Digit Permutation Without Apology

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Origins of My Fascination

As an undergraduate, I found the following problem in a book of puzzles.

Find numbers A, B, C, D, and E such that

$$\begin{array}{ccccc} A & B & C & D & E \\ \times & & & & 4 \\ \hline E & D & C & B & A \end{array}$$

Origins of My Fascination

The Solution

$$\begin{array}{ccccc} 2 & 1 & 9 & 7 & 8 \\ \times & & & & 4 \\ \hline 8 & 7 & 9 & 1 & 2 \end{array}$$

What G.H. Hardy Had to Say About Such Numbers...

W. W. Rouse Ball's "Mathematical Recreations and Essays" states that $8712 = 4 \cdot 2178$ and $9801 = 9 \cdot 1089$ are the only 4-digit numbers which are multiples of their reversals. In his "Apology" G.H. Hardy had this to say about Ball's inclusion of this problem:

"These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but **there is nothing in them which appeals to a mathematician***. The proofs are neither difficult nor interesting - merely tiresome. The theorems are not serious; and it is plain that one reason (though perhaps not the most important) is the extreme speciality of both the enunciations and proofs, which are* **not capable of any significant[1] generalization***."*

[1] See S. Weisgerber, Value judgments in mathematics: G. H. Hardy and the (non-)seriousness of mathematical theorems, *Global Philosophy* **34**:1 (2024)

Timeline

Below is an incomplete list of authors who have worked on finding numbers which are multiples of their reversals. Such numbers go by several names: palintiples, reverse multiples, and reverse divisors. This body of work seems to challenge Hardy's claims.

A. Sutcliffe, 1966
T. J. Kaczynski[2], 1968
L. F. Klosinski and D. C. Smolarski, 1969
A. L. Young, 1992,
D. J. Hoey, 1992
L. Pudwell[3], 2007
R. Webster and G. Williams, 2013
N. J. A. Sloane, 2014
B. V. Holt, 2014, 2016
L. H. Kendrick, 2015

[2] "Better known for other work."-Laura Pudwell
[3] Pudwell's paper is titled "Digit Reversal Without Apology"

Statement of Problem

In this talk, we consider a generalization of the digit-reversal problem.

We find numbers which are multiples of some permutation of their digits.

Permutiple Definition

We shall use $(d_k, d_{k-1}, \ldots, d_0)_b$ to denote the natural number $\sum_{j=0}^{k} d_j b^j$ where each $0 \leq d_j < b$.

Definition

Let n be a natural number and σ be a permutation on $\{0, 1, 2, \ldots, k\}$. We say that $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-*permutiple* provided

$$(d_k, d_{k-1}, \ldots, d_1, d_0)_b = n(d_{\sigma(k)}, d_{\sigma(k-1)}, \ldots, d_{\sigma(1)}, d_{\sigma(0)})_b.$$

A Basic Result

Theorem

Let $(d_k, d_{k-1}, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, d_{\sigma(k-1)}, \ldots, d_{\sigma(0)})_b$ *be an* (n, b, σ)*-permutiple, and let* c_j *be the* j*th carry. Then,*

$$bc_{j+1} - c_j = nd_{\sigma(j)} - d_j$$

for all $0 \leq j \leq k$.

A Basic Result

The carries of any permutiple are always less than the multiplier, n.

Theorem

Let $(d_k, d_{k-1}, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, d_{\sigma(k-1)}, \ldots, d_{\sigma(0)})_b$ *be an* (n, b, σ)*-permutiple, and let* c_j *be the* j*th carry. Then,* $c_j \leq n - 1$ *for all* $0 \leq j \leq k$*.*

Some $(4, 10)$-Permutiple Examples

$(4, 10, \tau)$-Example	π	τ
$(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$	ε	ρ
$(8, 7, 1, 9, 2)_{10} = 4 \cdot (2, 1, 7, 9, 8)_{10}$	$(1, 2)$	$(1, 2)\rho(1, 2)$
$(7, 9, 1, 2, 8)_{10} = 4 \cdot (1, 9, 7, 8, 2)_{10}$	ψ^4	$\psi^{-4}\rho\psi^4$
$(7, 1, 9, 2, 8)_{10} = 4 \cdot (1, 7, 9, 8, 2)_{10}$	$(1, 2)\psi^4$	$\psi^{-4}(1, 2)\rho(1, 2)\psi^4$

In the above table, ψ is the 5-cycle $(0, 1, 2, 3, 4)$, ρ is the reversal permutation, and ε is the identity permutation.

