ON PERMUTIPLES HAVING A FIXED SET OF DIGITS

Benjamin V. Holt
Department of Mathematics, Humboldt State University, Arcata, California
bvh6@humboldt.edu

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Abstract
A permutiple is the product of a digit-preserving multiplication, that is, a number which is an integer multiple of some permutation of its digits. Certain permutiple problems, particularly transposable, cyclic, and, more recently, palintiple numbers, have been well-studied. In this paper we study the problem of general digit-preserving multiplication. We show how the digits and carries of a permutiple are related and utilize these relationships to develop methods for finding new permutiple examples from old. In particular, we shall focus on the problem of finding new permutiples from a known example having the same set of digits.

1. Introduction

A *permutiple* is a natural number with the property of being an integer multiple of some permutation of its digits. Digit permutation problems are nothing new [2, 10] and have been a topic of study for both amateurs and professionals alike [5]. A relatively well-studied example of permutiples includes *palintiple* numbers, also known as reverse multiples [6, 9, 11], which are integer multiples of their digit reversals and include well-known base-10 examples such as $87912 = 4 \cdot 21978$ and $98901 = 9 \cdot 10989$. As noted by Sutcliffe [10] in his seminal paper on palintiple numbers, cyclic digit permutations such as $714285 = 5 \cdot 142857$ are also well-studied examples. We also note that 142857 is an example of a *cyclic number*; not only does multiplication by 5 permute the digits, but 2,3,4, and 6 also produce cyclic digit permutations.

Permutiples for which the digits are cyclically permuted are relatively well-understood, and their description is fairly straightforward in comparison to palintiples. The digits of cyclic permutiples are found in repeating base-b decimal expansions of a/p where $a < p$ and p is a prime which does not divide b [2, 5]. On the other hand, palintiples (digit-reversing permutations) admit quite a variety of classifications [6, 7] and are not nearly as well-understood. Young [11, 12], building upon the body of work of Sutcliffe [10] and others [1, 8], translates the palintiple problem into graph-theoretical language by representing an efficient palintiple search method as a tree-graph where the possible carries are represented as nodes and the potential digits are associated with the edges. Continuing the work of Young [11, 12], Sloane [9] modified Young's tree-graph representation into the *Young graph* which is a visualization of digit-carry palintiple structure. The paper identifies and studies several Young graph isomorphism classes which describe palintiple type. Furthering the work of Sloane [9], Kendrick [6] proves two of Sloane's [9] main conjectures involving Young graph isomorphism classes which describe two well-understood palintiple types. The work of Holt [3, 4] takes a more elementary approach and classifies palintiples according to patterns exhibited by their carries. This approach, as noted by Kendrick [6], seems to coincide with certain Young graph isomorphism classes with Holt [4] conjecturing that the class of palintiples characterized by the 1089 graph are precisely the collection of *symmetric palintiples* (the carry sequence is palindromic) described in [3]. Kendrick's work [6, 7] reveals the sheer multitude of palintiple types when classified according to Young graph isomorphism.

In this paper we establish some general properties of digit-preserving multiplication. We generalize the results for palintiple numbers found in Young [11], Sloane [9], Kendrick [6], and Holt [3, 4] to an arbitrary permutation. Using these results, we develop methods for finding new permutiples from old. In particular, we consider the problem of finding new base-b permutiples with multiplier n having a fixed set of digits from a single known example. Moreover, we find a condition under which our methods give us all permutiples of a particular base and multiplier having the same digits as a known example.

2. Permutiple Digits and Carries

We begin with a definition. We shall use $(d_k, d_{k-1}, \ldots, d_0)_b$ to denote the natural number $\sum_{j=0}^{k} d_j b^j$ where each $0 \leq d_j < b$.

Definition 1. Let n be a natural number and σ be a permutation on $\{0, 1, 2, \ldots, k\}$. We say that $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-*permutiple* provided

$$(d_k, d_{k-1}, \ldots, d_1, d_0)_b = n(d_{\sigma(k)}, d_{\sigma(k-1)}, \ldots, d_{\sigma(1)}, d_{\sigma(0)})_b.$$

Using the language established above, letting $\rho = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$, 87912 is a $(4, 10, \rho)$-permutiple since $87912 = 4 \cdot 21978$.

In order to avoid introducing extra digits when multiplying, it is assumed that $n < b$. We note, however, that in order to circumvent overly cumbersome theorem statements, we do allow for leading zeros. Letting ε be the identity permutation, every natural number is a $(1, b, \varepsilon)$-permutiple. Such trivial examples are ignored so that $n > 1$. Furthermore, $b = 2$ implies that $n = 1$. Therefore, we impose the additional restriction that $b \neq 2$. Thus, hereafter, we assume that n and b are natural numbers such that $1 < n < b$.

The following two theorems are an exceedingly straightforward generalization of the first and third theorems of Holt [3] which concern palintiple numbers. A description of single-digit multiplication in general is as follows: let p_j denote the jth digit of the product, c_j the jth carry, and q_j the jth digit of the number being multiplied by n. Then the iterative algorithm for single-digit multiplication is

$$\begin{aligned} c_0 &= 0 \\ p_j &= \lambda(nq_j + c_j) \\ c_{j+1} &= [nq_j + c_j - \lambda(nq_j + c_j)] \div b \end{aligned}$$

where λ gives the least non-negative residue modulo b. Since $(p_k, p_{k-1}, \dots, p_0)_b$ is a $(k+1)$-digit number, $c_{k+1} = 0$. For any (n, b, σ)-permutiple, $(d_k, d_{k-1}, \dots, d_0)_b$, $q_j = d_{\sigma(j)}$ so that $d_j = p_j = \lambda(nd_{\sigma(j)} + c_j)$. Hence, we have our first result.

Theorem 1. *Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ)-permutiple and let c_j be the jth carry. Then*

$$bc_{j+1} - c_j = nd_{\sigma(j)} - d_j$$

for all $0 \leq j \leq k$.

As is the case for palintiples, the following shows that the carries of any permutiple are less than the multiplier.

Theorem 2. *Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ)-permutiple and let c_j be the jth carry. Then $c_j \leq n - 1$ for all $0 \leq j \leq k$.*

Proof. The proof will proceed by induction. We have $c_0 = 0 \leq n - 1$. Now suppose that $c_j \leq n - 1$. For a contradiction suppose $c_{j+1} \geq n$. Then Theorem 1 implies $bc_{j+1} - c_j + d_j = nd_{\sigma(j)}$. By our inductive hypothesis we have $bn - (n - 1) = (b - 1)n + 1 \leq nd_{\sigma(j)}$. Therefore, $d_{\sigma(j)} > b - 1$ which is a contradiction. $\square$

The following is a converse to Theorem 1.

