

Origins of My Fascination

As an undergraduate, I found the following problem in a book of puzzles.

Find digits A , B , C , D , and E such that

$$\begin{array}{ccccc} A & B & C & D & E \\ \times & & & & 4 \\ \hline E & D & C & B & A \end{array}$$

Origins of My Fascination

The Solution

$$\begin{array}{r} 21978 \\ \times \quad 4 \\ \hline 87912 \end{array}$$

What G.H. Hardy Had to Say About Such Numbers...

W. W. Rouse Ball's "Mathematical Recreations and Essays" states that $8712 = 4 \cdot 2178$ and $9801 = 9 \cdot 1089$ are the only 4-digit numbers which are multiples of their reversals. In his "Apology" G.H. Hardy had this to say about Ball's inclusion of this problem:

*"These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but **there is nothing in them which appeals to a mathematician.** The proofs are neither difficult nor interesting - merely tiresome. The theorems are not serious; and it is plain that one reason (though perhaps not the most important) is the extreme speciality of both the enunciations and proofs, which are **not capable of any significant¹ generalization.**"*

¹See S. Weisgerber, Value judgments in mathematics: G. H. Hardy and the (non-)seriousness of mathematical theorems, *Global Philosophy* **34**:1 (2024)

Timeline

Numbers which are multiples of their reversals go by several names: *palintiples*, *reverse multiples*, and *reverse divisors*. The body of work on this problem seems to challenge Hardy's claims. Below is an incomplete list of authors:

- A. Sutcliffe, 1966
- T. J. Kaczynski², 1968
- L. F. Klosinski and D. C. Smolarski, 1969
- A. L. Young, 1992,
- D. J. Hoey, 1992
- L. Pudwell³, 2007
- R. Webster and G. Williams, 2013
- N. J. A. Sloane, 2014
- B. V. Holt, 2014, 2016
- L. H. Kendrick, 2015

² "Better known for other work." -Laura Pudwell

³ Pudwell's paper, "Digit Reversal Without Apology" inspired this talk's title.

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Permutiple Definition

We use $(d_k, d_{k-1}, \dots, d_0)_b$ to denote the natural number $\sum_{j=0}^k d_j b^j$ where each $0 \leq d_j < b$.

Definition

Let n be a natural number and σ be a permutation on $\{0, 1, 2, \dots, k\}$. We say that $(d_k, d_{k-1}, \dots, d_0)_b$ is an (n, b, σ) -permutiple provided

$$(d_k, d_{k-1}, \dots, d_1, d_0)_b = n \cdot (d_{\sigma(k)}, d_{\sigma(k-1)}, \dots, d_{\sigma(1)}, d_{\sigma(0)})_b.$$

Example

The example below is a $(4, 10, \rho)$ -permutiple, where ρ is the reversal permutation.

All the carries in the calculation below are less than $n = 4$.

$$\begin{array}{r} 03330 \\ 21978 \\ \times 4 \\ \hline 87912 \end{array}$$

The carries play a vital role in the overall scheme of things, but for the time being we will paying less attention to them.

Permutiple Graphs

Definition

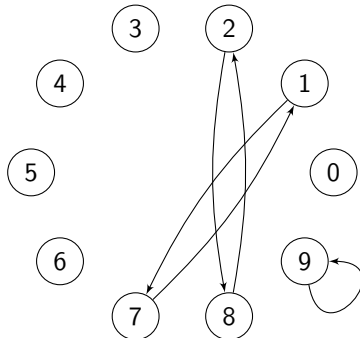
Let $p = (d_k, d_{k-1}, \dots, d_0)_b = n \cdot (d_{\sigma(k)}, d_{\sigma(k-1)}, \dots, d_{\sigma(0)})_b$ be an (n, b, σ) -permutiple. We define a directed graph, called *the graph of p* , denoted as G_p , to consist of the collection of base- b digits as vertices, and the collection of directed edges $E_p = \{(d_j, d_{\sigma(j)}) \mid 0 \leq j \leq k\}$. A graph, G , for which there is a permutiple, p , such that $G = G_p$ is called a *permutiple graph*.

For the remainder of this talk, all graphs will be directed graphs, and we may refer to a “directed graph” as simply a “graph,” or a “directed edge” as an edge.

Permutiple Graphs

All of the permutiples below have the same graph.

$(4, 10, \tau)$ -Example	π	τ
$(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$	ε	ρ
$(8, 7, 1, 9, 2)_{10} = 4 \cdot (2, 1, 7, 9, 8)_{10}$	$(1, 2)$	$(1, 2)\rho(1, 2)$
$(7, 9, 1, 2, 8)_{10} = 4 \cdot (1, 9, 7, 8, 2)_{10}$	ψ^4	$\psi^{-4}\rho\psi^4$
$(7, 1, 9, 2, 8)_{10} = 4 \cdot (1, 7, 9, 8, 2)_{10}$	$(1, 2)\psi^4$	$\psi^{-4}(1, 2)\rho(1, 2)\psi^4$



Permutiple Classes

Definition

Let p be an (n, b) -permutiple with graph G_p . We define the *class* of p to be the collection, C , of all (n, b) -permutiples, q , such that G_q is a subgraph of G_p . We also define the graph of the class to be G_p , which we will denote as G_C and will call the *graph of C* .

All of the permutiples below have the same graph, and are, therefore, members of the same class.

$(4, 10, \tau)$ -Example	π	τ
$(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$	ε	ρ
$(8, 7, 1, 9, 2)_{10} = 4 \cdot (2, 1, 7, 9, 8)_{10}$	$(1, 2)$	$(1, 2)\rho(1, 2)$
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More Examples...

More examples of permutiples in this same class include:

$$(8, 7, 1, 2)_{10} = 4 \cdot (2, 1, 7, 8)_{10},$$

$$(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10},$$

$$(8, 7, 9, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 9, 7, 8)_{10},$$

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$$(8, 7, 1, 2, 8, 7, 1, 2)_{10} = 4 \cdot (2, 1, 7, 8, 2, 1, 7, 8)_{10}.$$

The Mother Graph

Theorem

Let $p = (d_k, d_{k-1}, \dots, d_0)_b = n \cdot (d_{\sigma(k)}, d_{\sigma(k-1)}, \dots, d_{\sigma(0)})_b$ be an (n, b, σ) -permutiple with graph G_p . Then, for any edge, $(d_j, d_{\sigma(j)})$, of G_p , it must be that $\lambda(d_j + (b - n)d_{\sigma(j)}) \leq n - 1$ for all $0 \leq j \leq k$, where λ gives the least non-negative residue modulo b .