Permutiple Graphs

Definition

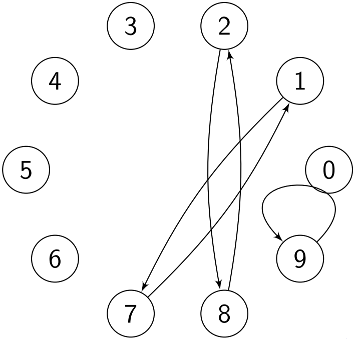
Let $p = (d_k, d_{k-1}, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, d_{\sigma(k-1)}, \ldots, d_{\sigma(0)})_b$ be an (n, b, σ)-permutiple. We define a directed graph, called *the graph of* p, denoted as G_p, to consist of the collection of base-b digits as vertices, and the collection of directed edges $E_p = \left\{ \left(d_j, d_{\sigma(j)}\right) \,|\, 0 \le j \le k \right\}$. A graph, G, for which there is a permutiple, p, such that $G = G_p$ is called a *permutiple graph*.

For the remainder of this talk, all graphs will be directed graphs, and we may refer to a "directed graph" as simply a "graph," or a "directed edge" as an edge.

Permutiple Graphs

All of the permutiples below have the same graph.

$(4,10,\tau)$-Example	π	τ
$(8,7,9,1,2)_{10} = 4 \cdot (2,1,9,7,8)_{10}$	ε	ρ
$(8,7,1,9,2)_{10} = 4 \cdot (2,1,7,9,8)_{10}$	$(1,2)$	$(1,2)\rho(1,2)$
$(7,9,1,2,8)_{10} = 4 \cdot (1,9,7,8,2)_{10}$	ψ^4	$\psi^{-4}\rho\psi^4$
$(7,1,9,2,8)_{10} = 4 \cdot (1,7,9,8,2)_{10}$	$(1,2)\psi^4$	$\psi^{-4}(1,2)\rho(1,2)\psi^4$



Permutiple Classes

Definition

Let p be an (n, b)-permutiple with graph G_p. We define the *class of* p to be the collection, C, of all (n, b)-permutiples, q, such that G_q is a subgraph of G_p. We also define the graph of the class to be G_p, which we will denote as G_C and will call the *graph of* C.

All of the permutiples below have the same graph, and are, therefore, members of the same class.

$(4, 10, \tau)$-Example	π	τ
$(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$	ε	ρ
$(8, 7, 1, 9, 2)_{10} = 4 \cdot (2, 1, 7, 9, 8)_{10}$	$(1, 2)$	$(1, 2)\rho(1, 2)$
$(7, 9, 1, 2, 8)_{10} = 4 \cdot (1, 9, 7, 8, 2)_{10}$	ψ^4	$\psi^{-4}\rho\psi^4$
$(7, 1, 9, 2, 8)_{10} = 4 \cdot (1, 7, 9, 8, 2)_{10}$	$(1, 2)\psi^4$	$\psi^{-4}(1, 2)\rho(1, 2)\psi^4$

More Examples...

More examples of permutiples in this same class include:

$$
\begin{aligned}
&(8,7,1,2)_{10} = 4 \cdot (2,1,7,8)_{10},\\
&(8,7,9,1,2)_{10} = 4 \cdot (2,1,9,7,8)_{10},\\
&(8,7,9,9,1,2)_{10} = 4 \cdot (2,1,9,9,7,8)_{10},\\
&(7,1,2,8)_{10} = 4 \cdot (1,7,8,2)_{10},\\
&(7,9,1,2,8)_{10} = 4 \cdot (1,9,7,8,2)_{10},\\
&(7,9,9,1,2,8)_{10} = 4 \cdot (1,9,9,7,8,2)_{10},\\
&(8,7,1,2,8,7,1,2)_{10} = 4 \cdot (2,1,7,8,2,1,7,8)_{10}.
\end{aligned}
$$

