,7,8,2)_{10}, \\ p_{\varphi_1\psi^2} &= (8,8,7,2,7,1,9,1,2)_{10} = 4 \cdot (2,2,1,8,1,7,9,7,8)_{10}, \\ p_{\varphi_1\psi^7} &= (7,1,9,1,2,8,8,7,2)_{10} = 4 \cdot (1,7,9,7,8,2,2,1,8)_{10}, \\ p_{\varphi_2\psi} &= (8,7,2,7,9,1,1,2,8)_{10} = 4 \cdot (2,1,8,1,9,7,7,8,2)_{10}, \\ p_{\varphi_2\psi^2} &= (8,8,7,2,7,9,1,1,2)_{10} = 4 \cdot (2,2,1,8,1,9,7,7,8)_{10}, \\ p_{\varphi_2\psi^7} &= (7,9,1,1,2,8,8,7,2)_{10} = 4 \cdot (1,9,7,7,8,2,2,1,8)_{10}. \end{aligned}$$

Here, we draw attention to the fact that all of the examples above have the state-transition sequence S, S_{φ_1}, or S_{φ_2}. That said, it is not difficult to find other symmetries of p which correspond to permutiple strings distinct from those listed above. We will have more to say about this later. For now, we consider the reflective siblings of p. Since $c_j = n - 1 = 3$ for $j = 3, 4, 5$, and 6, we know, by Corollary 19, that p has four reflective siblings,

$$\begin{aligned} \overline{p}_{\psi^3} &= (7,1,1,2,7,2,8,8,0)_{10} = 4 \cdot (1,7,7,8,1,8,2,2,0)_{10}, \\ \overline{p}_{\psi^4} &= (0,7,1,1,2,7,2,8,8)_{10} = 4 \cdot (0,1,7,7,8,1,8,2,2)_{10}, \\ \overline{p}_{\psi^5} &= (8,0,7,1,1,2,7,2,8)_{10} = 4 \cdot (2,0,1,7,7,8,1,8,2)_{10}, \\ \overline{p}_{\psi^6} &= (8,8,0,7,1,1,2,7,2)_{10} = 4 \cdot (2,2,0,1,7,7,8,1,8)_{10}. \end{aligned}$$

Applying φ_1 and φ_2 to the reflection of p, that is, $\overline{p} = (2,7,2,8,8,0,7,1,1)_{10}$, gives $\overline{p}_{\varphi_1} = (2,7,2,8,0,8,7,1,1)_{10}$ and $\overline{p}_{\varphi_2} = (2,7,2,0,8,8,7,1,1)_{10}$. The reader should note that these are not permutiples. However, applying rotations to $\overline{p}_{\varphi_1}$ and $\overline{p}_{\varphi_2}$, we do obtain new examples:

$$\begin{aligned} \overline{p}_{\varphi_1\psi^3} &= (7,1,1,2,7,2,8,0,8)_{10} = 4 \cdot (1,7,7,8,1,8,2,0,2)_{10}, \\ \overline{p}_{\varphi_1\psi^4} &= (8,7,1,1,2,7,2,8,0)_{10} = 4 \cdot (2,1,7,7,8,1,8,2,0)_{10}, \\ \overline{p}_{\varphi_1\psi^5} &= (0,8,7,1,1,2,7,2,8)_{10} = 4 \cdot (0,2,1,7,7,8,1,8,2)_{10}, \\ \overline{p}_{\varphi_1\psi^6} &= (8,0,8,7,1,1,2,7,2)_{10} = 4 \cdot (2,0,2,1,7,7,8,1,8)_{10}, \\ \overline{p}_{\varphi_2\psi^3} &= (7,1,1,2,7,2,0,8,8)_{10} = 4 \cdot (1,7,7,8,1,8,0,2,2)_{10}, \\ \overline{p}_{\varphi_2\psi^4} &= (8,7,1,1,2,7,2,0,8)_{10} = 4 \cdot (2,1,7,7,8,1,8,0,2)_{10}, \\ \overline{p}_{\varphi_2\psi^5} &= (8,8,7,1,1,2,7,2,0)_{10} = 4 \cdot (2,2,1,7,7,8,1,8,0)_{10}, \\ \overline{p}_{\varphi_2\psi^6} &= (0,8,8,7,1,1,2,7,2)_{10} = 4 \cdot (0,2,2,1,7,7,8,1,8)_{10}. \end{aligned}$$

As mentioned above, it is not difficult to find symmetries of p which do not correspond to a cyclic permutation of S. For example, taking

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 2 & 3 & 4 & 5 & 6 & 1 & 7 & 8 \end{pmatrix},$$

we obtain the permutiple string

$$s_\varphi = (8,2)(2,8)(9,9)(1,7)(1,7)(7,1)(8,2)(2,8)(7,1)$$

corresponding to the state-transition sequence

$$S_\varphi = \{(0,0),(0,3),(3,3),(3,3),(3,3),(3,0),(0,3),(0,0),(3,0)\},$$

which is not a cyclic permutation of the elements of S. The reader may verify the above by using the multi-image union, Δ_I, shown in Figure 6. The resulting permutiple, $p_\varphi = (7,2,8,7,1,1,9,2,8)_{10} = 4 \cdot (1,8,2,1,7,7,9,8,2)_{10}$, is not on the above list. The techniques which we applied to p may now be applied to p_φ. Continuing in this fashion, we may find all 72 examples belonging to the permutiple class containing p having the same multiset of digits. We leave this task to the ambitious reader.