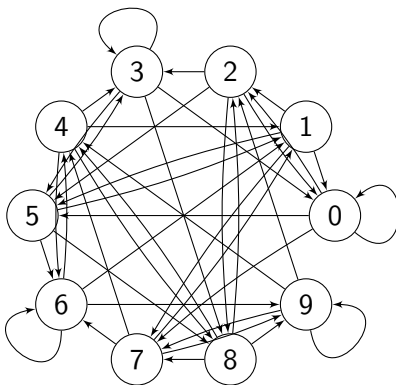
With the above, we gather all possible edges of a permutiple graph into a single graph.

Definition

The (n, b) -mother graph, denoted M , is the graph having all base- b digits as its vertices and the collection of edges, (d_1, d_2) , satisfying the inequality $\lambda(d_1 + (b - n)d_2) \leq n - 1$.

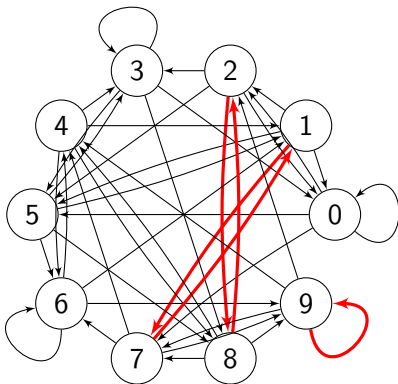
Example: The $(4, 10)$ -Mother Graph

The $(4, 10)$ -mother graph tells us which edges are possible for a base-10 permutiple when multiplying by 4.



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Constructing a Finite-State Machine

We now take our most basic results and construct a finite state machine which recognizes permutiples.

We will use the fact that for an (n, b, σ) -permutiple, $(d_k, d_{k-1}, \dots, d_0)_b = n \cdot (d_{\sigma(k)}, d_{\sigma(k-1)}, \dots, d_{\sigma(0)})_b$, we have

$$bc_{j+1} - c_j = nd_{\sigma(j)} - d_j$$

and that $c_j \leq n - 1$ for all $0 \leq j \leq k$.

The Hoey-Sloane Machine

Definition

Taking non-negative integers less than n as the collection of states, and the edges of the mother graph, M , as the input alphabet, the equation

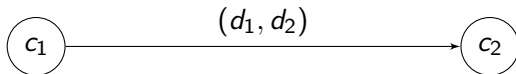
$$c_2 = [nd_2 - d_1 + c_1] \div b$$

defines a state-transition function from state c_1 to state c_2 with (d_1, d_2) serving as the input which induces the transition. The initial state is zero, and the only accepting state is zero. We name this construction the (n, b) -*Hoey-Sloane machine*.

Since the $c_0 = 0$ in any multiplication (see multiplication algorithm), the initial state must be zero. Also, for any multiplication of length ℓ , the final state, $c_{\ell+1}$ must be zero for the process to end. This is why zero is the only accepting state.

The Hoey-Sloane Graph

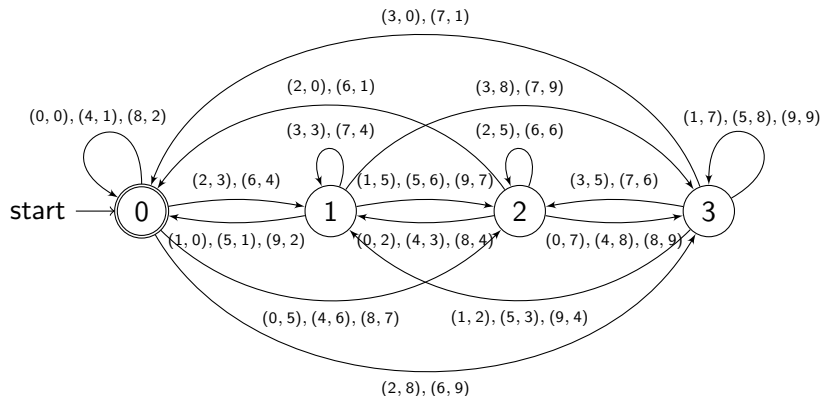
When an input (d_1, d_2) induces the transition from the state c_1 to state c_2 , this transition corresponds to a labeled edge on the state diagram as seen below.



An input which induces a transition from c_1 to c_2 is not generally unique. Thus, the edge label may consist of a collection of mother-graph edges.

The state diagram describes the (n, b) -Hoey-Sloane graph, which we will denote as Γ .

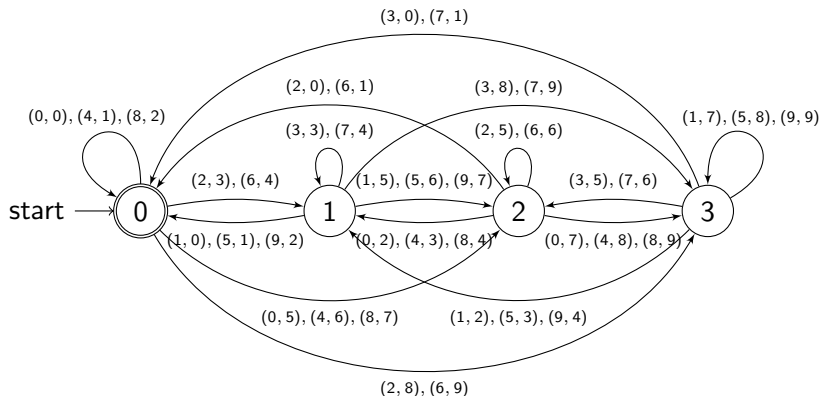
The (4, 10)-Hoey-Sloane Graph



An Example (NOT a Permutiple)

The input string $s = (2, 8)(7, 1)$ defines an L -walk on Γ .