The Mother Graph

Theorem

Let $p = (d_k, d_{k-1}, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, d_{\sigma(k-1)}, \ldots, d_{\sigma(0)})_b$ *be an* (n, b, σ)*-permutiple with graph* G_p*. Then, for any edge,* $(d_j, d_{\sigma(j)})$*, of* G_p*, it must be that* $\lambda\left(d_j + (b-n)d_{\sigma(j)}\right) \leq n-1$ *for all* $0 \leq j \leq k$*, where* λ *gives the least non-negative residue modulo* b*.*

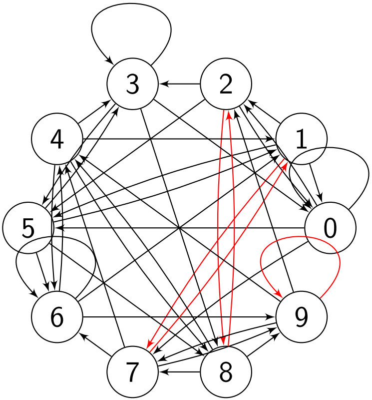
With the above, we gather all possible edges of a permutiple graph into a single graph.

Definition

The (n, b)-*mother graph*, denoted M, is the graph having all base-b digits as its vertices and the collection of edges, (d_1, d_2), satisfying the inequality $\lambda\left(d_1 + (b-n)d_2\right) \leq n-1$.

Example: The $(4, 10)$-Mother Graph

The $(4, 10)$-mother graph tells us which edges are possible for a base-10 permutiple when multiplying by 4.



Constructing a Finite-State Machine

We now take our most basic results and construct a finite state machine which recognizes permutiples.

We will use the fact that for an (n, b, σ)-permutiple, $(d_k, d_{k-1}, \ldots, d_0)_b = n \cdot (d_{\sigma(k)}, d_{\sigma(k-1)}, \ldots, d_{\sigma(0)})_b$, we have

$$bc_{j+1} - c_j = nd_{\sigma(j)} - d_j$$

and that $c_j \leq n - 1$ for all $0 \leq j \leq k$.

The Hoey-Sloane Machine

Definition

Taking non-negative integers less than n as the collection of states, and the edges of the mother graph, M, as the input alphabet, the equation

$$c_2 = [nd_2 - d_1 + c_1] \div b$$

defines a state-transition function from state c_1 to state c_2 with (d_1, d_2) serving as the input which induces the transition. The initial state is zero, and the only accepting state is zero. We name this construction the (n, b)-*Hoey-Sloane machine*.

Since the $c_0 = 0$ in any multiplication (see multiplication algorithm), the initial state must be zero. Also, for any multiplication of length ℓ, the final state, $c_{\ell+1}$ must be zero for the process to end. This is why zero is the only accepting state.

The Hoey-Sloane Graph

When the input (d_1, d_2) induces the transition from the state c_1 to state c_2, this transition corresponds to a labeled edge on the state diagram as seen below.

$$c_1 \xrightarrow{(d_1, d_2)} c_2$$

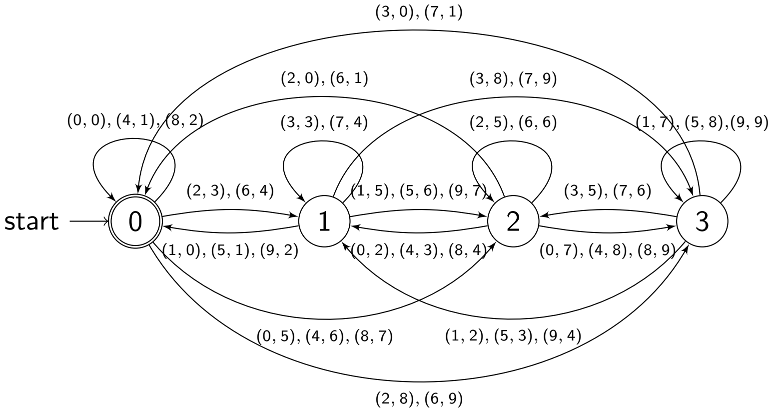
This state diagram is called the (n, b)-Hoey-Sloane graph, which we will denote as Γ.