Theorem 3. *Suppose $bc_{j+1} - c_j = nd_{\sigma(j)} - d_j$ for all $0 \leq j \leq k$ where $(d_k, d_{k-1}, \dots, d_0)$ is a $(k+1)$-tuple of base-b digits and $(c_k, c_{k-1}, \dots, c_0)$ is a $(k+1)$-tuple of base-n digits such that $c_0 = 0$. Then $(d_k, d_{k-1}, \dots, d_0)_b$ is an (n, b, σ)-permutiple with carries $c_k, c_{k-1}, \dots, c_0$.*

Proof. By a simple calculation,

$$\sum_{j=0}^{k}(nd_{\sigma(j)} - d_j)b^j = \sum_{j=0}^{k}(bc_{j+1} - c_j)b^j = 0.$$

Thus, $p = (d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-permutiple. Letting $\hat{c}_k$, $\hat{c}_{k-1}$, …, $\hat{c}_0$ be the carries obtained by multiplying p by n, an application of Theorem 1 and a simple induction argument establish that $\hat{c}_j = c_j$ for all $0 \leq j \leq k$. □

Letting ψ be the $(k+1)$-cycle $(0, 1, 2, \ldots, k)$, it is convenient to write the relations between the digits and the carries found in Theorems 1 and 3 in matrix form,

$$(bP_\psi - I)\mathbf{c} = (nP_\sigma - I)\mathbf{d}, \tag{1}$$

where I is the identity matrix, P_ψ and P_σ are permutation matrices, and $\mathbf{c}$ and $\mathbf{d}$ are column vectors containing the carries and digits, respectively. We note that these matrices are indexed from 0 to k rather than from 1.

We highlight that we will extensively use the fact that $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-permutiple with carries c_k, c_{k-1}, … , c_1, $c_0 = 0$ if and only if Equation 1 holds (a consequence of Theorems 1 and 3).

Multiplying both sides of Equation 1 by $\sum_{\ell=0}^{|\sigma|-1}(nP_\sigma)^\ell$, we can express the digits in terms of the carries as $\mathbf{d} = \frac{1}{n^{|\sigma|}-1}\sum_{\ell=0}^{m-1}(n^\ell P_{\sigma^\ell})(bP_\psi - I)\mathbf{c}$. Similarly, multiplying both sides by $\sum_{\ell=0}^{k}(bP_\psi)^\ell$, we can likewise express the carries in terms of the digits: $\mathbf{c} = \frac{1}{b^{k+1}-1}\sum_{\ell=0}^{k}(b^\ell P_{\psi^\ell})(nP_\sigma - I)\mathbf{d}$. In component form, we have the following result.

Theorem 4. *Let $(d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b, σ)-permutiple, and let c_j be the jth carry. Then*

$$d_j = \frac{1}{n^{|\sigma|} - 1}\sum_{\ell=0}^{|\sigma|-1}(bc_{\psi\sigma^\ell(j)} - c_{\sigma^\ell(j)})n^\ell$$

and

$$c_j = \frac{1}{b^{k+1} - 1}\sum_{\ell=0}^{k}(nd_{\sigma\psi^\ell(j)} - d_{\psi^\ell(j)})b^\ell$$

for all $0 \leq j \leq k$.

Remark. Bearing in mind that $|\psi| = k + 1$, we direct the reader's attention to the symmetry between the above equations expressing the digits in terms of the carries and vice versa. The above also generalizes the relationship between palintiple numbers and their carries found in Young [11], Sloane [9], Kendrick [6], and Holt [3, 4].

The next theorem places restrictions on the first digit of any permutiple.

Theorem 5. *For any nontrivial (n, b, σ)-permutiple, $(d_k, d_{k-1}, \dots, d_0)_b$, $\gcd(n, b)$ divides d_0.*

Proof. Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ)-permutiple where c_j is the jth carry. Then, for $j = 0$, Theorem 4 gives

$$(n^{|\sigma|} - 1)d_0 = b\left(\sum_{\ell=0}^{|\sigma|-1} c_{\psi\sigma^{\ell}(0)} n^{\ell}\right) - n\left(\sum_{\ell=1}^{|\sigma|-1} c_{\sigma^{\ell}(0)} n^{\ell-1}\right).$$

Thus, $\gcd(n, b)$ divides d_0 since n and $n^{|\sigma|} - 1$ are relatively prime. □

3. New Permutiples from Old

We shall now consider the problem of finding new permutiples from known examples. The approach taken here, as stated in the beginning, will be to restrict our attention to finding new permutiples having the same digits as our known example. The ultimate aim of our effort is to answer the question of whether or not all permutiples having the same digits can be found from a single example. If not, are there conditions under which it is possible? The following results give us some methods for constructing new permutiples from old.

Theorem 6. *Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ)-permutiple with carries c_k, c_{k-1}, $\dots$, c_0. Let μ be a permutation such that $c_{\mu(0)} = 0$. Then $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ is an $(n, b, \pi^{-1}\sigma\pi)$-permutiple with carries $c_{\mu(k)}$, $c_{\mu(k-1)}$, $\dots$, $c_{\mu(0)}$ if and only if $P_\pi(bP_\psi - I)\mathbf{c} = (bP_\psi - I)P_\mu\mathbf{c}$.*

Proof. By our hypothesis, Equation 1 is satisfied. Thus, if $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ is an $(n, b, \pi^{-1}\sigma\pi)$-permutiple with carries $c_{\mu(k)}$, $c_{\mu(k-1)}$, $\dots$, $c_{\mu(0)}$, then

$$P_\pi(bP_\psi - I)\mathbf{c} = P_\pi(nP_\sigma - I)\mathbf{d} = (nP_{\pi^{-1}\sigma\pi} - I)P_\pi\mathbf{d} = (bP_\psi - I)P_\mu\mathbf{c}.$$

Conversely, if $P_\pi(bP_\psi - I)\mathbf{c} = (bP_\psi - I)P_\mu\mathbf{c}$, then

$$(nP_{\pi^{-1}\sigma\pi} - I)P_\pi\mathbf{d} = P_\pi(nP_\sigma - I)\mathbf{d} = P_\pi(bP_\psi - I)\mathbf{c} = (bP_\psi - I)P_\mu\mathbf{c}.$$

□

Corollary 1. *Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ)-permutiple with carries c_k, c_{k-1}, $\dots$, c_0. If $c_j = 0$, then $(d_{\psi^j(k)}, d_{\psi^j(k-1)}, \dots, d_{\psi^j(1)}, d_{\psi^j(0)})_b$ is an $(n, b, \psi^{-j}\sigma\psi^j)$-permutiple with carries $c_{\psi^j(k)}$, $c_{\psi^j(k-1)}$, $\dots$, $c_{\psi^j(1)}$, $c_{\psi^j(0)} = c_j = 0$.*

Remark. We note that the above corollary follows either from Theorem 4 by setting $c_j = 0$, or from Theorem 6 by setting $\pi = \mu = \psi^j$.