To summarize, with Examples 2 and 3 in mind, manufacturing new permutiples of a specified length is a straightforward task when we have a multiset union of mother-graph cycles accompanied by its corresponding multi-image union, Δ_I, which satisfies the conditions of Corollary 7. If we already have a known example, p, in hand, we may easily construct Δ_I from the permutiple string and carries of p. From there, we may find the dihedral siblings of p by using Corollary 19 and Theorem 21. To p and each of its dihedral siblings, we may then apply Theorem 25 to find symmetries which fix the state-transition sequences of the dihedral siblings, but permute edge labels in a nontrivial way, as seen in Example 10. To find even more examples with the same multiset of digits as p, we may either reexamine Δ_I, or find non-cyclic permutations of S which begin and end with the zero state. If p is contained in a permutiple class, C, the above considerations, coupled with Corollary 28 in the next section, give us everything we need to find the collection of all permutiples in C which have the same multiset of digits as p. Additionally, we may find all permutiples in the reflected class, $\overline{C}$, having the same multiset of digits as the reflective siblings of p.

The conjugacy class given in Table 1 also puts the considerations of this section on direct display; all of the symmetries, π, are of the form $\varphi\psi^j$, where φ fixes the state-transition sequence.

6 Characterizing permutiple symmetries

We now relate permutiple symmetries to ideas encountered in previous work [11, 12, 13]. Theorem 3 tells us that conjugate permutiples must have the same graph, so that by Definition 4, they must also belong to the same permutiple class. To demonstrate the converse of these statements would require that we make certain assumptions about digit permutations in the presence of repeated digits. To circumvent such tedium, we define a less strict notion of permutiple conjugacy which we call *coarse conjugacy.*

Definition 14. Suppose $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-permutiple. Also, suppose $p_1 = (d_{\pi_1(k)}, d_{\pi_1(k-1)}, \ldots, d_{\pi_1(0)})_b$, is an (n, b, τ_1)-permutiple, and $p_2 = (d_{\pi_2(k)}, d_{\pi_2(k-1)}, \ldots, d_{\pi_2(0)})_b$ is an (n, b, τ_2)-permutiple. Then, we say that p_1 and p_2 are *coarsely conjugate* if $d_{\pi_1\tau_1\pi_1^{-1}(j)} = d_{\pi_2\tau_2\pi_2^{-1}(j)}$ for all $0 \leq j \leq k$.

To distinguish coarse conjugacy from other notions, we refer to permutiple conjugacy, as defined in Definition 3, as *fine* conjugacy. We see that fine and coarse conjugacy are equivalent definitions in the absence of repeated digits. In the case of repeated digits, however, fine conjugacy requires that we treat each repeated digit as distinct from other digits having the same value. That is, under fine conjugacy, two permutiple examples which are numerically equal can be considered distinct based upon how we choose to permute repeated digits [12, Example 2]. On the other hand, coarse conjugacy does not make this distinction; any two permutiples with distinct permutations, but give the same arrangement of repeated digits, are considered members of the same coarse conjugacy class. That is, within a multiset framework, permutiples are, quite literally, more coarsely partitioned into classes. More precisely, if σ_1 and σ_2 are permutations on the set $\{0, 1, \ldots, k\}$, and $p = (d_k, d_{k-1}, \ldots, d_0)_b = n \cdot (d_{\sigma_1(k)}, d_{\sigma_1(k-1)}, \ldots, d_{\sigma_1(0)})_b = n \cdot (d_{\sigma_2(k)}, d_{\sigma_2(k-1)}, \ldots, d_{\sigma_2(0)})_b$, then, if $p_1 = (d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$, is an (n, b, τ)-permutiple which is coarsely conjugate to p, we have that $d_{\sigma_1(j)} = d_{\sigma_2(j)} = d_{\pi\tau\pi^{-1}(j)}$ for all $0 \leq j \leq k$. If repeated digits are present, then it is not necessarily the case that $\sigma_1 = \sigma_2 = \pi\tau\pi^{-1}$, meaning that fine conjugacy depends on the choice of permutation. Coarse conjugacy does not depend on this choice. In this way, we see that coarse conjugacy class membership is more permissive than fine class membership, which further explains the choice of terminology.

The next two results set the stage for a characterization which involves permutiple symmetries. The first of these is a characterization of coarse conjugacy in terms of permutiple graphs.