The (n, b)-Hoey-Sloane Language

Definition

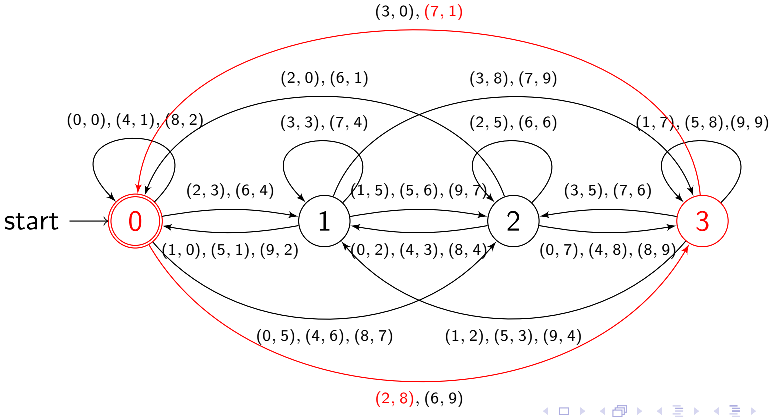
We shall denote language of input strings accepted by the (n, b)-Hoey-Sloane machine as L. We may describe L as finite sequences of edge-label inputs which define walks on Γ whose initial and final states are zero. For simplicity, we will call such walks *L-walks*. Members of L which produce permutiple numbers will be called (n, b)*-permutiple strings.*

The (4, 10)-Hoey-Sloane Graph



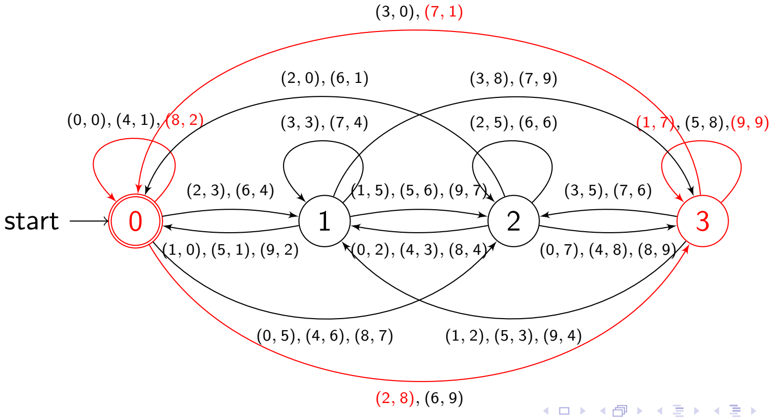
An Example

The input string $s = (2,8)(7,1)$ defines an L-walk on Γ. The corresponding mutilplication is $(7,2)_{10} = 4 \cdot (1,8)_{10}$, with state (carry) sequence $c_0 = 0$, and $c_1 = 3$. We see that s is a member of L, but is NOT a permutiple string.



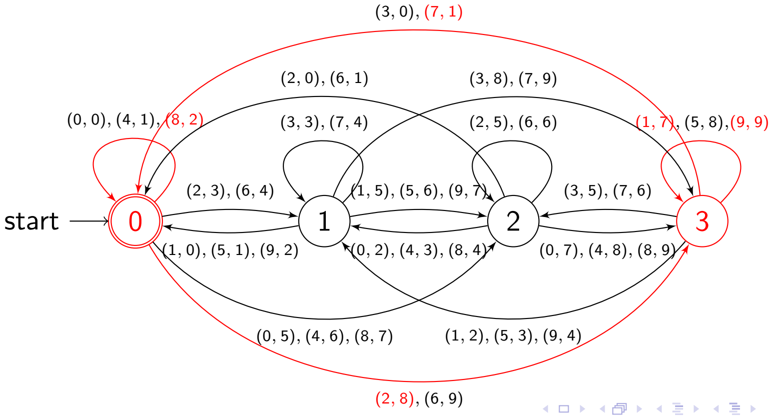
Another Example

The input string $s = (2,8)(1,7)(9,9)(7,1)(8,2)$ defines an L-walk on Γ. The corresponding mutilplication is $(8,7,9,1,2)_{10} = 4 \cdot (2,1,9,7,8)_{10}$. We see that s is not only a member of L, it's also a permutiple string!



More Examples!

Permutiple	Permutiple String
$(7,1,2,8)_{10} = 4 \cdot (1,7,8,2)_{10}$,	$s = (8,2)(2,8)(1,7)(7,1)$
$(7,9,1,2,8)_{10} = 4 \cdot (1,9,7,8,2)_{10}$,	$s = (8,2)(2,8)(1,7)(9,9)(7,1)$



Some Questions

If we have the (n, b)-Hoey-Sloane graph, how do we find permutiple strings?