Example 1. Consider the $(4, 10, \rho)$-permutiple $(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$. Performing routine multiplication, we see that the carries are $(c_4, c_3, c_2, c_1, c_0) = (0, 3, 3, 3, 0)$. Not surprisingly, applying Corollary 1 to $j = 0$ yields the original permutiple. However, for the case of $j = 4$, we have $(7, 9, 1, 2, 8)_{10} = 4 \cdot (1, 9, 7, 8, 2)_{10}$ with carries $(c_{\psi^4(4)}, c_{\psi^4(3)}, c_{\psi^4(2)}, c_{\psi^4(1)}, c_{\psi^4(0)}) = (c_3, c_2, c_1, c_0, c_4) = (3, 3, 3, 0, 0)$.

Applying Corollary 1 a bit more generally, if $(d_k, d_{k-1}, \ldots, d_0)_b$ is any (n, b)-palintiple such that $c_k = 0$ (this includes all symmetric, doubly-derived, and doubly-reverse-derived palintiples [4]), then $(d_{k-1}, d_{k-2}, \ldots, d_0, d_k)_b$ is an $(n, b, \psi^{-k}\rho\psi^k)$-permutiple.

Setting μ to the identity permutation in Theorem 6, we obtain another useful corollary.

Corollary 2. *Let $(d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b, σ)-permutiple with carries c_k, c_{k-1}, $\ldots$, c_0. Then $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ is an $(n, b, \pi^{-1}\sigma\pi)$-permutiple with carries c_k, c_{k-1}, $\ldots$, c_0 if and only if $P_\pi(bP_\psi - I)\boldsymbol{c} = (bP_\psi - I)\boldsymbol{c}$.*

Example 2. Consider again the base-10 palintiple $(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$ with carries $(c_4, c_3, c_2, c_1, c_0) = (0, 3, 3, 3, 0)$. With the above in mind, we calculate

$$(10P_\psi - I)\mathbf{c} = \left(10 \cdot \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 30 \\ 27 \\ 27 \\ -3 \\ 0 \end{bmatrix}.$$

Since the above column vector is unchanged by P_π, where π is the transposition $(1, 2)$, we see by Corollary 2 that $(8, 7, 1, 9, 2)_{10}$ is a $(4, 10, (1, 2)\rho(1, 2))$-permutiple with carries $(c_4, c_3, c_2, c_1, c_0) = (0, 3, 3, 3, 0)$ which may also be be confirmed by simple arithmetic.

Performing the same calculation as above, the $(4, 10, \psi^{-4}\rho\psi^4)$-permutiple from Example 1, $(7, 9, 1, 2, 8)_{10} = 4 \cdot (1, 9, 7, 8, 2)_{10}$, yields via Corollary 2 the new $(4, 10, (2, 3)\psi^{-4}\rho\psi^4(2, 3))$-permutiple $(7, 1, 9, 2, 8)_{10} = 4 \cdot (1, 7, 9, 8, 2)_{10}$ with carries $(c_4, c_3, c_2, c_1, c_0) = (3, 3, 3, 0, 0)$. We note that we arrive at the same result by applying Corollary 1 to the $(4, 10, (1, 2)\rho(1, 2))$-permutiple $(8, 7, 1, 9, 2)_{10}$ which is affirmed by the fact that $(1, 2)\psi^4 = \psi^4(2, 3)$.

At this point, several questions naturally present themselves. Can all permutiples having a particular set of digits be found by repeated use of Theorem 6 and its corollaries? One does not have to look far to see that the answer is no. If we consider the example $(7, 8, 9, 1, 2)_{10} = 4 \cdot (1, 9, 7, 2, 8)_{10}$ with carries $(c_4, c_3, c_2, c_1, c_0) = (3, 2, 1, 3, 0)$, we see that the results obtained thus far do not account for this example since the carries are different.

Another question is if we have an (n, b, σ)-permutiple, $(d_k, d_{k-1}, \dots, d_0)_b$, with carries c_k, c_{k-1}, $\dots$, c_0, and an (n, b, τ)-permutiple, $(d_{\pi(k)}, \dots, d_{\pi(0)})_b$, with permuted carries $c_{\mu(k)}$, $c_{\mu(k-1)}$, $\dots$, $c_{\mu(0)}$, must it be that $\tau = \pi^{-1}\sigma\pi$? Again, with a little effort, we can find an example which shows that this is not always the case. Consider $(4, 3, 5, 1, 2)_6 = 2 \cdot (2, 1, 5, 3, 4)_6$, a $(2, 6, \sigma)$-permutiple, and $(2, 5, 1, 3, 4)_6 = 2 \cdot (1, 2, 3, 4, 5)_6$, a $(2, 6, \tau)$-permutiple, both with the same carry vector $(0, 1, 1, 1, 0)$. Now $\sigma = (0, 4)(1, 3)$, $\pi = (0, 4)(3, 2, 1)$, and $\tau = (0, 3, 4, 2, 1)$, but $\tau \neq \pi^{-1}\sigma\pi$.

Thus, it is clear that our results so far do not account for every possibility. Therefore, we shall require some additional machinery in order to find every permutiple with the same digits as our known example. Again, for the purpose of less cumbersome theorem statements, we shall henceforth assume that $(d_k, d_{k-1}, \dots, d_0)_b$ is an (n, b, σ)-permutiple. We begin with a definition, motivated by the above, which will help us to organize and classify our new examples.

Definition 2. We say that an (n, b, τ_1)-permutiple, $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$, and an (n, b, τ_2)-permutiple, $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$, are *conjugate* if $\pi_1\tau_1\pi_1^{-1} = \pi_2\tau_2\pi_2^{-1}$.

Clearly, permutiple conjugacy defines an equivalence relation on the collection of all base-b permutiples with multiplier n having the same digits. From this fact, we need to establish some additional terminology. For any two (n, b, τ_1) and (n, b, τ_2)-permutiples of the same conjugacy class, $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ and $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$, we shall refer to the common permutation $\beta = \pi_1\tau_1\pi_1^{-1} = \pi_2\tau_2\pi_2^{-1}$ as the *base permutation* of the class. We emphasize that the base permutation of a conjugacy class might not necessarily be a digit permutation itself.

Our next result tells us that two permutiples in the same conjugacy class both have the same set of carries.

Theorem 7. *Let* $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ *and* $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$ *be* (n, b, τ_1) *and* (n, b, τ_2)*-permutiples, respectively, from the same conjugacy class with carries given by* c_k, c_{k-1}, $\dots$, c_0 *and* $\hat{c}_k$, $\hat{c}_{k-1}$, $\dots$, $\hat{c}_0$*, respectively. Then* $\hat{c}_j = c_{\pi_1^{-1}\pi_2(j)}$ *for all* $0 \leq j \leq k$.