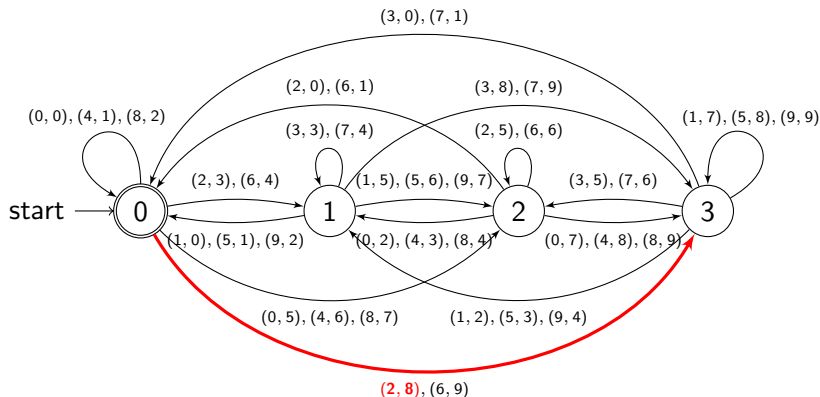
$$\begin{array}{r} 0 \quad 3 \quad 0 \\ 1 \quad 8 \\ \times \quad 4 \\ \hline 7 \quad 2 \end{array}$$



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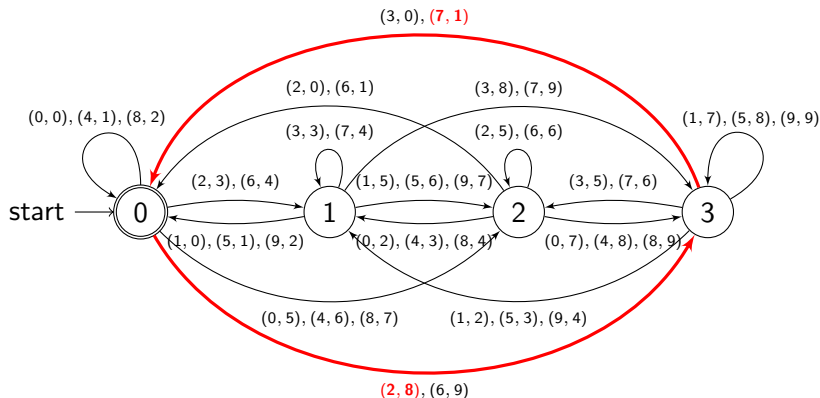
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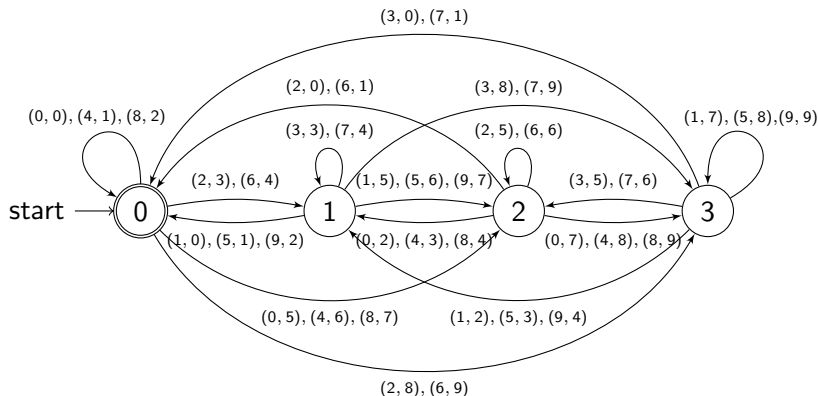
$$\begin{array}{r} 0 \quad 3 \quad 0 \\ 1 \quad 8 \\ \times \quad 4 \\ \hline 7 \quad 2 \end{array}$$



Another Example (IS a Permutiple)

$$s = (2, 8)(1, 7)(9, 9)(7, 1)(8, 2)$$

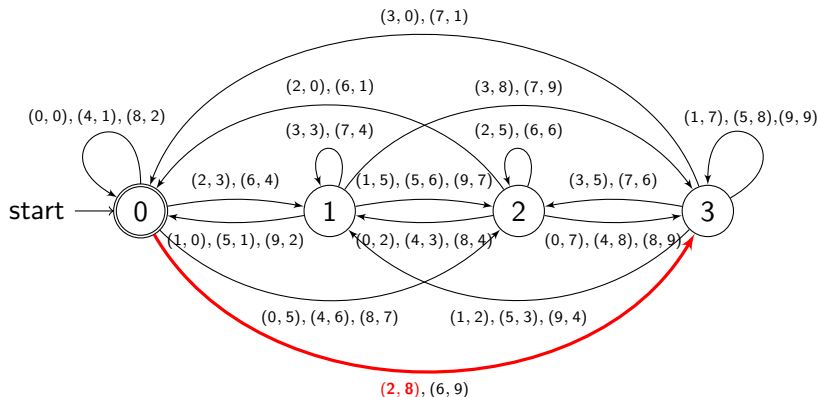
0	0	3	3	3	0
	2	1	9	7	8
×					4
	8	7	9	1	2



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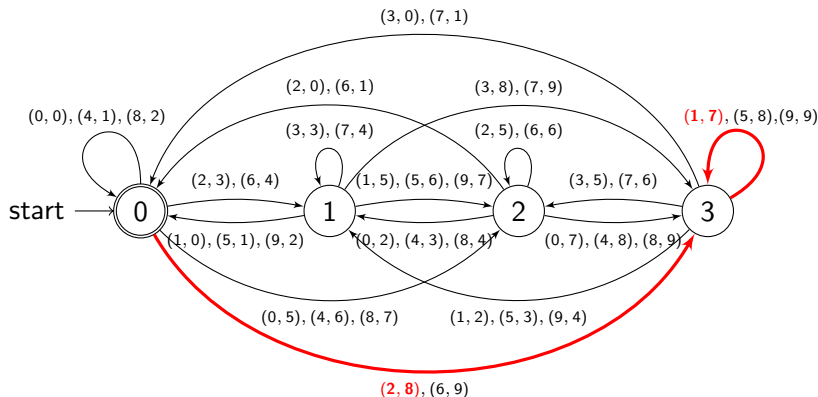
$$\begin{array}{r} 0 \quad 0 \quad 3 \quad 3 \quad 3 \quad 0 \\ 2 \quad 1 \quad 9 \quad 7 \quad 8 \\ \times \\ \hline 8 \quad 7 \quad 9 \quad 1 \quad 2 \end{array}$$