What makes a string, $s = (d_0, \hat{d}_0)(d_1, \hat{d}_1) \cdots (d_k, \hat{d}_k)$, in L a permutiple string?

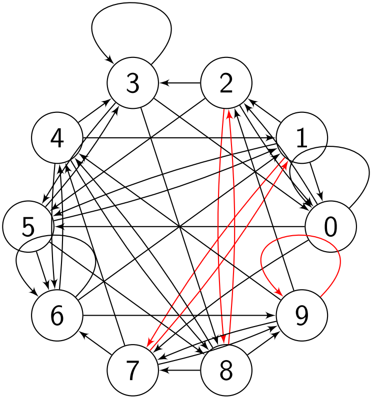
A Helpful Result

Theorem

Let $s = (d_0, \hat{d}_0)(d_1, \hat{d}_1) \cdots (d_k, \hat{d}_k)$ be a member of L. If s is a permutiple string, then the collection of ordered-pair inputs of s is a union of cycles of M.

Example

Consider the permutiple string $s = (2,8)(1,7)(9,9)(7,1)(8,2)$. The inputs of s form a union of the mother-graph cycles $C_0 = \{(9,9)\}$, $C_1 = \{(2,8),(8,2)\}$, and $C_2 = \{(1,7),(7,1)\}$.

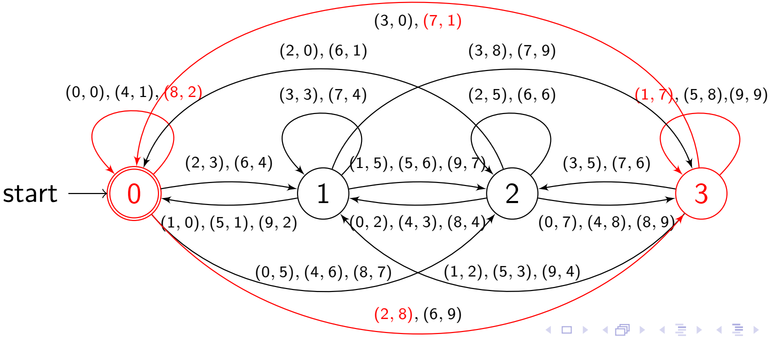


Example

For the mother-graph cycles $C_1 = \{(2,8),(8,2)\}$, and $C_2 = \{(1,7),(7,1)\}$ we may use Γ to order the multiset union

$$C_1 \uplus C_1 \uplus C_2 \uplus C_2 = \{(2,8),(8,2),(2,8),(8,2),(1,7),(7,1),(1,7),(7,1)\}$$

into $s = (8,2)(2,8)(1,7)(7,1)(2,8)(1,7)(7,1)(8,2)$ to produce examples such as $(8,7,1,2,7,1,2,8)_{10} = 4 \cdot (2,1,7,8,1,7,8,2)_{10}$.



Question

How do we begin to understand which multiset unions of mother-graph cycles can be ordered into permutiple strings?

A Partial Answer

For each mother-graph cycle, C_j, gather all the edges, (c_1, c_2), of the Hoey-Sloane graph for which the inputs in C_j are an edge label of (c_1, c_2).

The labeled subgraph, Γ_j, of Γ corresponding to the cycle C_j is called the *cycle image* of C_j.

A multiset union must correspond to a cycle-image union on which we can form L-walks. This is a necessary, but not sufficient, condition to form permutiple strings.

We interrupt this program to bring you a simpler example...

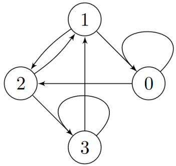
For reasons which will soon be apparent, we will return to our regularly scheduled base-10 example after considering a much simpler example:

Find all base-4 permutiples with multiplier 2.

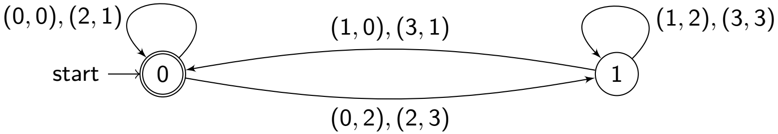
This example will set the stage for understanding the task of finding all $(4, 10)$-permutiples.