Proof. By assumption, we have both that $(nP_{\tau_1} - I)P_{\pi_1}\mathbf{d} = (bP_\psi - I)\mathbf{c}$ and that $(nP_{\tau_2} - I)P_{\pi_2}\mathbf{d} = (bP_\psi - I)\hat{\mathbf{c}}$. Then $P_{\pi_1}(nP_{\pi_1\tau_1\pi_1^{-1}} - I)\mathbf{d} = (bP_\psi - I)\mathbf{c}$ and $P_{\pi_2}(nP_{\pi_2\tau_2\pi_2^{-1}} - I)\mathbf{d} = (bP_\psi - I)\hat{\mathbf{c}}$. Since both permutiples are conjugate we have that $P_{\pi_1\tau_1\pi_1^{-1}} = P_{\pi_2\tau_2\pi_2^{-1}}$. It follows that $P_{\pi_1^{-1}}(bP_\psi - I)\mathbf{c} = P_{\pi_2^{-1}}(bP_\psi - I)\hat{\mathbf{c}}$. Reducing modulo b we have $P_{\pi_1^{-1}}\mathbf{c} \equiv P_{\pi_2^{-1}}\hat{\mathbf{c}} \mod b$, or $P_{\pi_1^{-1}\pi_2}\mathbf{c} \equiv \hat{\mathbf{c}} \mod b$. Theorem 2 then implies that $\hat{\mathbf{c}} = P_{\pi_1^{-1}\pi_2}\mathbf{c}$. $\square$

The above theorem gives us the following important result.

Theorem 8. *Let* $p = (d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ *be an* (n, b, τ_1)*-permutiple with carries* c_k*,* c_{k-1}*, ... ,* c_0*. If* $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$ *is an* (n, b, τ_2)*-permutiple from the same conjugacy class as* p*, then* $c_{\psi\pi_1^{-1}\pi_2(j)} = c_{\pi_1^{-1}\pi_2\psi(j)}$ *for all* $0 \le j \le k$*.*

Proof. Our first assumption in matrix form is $(nP_{\tau_1} - I)P_{\pi_1}\mathbf{d} = (bP_\psi - I)\mathbf{c}$, and by Theorem 7, our second assumption becomes $(nP_{\tau_2} - I)P_{\pi_2}\mathbf{d} = (bP_\psi - I)P_{\pi_1^{-1}\pi_2}\mathbf{c}$. Using this second equation, we have

$$(nP_{\pi_2\tau_2\pi_2^{-1}} - I)\mathbf{d} = P_{\pi_2^{-1}}(nP_{\tau_2} - I)P_{\pi_2}\mathbf{d} = P_{\pi_2^{-1}}(bP_\psi - I)P_{\pi_1^{-1}\pi_2}\mathbf{c},$$

which by conjugacy gives

$$(nP_{\pi_1\tau_1\pi_1^{-1}} - I)\mathbf{d} = P_{\pi_2^{-1}}(bP_\psi - I)P_{\pi_1^{-1}\pi_2}\mathbf{c}.$$

Multiplying by P_{π_1}, we then have

$$(nP_{\tau_1} - I)P_{\pi_1}\mathbf{d} = P_{\pi_1}P_{\pi_2^{-1}}(bP_\psi - I)P_{\pi_1^{-1}\pi_2}\mathbf{c},$$

which by the first relation above becomes

$$(bP_\psi - I)\mathbf{c} = P^{-1}_{\pi_1^{-1}\pi_2}(bP_\psi - I)P_{\pi_1^{-1}\pi_2}\mathbf{c}.$$

The above reduces to

$$P_{\pi_1^{-1}\pi_2}P_\psi\mathbf{c} = P_\psi P_{\pi_1^{-1}\pi_2}\mathbf{c},$$

and the proof is complete. □

With the above theorem, we may determine a list of candidate permutations, π, within a particular conjugacy class. The next theorem tells us that every item on this list yields a permutiple.

Theorem 9. *Let* $p = (d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ *be an* (n, b, τ_1)*-permutiple with carries* c_k*,* c_{k-1}*, ... ,* c_0*. If* π_2 *is a permutation such that* $c_{\pi_1^{-1}\pi_2(0)} = 0$*, and* $c_{\psi\pi_1^{-1}\pi_2(j)} = c_{\pi_1^{-1}\pi_2\psi(j)}$ *for all* $0 \le j \le k$*, then* $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$ *is an* (n, b, τ_2)*-permutiple from the same conjugacy class as* p*, where* $\tau_2 = \pi_2^{-1}\pi_1\tau_1\pi_1^{-1}\pi_2$*.*

Proof. By assumption, we have that $(nP_{\tau_1} - I)P_{\pi_1}\mathbf{d} = (bP_\psi - I)\mathbf{c}$ and $P_{\pi_1^{-1}\pi_2}P_\psi\mathbf{c} = P_\psi P_{\pi_1^{-1}\pi_2}\mathbf{c}$. Multiplying the first of the above equations by $P_{\pi_1^{-1}\pi_2}$, and then using the second equation, yields $P_{\pi_1^{-1}\pi_2}(nP_{\tau_1} - I)P_{\pi_1}\mathbf{d} = (bP_\psi - I)P_{\pi_1^{-1}\pi_2}\mathbf{c}$. Our first assumption and a routine calculation then show that

$$(nP_{\pi_2^{-1}\pi_1\tau_1\pi_1^{-1}\pi_2} - I)P_{\pi_2}\mathbf{d} = (bP_\psi - I)P_{\pi_1^{-1}\pi_2}\mathbf{c},$$

and the proof is complete. □

If we take the point of view that there is a reference permutiple, p, in every conjugacy class (not necessarily our initial example), we can simply take π_1 in the above to be the identity, and $\pi = \pi_2$ to be any suitable permutation of the digits of p. The above theorem then implies that $P_\pi P_\psi \mathbf{c} = P_\psi P_\pi \mathbf{c}$, or, $P_\psi \mathbf{c} = P_{\pi\psi\pi^{-1}}\mathbf{c}$. We state the above as a single corollary to Theorems 8 and 9.

Corollary 3. *Let $p = (d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b, τ_1)-permutiple with carries c_k, c_{k-1}, ... , c_0. Then the following hold:*

1. *If $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ is an (n, b, τ_2)-permutiple from the same conjugacy class as p, then $c_{\psi(j)} = c_{\pi\psi\pi^{-1}(j)}$ for all $0 \leq j \leq k$.*
2. *If π is a permutation such that $c_{\pi(0)} = 0$ and $c_{\psi(j)} = c_{\pi\psi\pi^{-1}(j)}$ for all $0 \leq j \leq k$, then $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ is an (n, b, τ_2)-permutiple from the same conjugacy class as p.*

The above corollary, together with $\pi\psi\pi^{-1} = (\pi(0), \pi(1), \ldots, \pi(k))$, enables us to construct the entire collection of all permutations, π, of a reference permutiple's digits within a particular conjugacy class. The next example illustrates this process.

Example 3. Consider an earlier example, $(d_4, d_3, d_2, d_1, d_0) = (4, 3, 5, 1, 2)_6 = 2\cdot(2, 1, 5, 3, 4)_6$, a $(2, 6, \sigma)$-permutiple whose carry vector is $(0, 1, 1, 1, 0)$. The reader will note that this example is a $(2, 6)$-palintiple.