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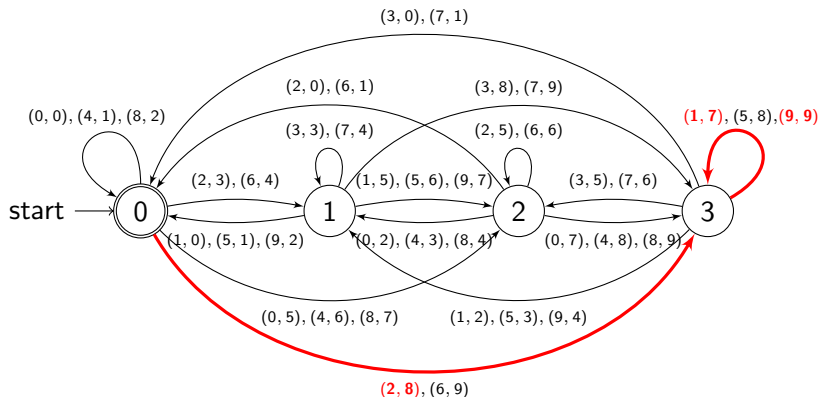
0	0	3	3	3	0
	2	1	9	7	8
×					4
					4
8	7	9	1	2	



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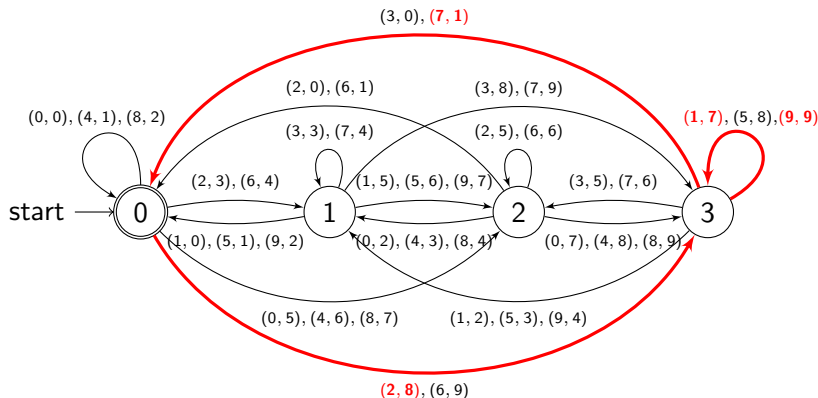
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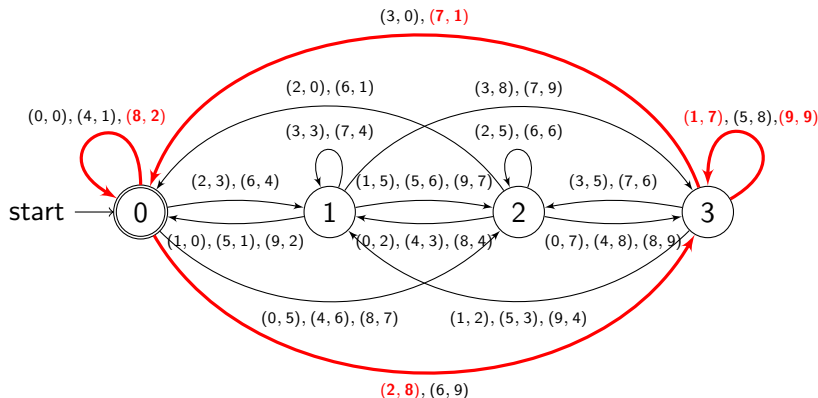
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×					4
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Another Example (IS a Permutiple)

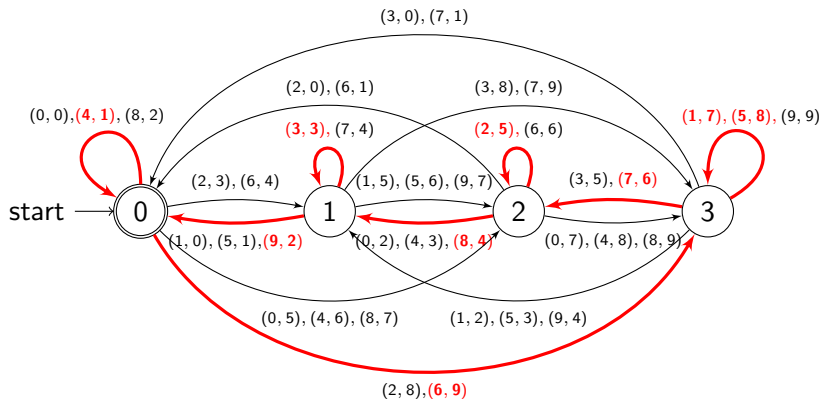
$$s = (2, 8)(1, 7)(9, 9)(7, 1)(8, 2)$$

$$\begin{array}{r} 0 \quad 0 \quad 3 \quad 3 \quad 3 \quad 0 \\ 2 \quad 1 \quad 9 \quad 7 \quad 8 \\ \times 4 \\ \hline 8 \quad 7 \quad 9 \quad 1 \quad 2 \end{array}$$



Yet Another Complete Example

$$\begin{array}{r}
 0 \ 0 \ 1 \ 1 \ 2 \ 2 \ 3 \ 3 \ 3 \ 0 \\
 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \\
 \times 4 \\
 \hline
 4 \ 9 \ 3 \ 8 \ 2 \ 7 \ 1 \ 5 \ 6
 \end{array}$$



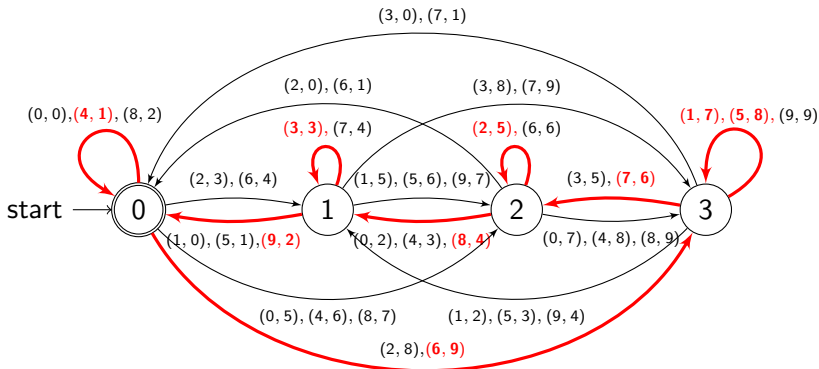
New Examples from Old

$$493827156 = 4 \cdot 123456789$$

$$493827516 = 4 \cdot 123456879$$

$$938271564 = 4 \cdot 234567891$$

$$938275164 = 4 \cdot 234568791$$



Some Questions

If we have the (n, b) -Hoey-Sloane graph, how do we find permutiple strings?

What makes a string, $s = (d_0, \hat{d}_0)(d_1, \hat{d}_1) \cdots (d_k, \hat{d}_k)$, in L a permutiple string?