$(2,4)$-Mother Graph, its Cycles, and Hoey-Sloane Graph

For each mother-graph cycle, C_j, gather all the edges, (c_1, c_2), of the Hoey-Sloane graph for which the inputs in C_j are an edge label of (c_1, c_2).



$$
\begin{aligned}
C_0 &= (0) = \{(0,0)\},\\
C_1 &= (3) = \{(3,3)\},\\
C_2 &= (1,2) = \{(1,2),(2,1)\},\\
C_3 &= (0,2,1) = \{(0,2),(2,1)(1,0)\},\\
C_4 &= (1,2,3) = \{(1,2),(2,3),(3,1)\},\\
C_5 &= (0,2,3,1) = \{(0,2),(2,3),(3,1),(1,0)\}.
\end{aligned}
$$



$(2,4)$-Mother Graph, its Cycles, and Hoey-Sloane Graph

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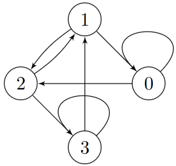

$$
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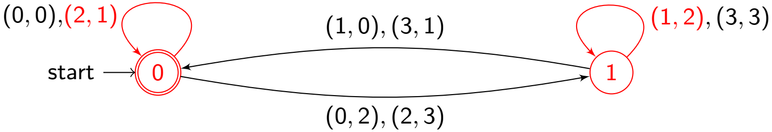
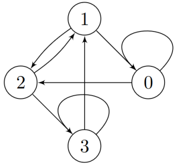
\end{aligned}
$$



$(2,4)$-Mother Graph, its Cycles, and Hoey-Sloane Graph

For each mother-graph cycle, C_j, gather all the edges, (c_1, c_2), of the Hoey-Sloane graph for which the inputs in C_j are an edge label of (c_1, c_2).



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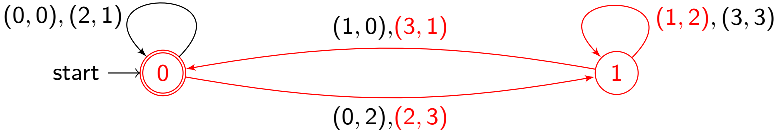
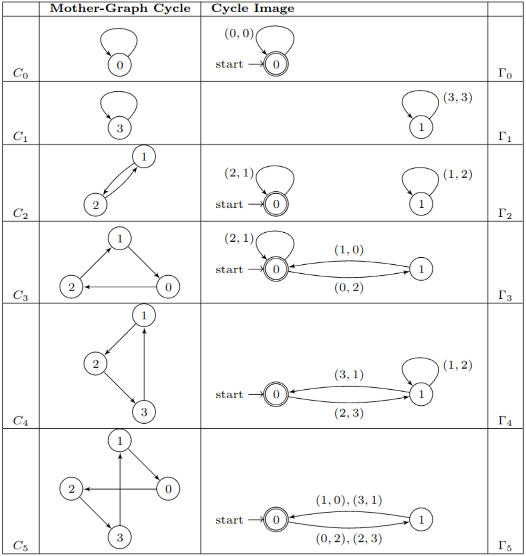
C_4 &= (1,2,3) = \{(1,2),(2,3),(3,1)\},\\
C_5 &= (0,2,3,1) = \{(0,2),(2,3),(3,1),(1,0)\}.
\end{aligned}
$$



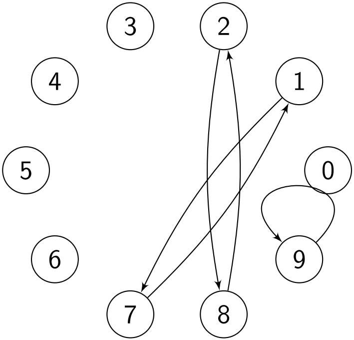
$s = (2,3)(1,2)(3,1)$ is a permutiple string; $(3,1,2)_4 = 2 \cdot (1,2,3)_4$

The Full Collection of Cycle Images

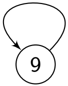
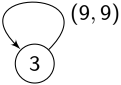
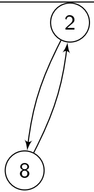
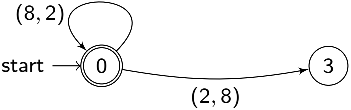
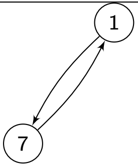
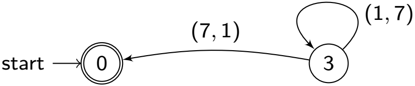


We now return to our base-10 example...