We shall use Corollary 3 to find all permutiples conjugate to $(d_4, d_3, d_2, d_1, d_0)$. By Corollary 3, we know that if $(d_4, d_3, d_2, d_1, d_0)$ and $(d_{\pi(4)}, d_{\pi(2)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})$ are conjugate, then π necessarily satisfies

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = P_\psi \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = P_{\pi\psi\pi^{-1}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Restating the above in component form, we have

$$\begin{aligned} c_{\pi\psi\pi^{-1}(0)} &= 1 = c_1, c_2, \text{ or, } c_3, \\ c_{\pi\psi\pi^{-1}(1)} &= 1 = c_1, c_2, \text{ or, } c_3, \\ c_{\pi\psi\pi^{-1}(2)} &= 1 = c_1, c_2, \text{ or, } c_3, \\ c_{\pi\psi\pi^{-1}(3)} &= 0 = c_0 \text{ or } c_4, \\ c_{\pi\psi\pi^{-1}(4)} &= 0 = c_0 \text{ or } c_4. \end{aligned}$$

The list of possible candidates for $\pi\psi\pi^{-1}$ from above then consists of any permutation which can be constructed from

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1, 2 \text{ or, } 3 & 1, 2 \text{ or, } 3 & 1, 2 \text{ or, } 3 & 0 \text{ or } 4 & 0 \text{ or } 4 \end{pmatrix}.$$

Since $\pi\psi\pi^{-1}$ is a 5-cycle, we can eliminate any fixed points so that the above permutation must have the form

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1,2 \text{ or}, 3 & 2 \text{ or } 3 & 1 \text{ or } 3 & 4 & 0 \end{pmatrix}.$$

Therefore, $\pi\psi\pi^{-1} = (\pi(0), \pi(1), \pi(2), \pi(3), \pi(4))$ must equal either $(0, 1, 2, 3, 4)$ or $(0, 2, 1, 3, 4)$. Since $c_0 = c_4 = 0$, we have four permutations which satisfy the conditions of Corollary 3: the identity $\varepsilon = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$, $\psi^4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}$, $(1, 2) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 3 & 4 \end{pmatrix}$, and $(1, 2)\psi^4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 2 & 1 & 3 \end{pmatrix}$. These permutations give us the entire conjugacy class listed in the table below.

$(d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_6$	π	τ	$(c_{\pi(4)}, c_{\pi(3)}, c_{\pi(2)}, c_{\pi(1)}, c_{\pi(0)})$
$(4, 3, 5, 1, 2)_6$	ε	ρ	$(0, 1, 1, 1, 0)$
$(4, 3, 1, 5, 2)_6$	$(1, 2)$	$(1, 2)\rho(1, 2)$	$(0, 1, 1, 1, 0)$
$(3, 5, 1, 2, 4)_6$	ψ^4	$\psi^{-4}\rho\psi^4$	$(1, 1, 1, 0, 0)$
$(3, 1, 5, 2, 4)_6$	$(1, 2)\psi^4$	$\psi^{-4}(1, 2)\rho(1, 2)\psi^4$	$(1, 1, 1, 0, 0)$

Table 1: The conjugacy class corresponding to ρ.

We shall now look between conjugacy classes. The converse of Theorem 7 does not hold in general. However, assuming its consequent does yield a useful theorem which gives us a list of base permutation candidates from every conjugacy class with the same set of carries as the original example.

Theorem 10. *Let $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \ldots, d_{\pi_1(0)})_b$ and $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \ldots, d_{\pi_2(0)})_b$ be (n, b, τ_1) and (n, b, τ_2)-permutiples, respectively, with carries given by c_k, c_{k-1}, $\ldots$, c_0 and $\hat{c}_k$, $\hat{c}_{k-1}$, $\ldots$, $\hat{c}_0$, respectively. If $\hat{c}_j = c_{\pi_1^{-1}\pi_2(j)}$ for all $0 \le j \le k$, then $nd_{\pi_1\tau_1\pi_1^{-1}(j)} \equiv nd_{\pi_2\tau_2\pi_2^{-1}(j)} \mod b$ for all $0 \le j \le k$.*

Proof. Our assumptions in matrix form are $(nP_{\tau_1} - I)P_{\pi_1}\mathbf{d} = (bP_\psi - I)\mathbf{c}$ and $(nP_{\tau_2} - I)P_{\pi_2}\mathbf{d} = (bP_\psi - I)P_{\pi_1^{-1}\pi_2}\mathbf{c}$. Reducing modulo b, we have both that $(nP_{\tau_1} - I)P_{\pi_1}\mathbf{d} \equiv -\mathbf{c} \mod b$ and $(nP_{\tau_2} - I)P_{\pi_2}\mathbf{d} \equiv -P_{\pi_1^{-1}\pi_2}\mathbf{c} \mod b$. It follows that $P_{\pi_1}(nP_{\pi_1\tau_1\pi_1^{-1}} - I)\mathbf{d} \equiv -\mathbf{c} \mod b$ and $P_{\pi_2}(nP_{\pi_2\tau_2\pi_2^{-1}} - I)\mathbf{d} \equiv -P_{\pi_1^{-1}\pi_2}\mathbf{c} \mod b$, from which we obtain $(nP_{\pi_1\tau_1\pi_1^{-1}} - I)\mathbf{d} \equiv -P_{\pi_1^{-1}}\mathbf{c} \equiv (nP_{\pi_2\tau_2\pi_2^{-1}} - I)\mathbf{d} \mod b$. Thus, $nP_{\pi_1\tau_1\pi_1^{-1}}\mathbf{d} \equiv nP_{\pi_2\tau_2\pi_2^{-1}}\mathbf{d} \mod b$. □

Letting π_1 be the identity and τ_1 be σ in the above theorem, we obtain a result which relates any (n, b, τ)-permutiple with the same set of carries to our known example.

Corollary 4. *Let $(d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b, σ)-permutiple with carries c_k, c_{k-1}, $\ldots$, c_0, and let $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ be an (n, b, τ)-permutiple with carries $c_{\pi(k)}$, $c_{\pi(k-1)}$, $\ldots$, $c_{\pi(0)}$. Then $nd_{\sigma(j)} \equiv nd_{\pi\tau\pi^{-1}(j)} \mod b$ for all $0 \le j \le k$.*

The above corollary will, under certain conditions, enable us to find all base permutations $\beta = \pi\tau\pi^{-1}$ for every possible conjugacy class.