Some Helpful Results

Theorem

Let C be any (n, b) -permutable class. Then, G_C is a union of cycles of M .

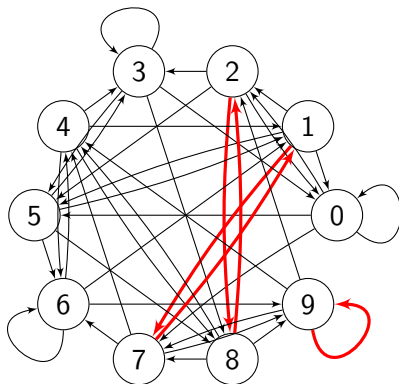
Restating the above within our new framework, we have the following result.

Corollary

Let $s = (d_0, \hat{d}_0)(d_1, \hat{d}_1) \cdots (d_k, \hat{d}_k)$ be a member of L . If s is a permutable string, then the collection of ordered-pair inputs of s is a union of cycles of M .

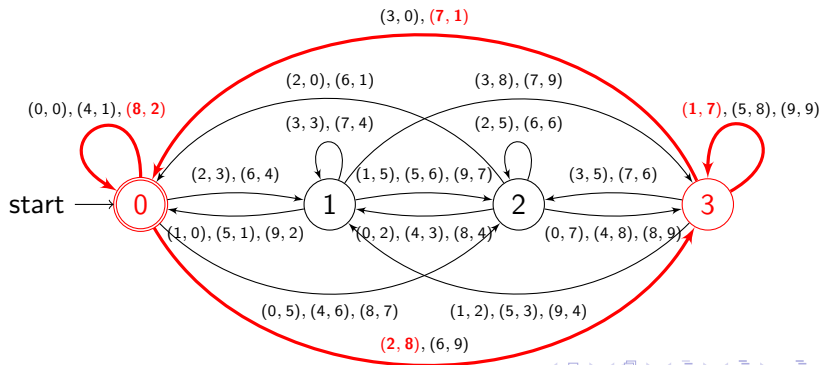
Example

Consider the permutiple string $s = (2, 8)(1, 7)(9, 9)(7, 1)(8, 2)$.
The inputs of s form a union of the mother-graph cycles
 $C_0 = \{(9, 9)\}$, $C_1 = \{(2, 8), (8, 2)\}$, and $C_2 = \{(1, 7), (7, 1)\}$.



Example

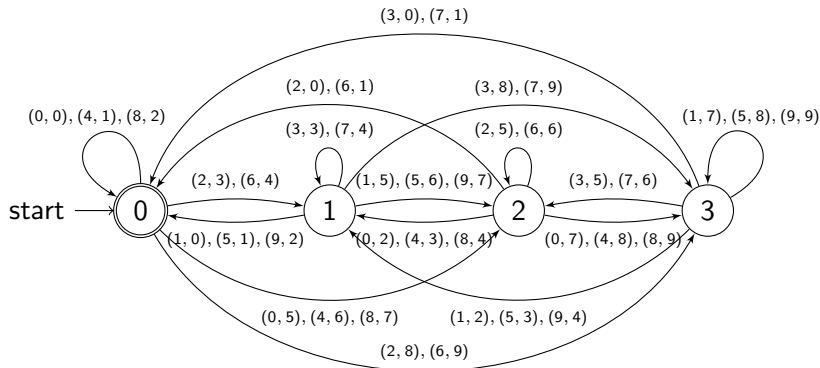
For the mother-graph cycles $C_1 = \{(2, 8), (8, 2)\}$, and $C_2 = \{(1, 7), (7, 1)\}$ we may use Γ to order the **multiset** union $C_1 \uplus C_1 \uplus C_2 \uplus C_2 = \{(2, 8), (8, 2), (2, 8), (8, 2), (1, 7), (7, 1), (1, 7), (7, 1)\}$ into $s = (8, 2)(2, 8)(1, 7)(7, 1)(2, 8)(1, 7)(7, 1)(8, 2)$ to produce examples such as $(8, 7, 1, 2, 7, 1, 2, 8)_{10} = 4 \cdot (2, 1, 7, 8, 1, 7, 8, 2)_{10}$.



An Observation

Not every multiset union of mother-graph cycles can be ordered to determine an L -walk on Γ . For example, consider

$$C_1 \uplus C_1 \uplus C_2 = \{(2, 8), (8, 2), (2, 8), (8, 2), (1, 7), (7, 1)\}.$$



Question

How do we begin to understand which multiset unions of mother-graph cycles can be ordered into permutiple strings?

Two More Basic Facts

An edge label (d_1, d_2) cannot appear on distinct edges of Γ .

Theorem

If $c_1, c_2, \hat{c}_1$, and $\hat{c}_2$ are states on Γ , and (d_1, d_2) is an input associated with the transitions (c_1, c_2) and $(\hat{c}_1, \hat{c}_2)$, then $(c_1, c_2) = (\hat{c}_1, \hat{c}_2)$.

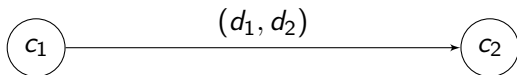
Any edge (d_1, d_2) of M induces some transition (c_1, c_2) on Γ .

Theorem

Let (d_1, d_2) be an edge of M . Then, there are integers $0 \leq c_1 \leq n-1$ and $0 \leq c_2 \leq n-1$ such that $bc_2 - c_1 = nd_2 - d_1$.

Toward an Answer

For each edge, (d_1, d_2) , of the mother graph, we associate a single, labeled **multi-edge**, $((c_1, c_2), (d_1, d_2))$, as seen below.

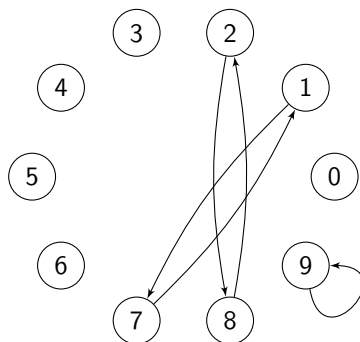


For each mother-graph cycle, C_j , gather all these labeled multi-edges, $((c_1, c_2), (d_1, d_2))$, into a labeled **multigraph** which we will call Δ_j .

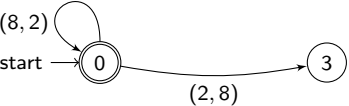
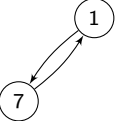
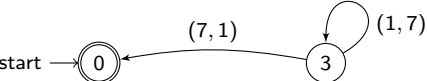
We will call Δ_j the *multi-image* of C_j .