The class, C, of $(4, 10)$-permutiples with considered above may be built from unions of cycles of G_C whose corresponding cycle-image union allows for the formation of L-walks.



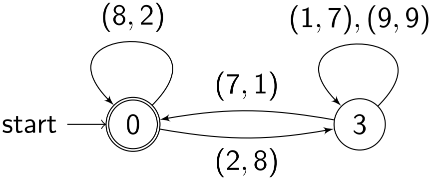
Cycles and Cycle Images of G_C

	Cycle of G_C	Cycle Image	
C_0			Γ_0
C_1			Γ_1
C_2			Γ_2

One Final Example

$$\begin{array}{cccccc}
0 & 3 & 3 & 3 & 0 & 0 \\
 & 1 & 9 & 7 & 8 & 2 \\
\times & & & & & 4 \\
\hline
 & 7 & 9 & 1 & 2 & 8
\end{array}$$

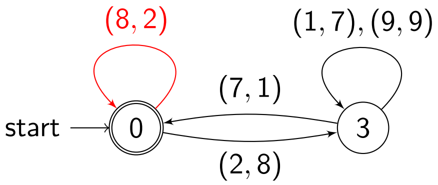
$$s = \text{“ ”}$$



One Final Example

$$\begin{array}{cccccc} 0 & 3 & 3 & 3 & 0 & 0 \\ & 1 & 9 & 7 & 8 & 2 \\ \times & & & & & 4 \\ \hline & 7 & 9 & 1 & 2 & 8 \end{array}$$

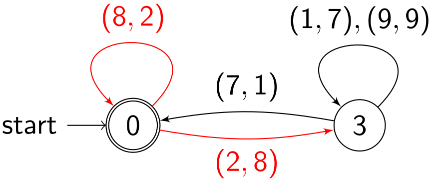
$$s = (8, 2)$$



One Final Example

$$
\begin{array}{cccccc}
0 & 3 & 3 & 3 & 0 & 0 \\
 & 1 & 9 & 7 & 8 & 2 \\
\times & & & & & 4 \\
\hline
 & 7 & 9 & 1 & 2 & 8
\end{array}
$$

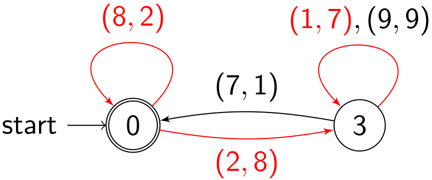
$$s = (8,2)(2,8)$$



One Final Example

$$\begin{array}{cccccc} 0 & 3 & 3 & 3 & 0 & 0 \\ & 1 & 9 & 7 & 8 & 2 \\ \times & & & & & 4 \\ \hline & 7 & 9 & 1 & 2 & 8 \end{array}$$

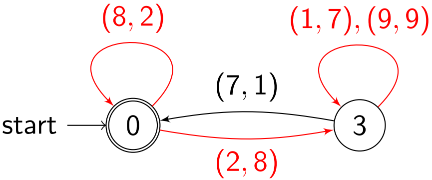
$$s = (8,2)(2,8)(1,7)$$



One Final Example

$$\begin{array}{cccccc}
0 & 3 & 3 & 3 & 0 & 0 \\
 & 1 & 9 & 7 & 8 & 2 \\
\times & & & & & 4 \\
\hline
 & 7 & 9 & 1 & 2 & 8
\end{array}$$

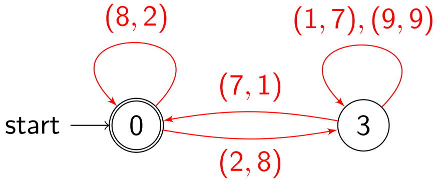
$$s = (8,2)(2,8)(1,7)(9,9)$$



One Final Example

$$\begin{array}{cccccc} 0 & 3 & 3 & 3 & 0 & 0 \\ & 1 & 9 & 7 & 8 & 2 \\ \times & & & & & 4 \\ \hline & 7 & 9 & 1 & 2 & 8 \end{array}$$

$$s = (8,2)(2,8)(1,7)(9,9)(7,1)$$



Thank you!

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