Our next result gives us conditions for the existence of a bijective correspondence between permutiples, namely,

$$(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b \mapsto (d_{\alpha\pi(k)}, d_{\alpha\pi(k-1)}, \ldots, d_{\alpha\pi(0)})_b.$$

Theorem 11. *Let* $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \ldots, d_{\pi_1(0)})_b$ *and* $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \ldots, d_{\pi_2(0)})_b$ *be* (n, b, τ_1) *and* (n, b, τ_2)*-permutiples, respectively. If there exists an* α *such that* $(nP_{\tau_2} - I)P_\alpha \boldsymbol{d} = (nP_{\tau_1} - I)\boldsymbol{d}$*, then* $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ *is an* $(n, b, \pi^{-1}\tau_1\pi)$*-permutiple if and only if* $(d_{\alpha\pi(k)}, d_{\alpha\pi(k-1)}, \ldots, d_{\alpha\pi(0)})_b$ *is an* $(n, b, \pi^{-1}\tau_2\pi)$*-permutiple.*

Proof. If $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ is an $(n, b, \pi^{-1}\tau_1\pi)$-permutiple with carries c_k, c_{k-1}, ... , c_0, then $(nP_{\pi^{-1}\tau_1\pi} - I)P_\pi\mathbf{d} = (bP_\psi - I)\mathbf{c}$. Now, by the theorem's hypothesis, we have

$$(nP_{\pi^{-1}\tau_1\pi} - I)P_\pi\mathbf{d} = P_\pi(nP_{\tau_1} - I)\mathbf{d} = P_\pi(nP_{\tau_2} - I)P_\alpha\mathbf{d} = (nP_{\pi^{-1}\tau_2\pi} - I)P_{\alpha\pi}\mathbf{d}.$$

Then $(nP_{\pi^{-1}\tau_2\pi} - I)P_{\alpha\pi}\mathbf{d} = (bP_\psi - I)\mathbf{c}$. By Theorem 3, the forward implication holds. The reverse implication follows in similar fashion. $\square$

Thus, the above gives us a bijection between conjugacy classes provided that $\pi_1\tau_1\pi_1^{-1} \neq \pi_2\tau_2\pi_2^{-1}$. Also, the reader should note that according to the above argument, the carries of $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ and $(d_{\alpha\pi(k)}, d_{\alpha\pi(k-1)}, \ldots, d_{\alpha\pi(0)})_b$ must be the same.

At this point, the big question is whether or not the results given thus far can give us all the examples we seek. If we recall our initial examples which motivated Definition 2, we can see that, in general, the answer is no. However, there is a condition which guarantees that we have found all of the desired examples. The next theorem tells us that if n divides b, then all permutiples having the same digits as our known example have the same set of carries as the known example.

Theorem 12. *Let* $(d_k, d_{k-1}, \ldots, d_0)_b$ *be an* (n, b, σ)*-permutiple with carries* c_k, c_{k-1}, ... , c_0*, and let* $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ *be an* (n, b, τ)*-permutiple with carries* $\hat{c}_k$, $\hat{c}_{k-1}$, ... , $\hat{c}_0$*. If* n *divides* b*, then* $\hat{c}_j = c_{\pi(j)}$ *for all* $0 \leq j \leq k$*.*

Proof. By Equation 1, we have both $(bP_\psi - I)\mathbf{c} = (nP_\sigma - I)\mathbf{d}$ and $(bP_\psi - I)\hat{\mathbf{c}} = (nP_\tau - I)P_\pi\mathbf{d}$. Since n divides b, it follows that $\mathbf{c} \equiv \mathbf{d} \mod n$ and $\hat{\mathbf{c}} \equiv P_\pi\mathbf{d} \mod n$. Thus, $\hat{\mathbf{c}} \equiv P_\pi\mathbf{c} \mod n$. By Theorem 2, it follows that $\hat{\mathbf{c}} = P_\pi\mathbf{c}$, which establishes the theorem. $\square$

Thus, when n divides b, Corollary 4 enables us to find every possible base permutation $\beta = \pi\tau\pi^{-1}$. From there, we may use either Theorem 1 or other techniques (such as those in the following example) to find a reference permutiple from each conjugacy class. Then, using Corollary 3, we find all π within each conjugacy class. Thus, when n divides b, finding all permutiples with the same digits as a known permutiple becomes considerably easier. The next two examples illustrate the above approach.

Example 4. Using our results, we find all 5-digit $(2, 6, \sigma)$-permutiples starting from the $(2, 6, \rho)$-permutiple $(d_4, d_3, d_2, d_1, d_0)_6 = (4, 3, 5, 1, 2)_6 = 2 \cdot (2, 1, 5, 3, 4)_6$ with carries $(c_4, c_3, c_2, c_1, c_0) = (0, 1, 1, 1, 0)$ which we considered in Example 3.

By Corollary 4 and Theorem 12, any suitable base permutation, β, necessarily satisfies $2d_{\rho(j)} \equiv 2d_{\beta(j)} \mod 6$ for all $0 \leq j \leq 4$, which, since 2 divides 6, becomes $d_{\rho(j)} \equiv d_{\beta(j)} \mod 3$ for all $0 \leq j \leq 4$. In matrix form we have

$$\begin{bmatrix} d_{\beta(0)} \\ d_{\beta(1)} \\ d_{\beta(2)} \\ d_{\beta(3)} \\ d_{\beta(4)} \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \mod 3.$$

Expressing the above in component form gives us

$$\begin{aligned} d_{\beta(0)} &= 1 = d_1 \text{ or } d_4, \\ d_{\beta(1)} &= 0 = d_3, \\ d_{\beta(2)} &= 2 = d_0 \text{ or } d_2, \\ d_{\beta(3)} &= 1 = d_1 \text{ or } d_4, \\ d_{\beta(4)} &= 2 = d_0 \text{ or } d_2. \end{aligned}$$

Then, any base permutation, β, necessarily has the form

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 \text{ or } 4 & 3 & 0 \text{ or } 2 & 1 \text{ or } 4 & 0 \text{ or } 2 \end{pmatrix}.$$

Thus, there are four candidate base permutations: $\beta_1 = \rho$, $\beta_2 = (4, 2, 0, 1, 3)$, $\beta_3 = (4, 2, 0)(1, 3)$, and $\beta_4 = (4, 0, 1, 3)$.

For $\beta_1 = \rho$, the solution corresponding to our known example, we note that we already determined its conjugacy class in Example 3.