Example

We again consider the $(4, 10)$ -permutiples class, C , whose graph G_C is seen below.

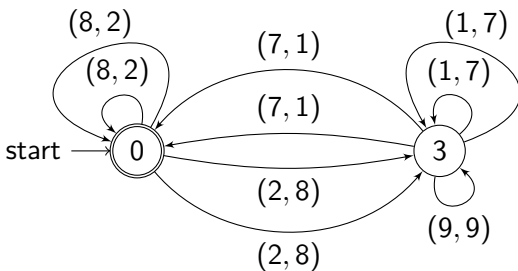


Cycles of G_C and Corresponding Multi-images

	Cycle of G_C	Cycle Multi-image	
C_0			Δ_0
C_1			Δ_1
C_2			Δ_2

Example

Below is a multi-image union $\Delta_0 \uplus \Delta_1 \uplus \Delta_1 \uplus \Delta_2 \uplus \Delta_2$ corresponding to the multiset union $C_0 \uplus C_1 \uplus C_1 \uplus C_2 \uplus C_2$ of mother-graph cycles.



L-Eulerian Multi-image Unions

Definition

A multigraph union $\Delta_I = \biguplus_{j \in I} \Delta_j$ of cycle multi-images is called *L-Eulerian* if the following conditions are met.

- 1 Δ_I contains the zero state.
- 2 Δ_I is strongly connected.
- 3 The indegree is equal to the outdegree at each vertex of Δ_I .

The multi-image union from the previous slide satisfies all three of the above criteria and is, therefore, *L-Eulerian*.

An Answer to our Question

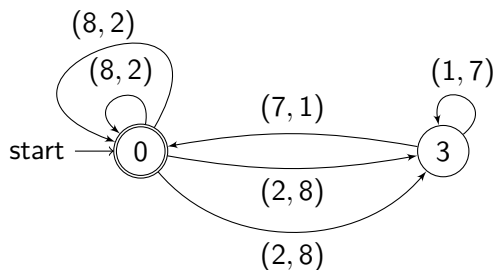
Theorem

Let $\{C_0, C_1, \dots, C_m\}$ be the collection of cycles of M , and let Δ_j be the corresponding multi-image of C_j . Also, let I be a multiset whose support is a subset of $\{0, 1, \dots, m\}$. Then, the multiset union of mother-graph cycles $C_I = \biguplus_{j \in I} C_j$ may be ordered into a permutiple string if and only if the corresponding multigraph union of the cycle multi-images $\Delta_I = \biguplus_{j \in I} \Delta_j$ is L -Eulerian.

The multi-image union from the slide before last is L -Eulerian.
Therefore, it may be ordered into permutiple strings.

Example

Below is a multi-image union $\Delta_1 \uplus \Delta_1 \uplus \Delta_2$ corresponding to the multiset union $C_1 \uplus C_1 \uplus C_2$ of mother-graph cycles.

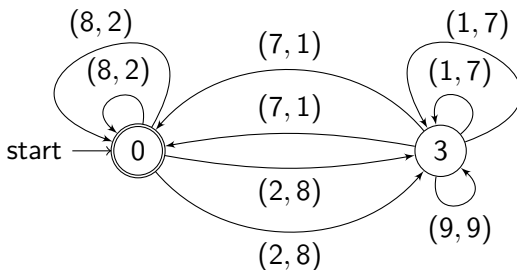


Since the third criterion of our definition is not satisfied, the graph is not L -Eulerian, and so, by the above theorem, $C_1 \uplus C_1 \uplus C_2 = \{(2, 8), (8, 2), (2, 8), (8, 2), (1, 7), (7, 1)\}$ cannot be ordered into a permutiple string.

An L -Eulerian Tour

$$\begin{array}{r}
 0 \ 0 \ 3 \ 3 \ 0 \ 3 \ 3 \ 3 \ 0 \ 0 \\
 2 \ 1 \ 7 \ 8 \ 1 \ 9 \ 7 \ 8 \ 2 \\
 \times \\
 \hline
 8 \ 7 \ 1 \ 2 \ 7 \ 9 \ 1 \ 2 \ 8
 \end{array}$$

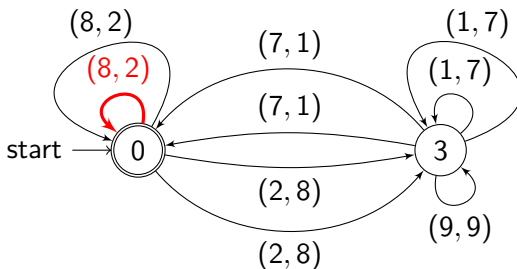
$s = "$



An L -Eulerian Tour

$$\begin{array}{r}
 0 \ 0 \ 3 \ 3 \ 0 \ 3 \ 3 \ 3 \ 0 \ 0 \\
 2 \ 1 \ 7 \ 8 \ 1 \ 9 \ 7 \ 8 \ 2 \\
 \times \\
 \hline
 8 \ 7 \ 1 \ 2 \ 7 \ 9 \ 1 \ 2 \ 8
 \end{array}$$

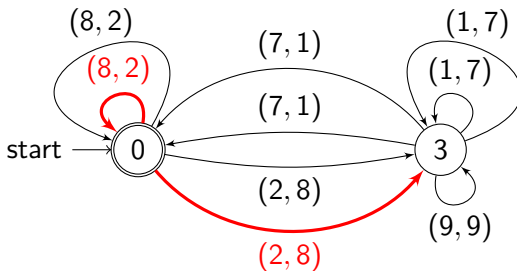
$$s = (8, 2)$$



An L -Eulerian Tour

$$\begin{array}{r}
 0 \ 0 \ 3 \ 3 \ 0 \ 3 \ 3 \ 3 \ 0 \ 0 \\
 2 \ 1 \ 7 \ 8 \ 1 \ 9 \ 7 \ 8 \ 2 \\
 \times \\
 \hline
 8 \ 7 \ 1 \ 2 \ 7 \ 9 \ 1 \ 2 \ 8
 \end{array}$$