Theorem 26. *Let $(d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b, σ)-permutiple. Also, suppose that $p_1 = (d_{\pi_1(k)}, d_{\pi_1(k-1)}, \ldots, d_{\pi_1(0)})_b$ is an (n, b, τ_1)-permutiple, and $p_2 = (d_{\pi_2(k)}, d_{\pi_2(k-1)}, \ldots, d_{\pi_2(0)})_b$ is an (n, b, τ_2)-permutiple. Then, $G_{p_1} = G_{p_2}$ if and only if p_1 and p_2 are coarsely conjugate.*

Proof. If $G_{p_1} = G_{p_2}$, then $E_{p_1} = E_{p_2}$. Thus,

$$\{(d_{\pi_1(j)}, d_{\pi_1\tau_1(j)}) \mid 0 \leq j \leq k\} = \{(d_{\pi_2(j)}, d_{\pi_2\tau_2(j)}) \mid 0 \leq j \leq k\},$$

from which we have

$$\{(d_j, d_{\pi_1\tau_1\pi_1^{-1}(j)}) \mid 0 \leq j \leq k\} = \{(d_j, d_{\pi_2\tau_2\pi_2^{-1}(j)}) \mid 0 \leq j \leq k\}.$$

It follows that $d_{\pi_1\tau_1\pi_1^{-1}(j)} = d_{\pi_2\tau_2\pi_2^{-1}(j)}$ for all $0 \le j \le k$. The converse is argued similarly. $\square$

The symmetries of a permutiple, p, characterize the graph of p.

Theorem 27. *Suppose $p = (d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-permutiple, and suppose $p_\pi = (d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ is an (n, b, τ)-permutiple. Then, π is a symmetry of p if and only if $G_p = G_{p_\pi}$.*

Proof. Suppose π is a symmetry of p, and let

$$s = (d_0, d_{\sigma(0)})(d_1, d_{\sigma(1)}) \cdots (d_{k-1}, d_{\sigma(k-1)})(d_k, d_{\sigma(k)})$$

be the permutiple string corresponding to p. Also, let

$$s_\pi = (d_{\pi(0)}, d_{\tau\pi(0)})(d_{\pi(1)}, d_{\tau\pi(1)}) \cdots (d_{\pi(k-1)}, d_{\tau\pi(k-1)})(d_{\pi(k)}, d_{\tau\pi(k)})$$

be the permutiple string of p_π. Since π is a symmetry of p, we have, by definition, that

$$\widehat{s}_\pi = (d_{\pi(0)}, d_{\sigma\pi(0)})(d_{\pi(1)}, d_{\sigma\pi(1)}) \cdots (d_{\pi(k-1)}, d_{\sigma\pi(k-1)})(d_{\pi(k)}, d_{\sigma\pi(k)})$$

is a permutiple string. This is to say that

$$\begin{aligned} p_\pi &= n \cdot (d_{\tau\pi(k)}, d_{\tau\pi(k-1)}, \ldots, d_{\tau\pi(0)})_b \\ &= (d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b \\ &= n \cdot (d_{\sigma\pi(k)}, d_{\sigma\pi(k-1)}, \ldots, d_{\sigma\pi(0)})_b \\ &= \widehat{p}_\pi. \end{aligned}$$

Hence, the permutiple strings s, $\widehat{s}_\pi$, and s_π contain the same collection of inputs. From this fact, it follows that $G_p = G_{p_\pi}$.

Now suppose that the graphs G_p and G_{p_π} are the same. Since p and p_π have the same collection of digits, the multiset of inputs in the strings s and s_π must be identical. Since these are both permutiple strings, π must be a symmetry of p. $\square$

The above definitions and results give us a characterization of what it means for two permutiples having the same multiset of digits to be members of the same permutiple class.

Corollary 28. *Suppose $p = (d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-permutiple, and suppose $p_\pi = (d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ is an (n, b, τ)-permutiple. Then, the following statements are equivalent:*

1. *π is a symmetry of p.*
2. *$G_p = G_{p_\pi}$.*
3. *p and p_π are coarsely conjugate.*
4. *p and p_π belong to the same permutiple class.*

7 Concluding remarks

This article summarizes a body of research pertaining to a form of digit-preserving multiplication known as the permutiple problem. It also details how the reflective symmetry of the mother graph is inherited by the Hoey-Sloane graph and its relevant subgraphs. Utilizing these results, we are able to construct new permutiples from known examples. From the above investigations, it is revealed that the elements of permutiple classes are bound together by a larger framework where symmetry plays a key role.

Although the word symmetry does not appear in Hardy's "Apology," we believe he would agree that the concept is a general, serious, significant, deep, and beautiful notion in mathematics. For this reason, we also believe that the generalizations presented here, at the very least, touch upon deeper mathematical notions. If the reader is inclined to agree, then perhaps there is something in the "odd fact" which appeals to a mathematician.

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