We now consider the conjugacy class for $\beta_2 = (4, 2, 0, 1, 3)$. Using Theorem 11, we shall find this class by finding a bijection from the class with base permutation ρ which we found in Example 3. But first we must find an example from the conjugacy class with base permutation β_2. Provided a suitable permutation α exists, the bijection guaranteed by Theorem 11 maps permutiples with carry vector $\mathbf{c}$ to other permutiples with the same carry vector. Therefore, if such an α exists, we know that our known example $(d_k, d_{k-1}, \ldots, d_0)$ will map to $(d_{\alpha(k)}, d_{\alpha(k-1)}, \ldots, d_{\alpha(0)})_b$. Then, by Theorem 12, α fixes the carry vector $(0, 1, 1, 1, 0)$ of our known $(2, 6, \rho)$ example. Thus, α must contain a factor of either the identity or $(4, 0)$, and a factor of $(3, 2, 1)$, $(1, 2, 3)$, $(3, 1, 2)$, $(1, 2)$, $(1, 2)$, $(1, 3)$, or $(2, 3)$. Using simple base-6 arithmetic to check the possibilities which are not already listed in the class with base permutation ρ, we see that either $\alpha = (4,0)(3,2,1) = (1,2)\rho$ or $\alpha = (4,0)(3,2) = (1,2)\rho(1,2)$. Respectively, these values give us the permutiples $(2,5,1,3,4)_6 = 2 \cdot (1,2,3,4,5)_6$, for which $\tau_2 = (0,1)\beta_2(0,1) = (4,2,1,0,3)$, and $(2,5,3,1,4)_6 = 2 \cdot (1,2,4,3,5)_6$, for which $\tau_2 = (1,2)(0,1)\beta_2(0,1)(1,2) = (4,1,2,0,3)$. The reader may check that both values of α yield a bijection. Using the first value, $\alpha = (1,2)\rho$ with $\tau_2 = (0,1)\beta_2(0,1) = (4,2,1,0,3)$, Theorem 11 easily gives us the rest of the permutiples in this class listed below.

$(d_{\pi(4)}, d_{\pi(3)}, \ldots, d_{\pi(0)})_6$	π	τ	$(c_{\pi(4)}, c_{\pi(3)}, \ldots, c_{\pi(0)})$
$(2,5,1,3,4)_6$	$(1,2)\rho$	$\tau_2 = (4,2,1,0,3)$	$(0,1,1,1,0)$
$(2,5,3,1,4)_6$	$(1,2)\rho(1,2)$	$(1,2)\tau_2(1,2)$	$(0,1,1,1,0)$
$(5,1,3,4,2)_6$	$(1,2)\rho\psi^4$	$\psi^{-4}\tau_2\psi^4$	$(1,1,1,0,0)$
$(5,3,1,4,2)_6$	$(1,2)\rho(1,2)\psi^4$	$\psi^{-4}(1,2)\tau_2(1,2)\psi^4$	$(1,1,1,0,0)$

Table 2: The conjugacy class corresponding to $\beta_2 = (4,2,0,1,3)$.

To determine the conjugacy class having the base permutation $\beta_3 = (4,2,0)(3,1)$, we find an example from this class. We shall attempt to find a $(2,6,\beta_3)$-permutiple $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$. Then $\pi\beta_3\pi^{-1} = \beta_3$ so that

$$(\pi(4), \pi(2), \pi(0))(\pi(3), \pi(1)) = (4,2,0)(3,1).$$

Since the first carry of any permutiple must always be zero, we know that $\pi(0)$ equals either 0 or 4. Hence, provided π commutes with β_3, there are eight possibilities, among which, $\pi = \beta_3$ provides us with a solution. Therefore, by Theorem 12, with $\pi = (4,2,0)(1,3)$, we have that $(c_{\pi(4)}, c_{\pi(3)}, c_{\pi(2)}, c_{\pi(1)}, c_{\pi(0)}) = (1,1,0,1,0)$. So the new initial carry vector is given by

$$\hat{\mathbf{c}} = P_\pi \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

In order to use Corollary 3, and for the sake of cleaner notation, we shall take $(\hat{d}_4, \hat{d}_3, \hat{d}_2, \hat{d}_1, \hat{d}_0)_6$ to mean $(d_{\pi(4)}, d_{\pi(2)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_6 = (5,1,2,3,4)_6$. Then Corollary 3 tells us that if both $(\hat{d}_4, \hat{d}_3, \hat{d}_2, \hat{d}_1, \hat{d}_0)_6$ and $(\hat{d}_{\pi(4)}, \hat{d}_{\pi(2)}, \hat{d}_{\pi(2)}, \hat{d}_{\pi(1)}, \hat{d}_{\pi(0)})_6$ are conjugate, then π necessarily satisfies

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = P_\psi \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = P_{\pi\psi\pi^{-1}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Restating the above in component form, we have

$$\begin{aligned}\hat{c}_{\pi\psi\pi^{-1}(0)} &= 1 = \hat{c}_1, \hat{c}_3, \text{ or, } \hat{c}_4,\\ \hat{c}_{\pi\psi\pi^{-1}(1)} &= 0 = \hat{c}_0 \text{ or } \hat{c}_2,\\ \hat{c}_{\pi\psi\pi^{-1}(2)} &= 1 = \hat{c}_1, \hat{c}_3, \text{ or, } \hat{c}_4,\\ \hat{c}_{\pi\psi\pi^{-1}(3)} &= 1 = \hat{c}_1, \hat{c}_3, \text{ or, } \hat{c}_4,\\ \hat{c}_{\pi\psi\pi^{-1}(4)} &= 0 = \hat{c}_0 \text{ or } \hat{c}_2,\end{aligned}$$

which provides a list of possible values of $\pi\psi\pi^{-1}$, which can be any permutation of the form

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1,3 \text{ or, } 4 & 0 \text{ or } 2 & 1,3 \text{ or, } 4 & 1 \text{ or } 4 & 0 \text{ or } 2 \end{pmatrix}.$$

Examining all eight possibilities, $\pi\psi\pi^{-1} = (\pi(0), \pi(1), \pi(2), \pi(3), \pi(4))$ must equal $(0,1,2,3,4)$, $(0,3,4,2,1)$, $(0,4,2,3,1)$, or $(0,3,1,2,4)$. Since $\hat{c}_0 = \hat{c}_2 = 0$, we have the following eight permutations which satisfy Corollary 3: the identity $\varepsilon = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$, $\pi_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 4 & 2 & 1 \end{pmatrix}$, $\pi_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 2 & 3 & 1 \end{pmatrix}$, $\pi_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 2 & 4 \end{pmatrix}$, $\pi_5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \end{pmatrix}$, $\pi_6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 3 & 4 \end{pmatrix}$, $\pi_7 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 0 & 4 \end{pmatrix}$, and $\pi_8 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 4 & 0 & 3 & 1 \end{pmatrix}$. These permutations give us the entire conjugacy class corresponding to $\beta_3 = (4,2,0)(3,1)$.