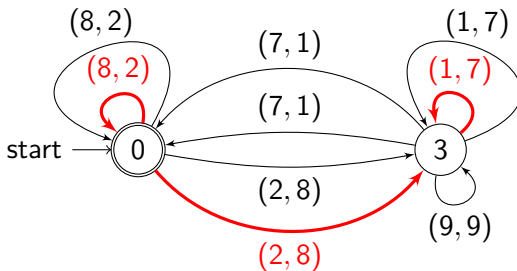
$$s = (8, 2)(2, 8)$$



An L -Eulerian Tour

$$\begin{array}{r}
 0 \ 0 \ 3 \ 3 \ 0 \ 3 \ 3 \ 3 \ 0 \ 0 \\
 2 \ 1 \ 7 \ 8 \ 1 \ 9 \ 7 \ 8 \ 2 \\
 \times \\
 \hline
 8 \ 7 \ 1 \ 2 \ 7 \ 9 \ 1 \ 2 \ 8
 \end{array}$$

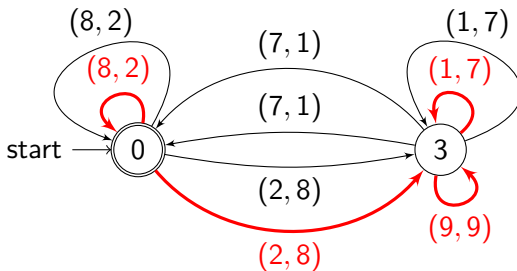
$$s = (8, 2)(2, 8)(1, 7)$$



An L -Eulerian Tour

$$\begin{array}{r}
 0 \ 0 \ 3 \ 3 \ 0 \ 3 \ 3 \ 3 \ 0 \ 0 \\
 2 \ 1 \ 7 \ 8 \ 1 \ 9 \ 7 \ 8 \ 2 \\
 \times \\
 \hline
 8 \ 7 \ 1 \ 2 \ 7 \ 9 \ 1 \ 2 \ 8
 \end{array}$$

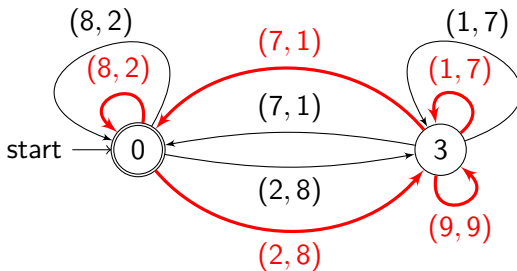
$$s = (8, 2)(2, 8)(1, 7)(9, 9)$$



An L -Eulerian Tour

$$\begin{array}{r}
 0 \ 0 \ 3 \ 3 \ 0 \ 3 \ 3 \ 3 \ 0 \ 0 \\
 2 \ 1 \ 7 \ 8 \ 1 \ 9 \ 7 \ 8 \ 2 \\
 \times \\
 \hline
 8 \ 7 \ 1 \ 2 \ 7 \ 9 \ 1 \ 2 \ 8
 \end{array}$$

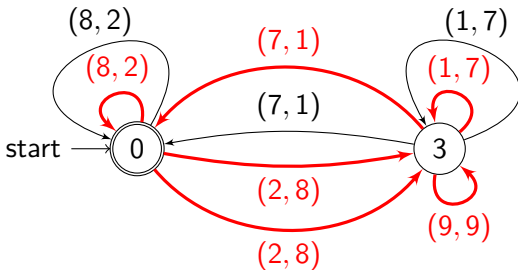
$$s = (8, 2)(2, 8)(1, 7)(9, 9)(7, 1)$$



An L -Eulerian Tour

0	0	3	3	0	3	3	3	0	0
	2	1	7	8	1	9	7	8	2
×									4
	8	7	1	2	7	9	1	2	8

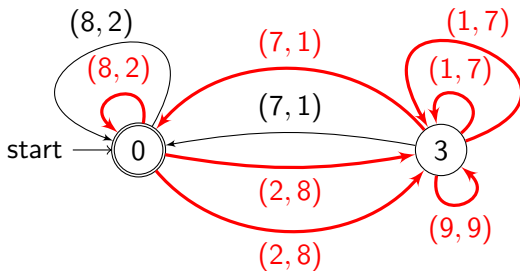
$$s = (8, 2)(2, 8)(1, 7)(9, 9)(7, 1)(2, 8)$$



An L -Eulerian Tour

$$\begin{array}{r}
 0 \quad 0 \quad 3 \quad 3 \quad 0 \quad 3 \quad 3 \quad 3 \quad 0 \quad 0 \\
 2 \quad 1 \quad 7 \quad 8 \quad 1 \quad 9 \quad 7 \quad 8 \quad 2 \\
 \times \\
 \hline
 8 \quad 7 \quad 1 \quad 2 \quad 7 \quad 9 \quad 1 \quad 2 \quad 8
 \end{array}$$

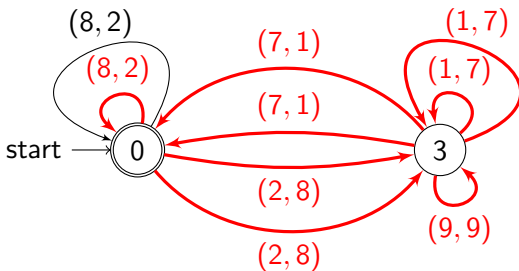
$$s = (8, 2)(2, 8)(1, 7)(9, 9)(7, 1)(2, 8)(1, 7)$$



An L -Eulerian Tour

$$\begin{array}{r}
 0 \quad 0 \quad 3 \quad 3 \quad 0 \quad 3 \quad 3 \quad 3 \quad 0 \quad 0 \\
 2 \quad 1 \quad 7 \quad 8 \quad 1 \quad 9 \quad 7 \quad 8 \quad 2 \\
 \times \hspace{10em} 4 \\
 \hline
 8 \quad 7 \quad 1 \quad 2 \quad 7 \quad 9 \quad 1 \quad 2 \quad 8
 \end{array}$$

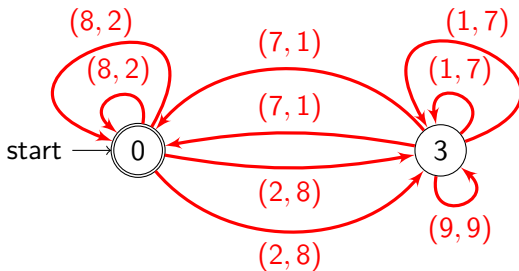
$$s = (8, 2)(2, 8)(1, 7)(9, 9)(7, 1)(2, 8)(1, 7)(7, 1)$$



An L -Eulerian Tour

$$\begin{array}{r}
 0 \ 0 \ 3 \ 3 \ 0 \ 3 \ 3 \ 3 \ 0 \ 0 \\
 2 \ 1 \ 7 \ 8 \ 1 \ 9 \ 7 \ 8 \ 2 \\
 \times \qquad \qquad \qquad 4 \\
 \hline
 8 \ 7 \ 1 \ 2 \ 7 \ 9 \ 1 \ 2 \ 8
 \end{array}$$

$$s = (8, 2)(2, 8)(1, 7)(9, 9)(7, 1)(2, 8)(1, 7)(7, 1)(8, 2)$$



Thank you!

Selected References

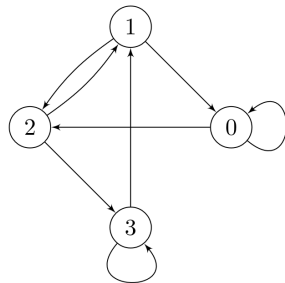


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Bonus Example

Find all $(2, 4)$ -permutiples.

(2, 4)-Mother Graph, its Cycles, and Hoey-Sloane Graph



$$C_0 = (0) = \{(0, 0)\},$$

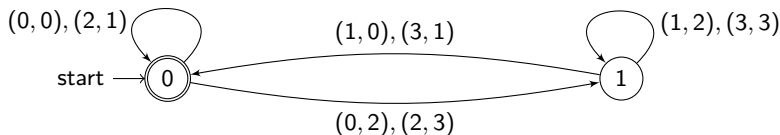
$$C_1 = (3) = \{(3, 3)\},$$

$$C_2 = (1, 2) = \{(1, 2), (2, 1)\},$$

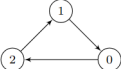
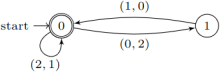
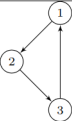
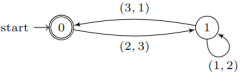
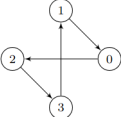
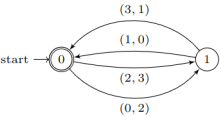
$$C_3 = (0, 2, 1) = \{(0, 2), (2, 1), (1, 0)\},$$

$$C_4 = (1, 2, 3) = \{(1, 2), (2, 3), (3, 1)\},$$

$$C_5 = (0, 2, 3, 1) = \{(0, 2), (2, 3), (3, 1), (1, 0)\}.$$

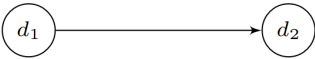
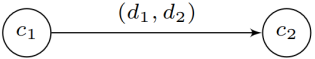
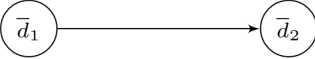
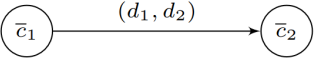


The Full Collection of Multi-images

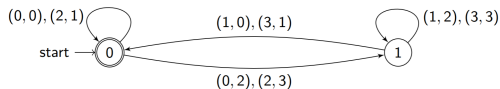
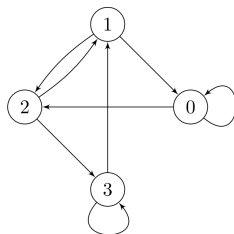
	Mother-Graph Cycle	Cycle Multi-Image	
C_0		$(0, 0)$ start 	Δ_0
C_1		$(3, 3)$ 	Δ_1
C_2		start 	Δ_2
C_3		start 	Δ_3
C_4		start 	Δ_4
C_5		start 	Δ_5

Reflective Symmetry of the Mother and Hoey-Sloane Graphs

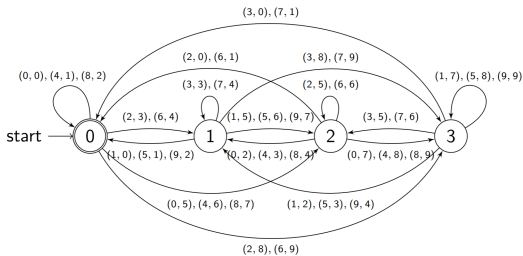
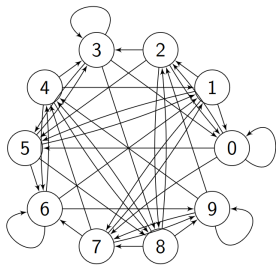
Defining digit and state “reflections” as $\bar{d}_j = b - 1 - d_j$ and $\bar{c}_j = n - 1 - c_j$, respectively, the following “reflective” symmetries hold.

Edge on Mother Graph	Labeled Edge on Hoey-Sloane Graph
	
	

Reflective Symmetry of the Mother and Hoey-Sloane Graphs



Reflective Symmetry of the Mother and Hoey-Sloane Graphs



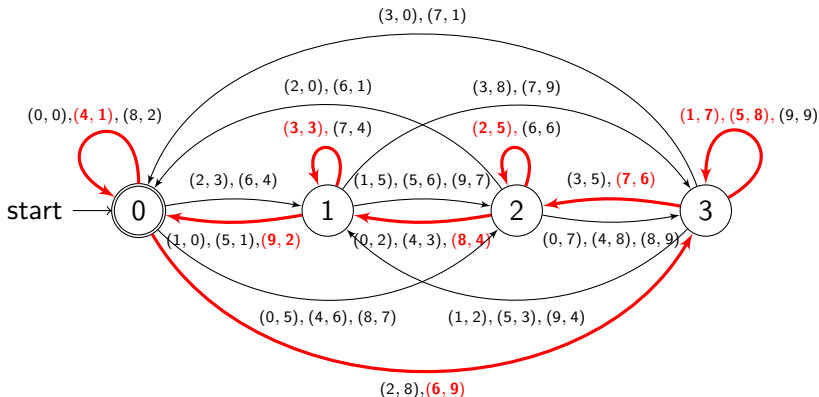
New Examples from a Reflected L -Walk

$$493827156 = 4 \cdot 123456789$$

$$493827516 = 4 \cdot 123456879$$

$$938271564 = 4 \cdot 234567891$$

$$938275164 = 4 \cdot 234568791$$



New Examples from a Reflected L -Walk

$$843506172 = 4 \cdot 210876543 \quad 435061728 = 4 \cdot 108765432$$

$$483506172 = 4 \cdot 120876543 \quad 835061724 = 4 \cdot 208765431$$

$$350617284 = 4 \cdot 087654321$$

$$350617248 = 4 \cdot 087654312$$