$(\hat{d}_{\pi(4)}, \hat{d}_{\pi(3)}, \hat{d}_{\pi(2)}, \hat{d}_{\pi(1)}, \hat{d}_{\pi(0)})_6$	π	τ	$(\hat{c}_{\pi(4)}, \hat{c}_{\pi(3)}, \hat{c}_{\pi(2)}, \hat{c}_{\pi(1)}, \hat{c}_{\pi(0)})$
$(5,1,2,3,4)_6$	ε	$\beta_3 = (4,2,0)(1,3)$	$(1,1,0,1,0)$
$(3,2,5,1,4)_6$	π_2	$\pi_2^{-1}\beta_3\pi_2$	$(1,0,1,1,0)$
$(3,1,2,5,4)_6$	π_3	$\pi_3^{-1}\beta_3\pi_3$	$(1,1,0,1,0)$
$(5,2,3,1,4)_6$	π_4	$\pi_4^{-1}\beta_3\pi_4$	$(1,0,1,1,0)$
$(3,4,5,1,2)_6$	π_5	$\pi_5^{-1}\beta_3\pi_5$	$(1,0,1,1,0)$
$(5,1,4,3,2)_6$	π_6	$\pi_6^{-1}\beta_3\pi_6$	$(1,1,0,1,0)$
$(5,4,3,1,2)_6$	π_7	$\pi_7^{-1}\beta_3\pi_7$	$(1,0,1,1,0)$
$(3,1,4,5,2)_6$	π_8	$\pi_8^{-1}\beta_3\pi_8$	$(1,1,0,1,0)$

Table 3: The conjugacy class corresponding to $\beta_3 = (4,2,0)(3,1)$.

We now consider the final candidate $\beta_4 = (4,0,1,3)$. By Theorem 12, any suitable π in this class must satisfy

$$P_\pi(2P_{\beta_4} - I)\mathbf{d} = (2P_{\pi^{-1}\beta_4\pi} - I)P_\pi\mathbf{d} = (6P_\psi - I)P_\pi\mathbf{c},$$

or,

$$P_\pi(2P_{\beta_4} - I)\begin{bmatrix}2\\1\\5\\3\\4\end{bmatrix} = (6P_\psi - I)P_\pi\begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix},$$

which simplifies to

$$P_\pi \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = P_\psi P_\pi \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We plainly see that P_ψ must fix the column vector

$$P_\pi \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Since there is no permutation, π, which makes this statement true, there is no π for which $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ is a $(2, 6, \pi^{-1}\beta_4\pi)$-permutiple.

We shall now apply the above techniques to another example with more digits and a larger base. However, instead of beginning with a palintiple as in Examples 1, 2, 3, and 4, we shall find all permutiples having the same digits as the base-12 cyclic number $(1, 8, 6, 10, 3, 5)_{12}$. In particular, we shall find all $(3, 12)$-permutiples from the example $(5, 1, 8, 6, 10, 3)_{12} = 3 \cdot (1, 8, 6, 10, 3, 5)_{12}$ with carries $(c_5, c_4, c_3, c_2, c_1, c_0) = (2, 1, 2, 0, 1, 0)$.

Example 5. We begin by computing the conjugacy class containing

$$(d_5, d_4, d_3, d_2, d_1, d_0) = (5, 1, 8, 6, 10, 3)_{12} = 3 \cdot (1, 8, 6, 10, 3, 5)_{12}.$$

The carries are given by $(c_5, c_4, c_3, c_2, c_1, c_0) = (2, 1, 2, 0, 1, 0)$, so by Corollary 3, any permutation, π, of the digits of our initial example must satisfy

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} = P_{\pi\psi\pi^{-1}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

Translating the above to component form, we have

$$\begin{aligned} c_{\pi\psi\pi^{-1}(0)} &= 1 = c_1 \text{ or } c_4, \\ c_{\pi\psi\pi^{-1}(1)} &= 0 = c_0 \text{ or } c_2, \\ c_{\pi\psi\pi^{-1}(2)} &= 2 = c_3 \text{ or } c_5, \\ c_{\pi\psi\pi^{-1}(3)} &= 1 = c_1 \text{ or } c_4, \\ c_{\pi\psi\pi^{-1}(4)} &= 2 = c_3 \text{ or } c_5, \\ c_{\pi\psi\pi^{-1}(5)} &= 0 = c_0 \text{ or } c_2. \end{aligned}$$

Then, it must be that $\pi\psi\pi^{-1}$ has the form

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 \text{ or } 4 & 0 \text{ or } 2 & 3 \text{ or } 5 & 4 \text{ or } 1 & 5 \text{ or } 3 & 0 \text{ or } 2 \end{pmatrix}.$$

From the above, there are three possible 6-cycles which yield acceptable permutations $\pi\psi\pi^{-1} = (\pi(0), \pi(1), \pi(2), \pi(3), \pi(4), \pi(5))$, namely, $(0, 1, 2, 3, 4, 5)$, $(0, 4, 5, 2, 3, 1)$, and $(0, 4, 3, 1, 2, 5)$. Since $c_0 = c_2 = 0$, we have six values of π, namely, the identity ε, $\pi_2 = \psi^2$, $\pi_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}$, $\pi_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 0 & 4 & 5 \end{pmatrix}$, $\pi_5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 4 & 3 & 1 & 2 & 5 \end{pmatrix}$, and $\pi_6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 0 & 4 & 3 & 1 \end{pmatrix}$, which yield the permutiple conjugacy class listed below.

$(d_{\pi(5)}, d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_{12}$	π	τ	$(c_{\pi(5)}, c_{\pi(4)}, c_{\pi(3)}, c_{\pi(2)}, c_{\pi(1)}, c_{\pi(0)})$
$(5, 1, 8, 6, 10, 3)_{12}$	ε	ψ^5	$(2, 1, 2, 0, 1, 0)$
$(10, 3, 5, 1, 8, 6)_{12}$	ψ^2	ψ^5	$(1, 0, 2, 1, 2, 0)$
$(10, 8, 6, 5, 1, 3)_{12}$	π_3	$\pi_3^{-1}\psi^5\pi_3$	$(1, 2, 0, 2, 1, 0)$
$(5, 1, 3, 10, 8, 6)_{12}$	π_4	$\pi_4^{-1}\psi^5\pi_4$	$(2, 1, 0, 1, 2, 0)$
$(5, 6, 10, 8, 1, 3)_{12}$	π_5	$\pi_5^{-1}\psi^5\pi_5$	$(2, 0, 1, 2, 1, 0)$
$(10, 8, 1, 3, 5, 6)_{12}$	π_6	$\pi_6^{-1}\psi^5\pi_6$	$(1, 2, 1, 0, 2, 0)$

Table 4: The conjugacy class corresponding to ψ^5.

We now set to the task of finding all candidate base permutations. By Corollary 4, we have for any base permutation, β, that

$$\begin{bmatrix} d_{\beta(0)} \\ d_{\beta(1)} \\ d_{\beta(2)} \\ d_{\beta(3)} \\ d_{\beta(4)} \\ d_{\beta(4)} \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \mod 4.$$

The above expressed in component form gives us

$$\begin{aligned} d_{\beta(0)} &= 1 = d_4 \text{ or } d_5, \\ d_{\beta(1)} &= 3 = d_0, \\ d_{\beta(2)} &= 2 = d_1 \text{ or } d_2, \\ d_{\beta(3)} &= 2 = d_1 \text{ or } d_2, \\ d_{\beta(4)} &= 0 = d_3, \\ d_{\beta(5)} &= 1 = d_4 \text{ or } d_5. \end{aligned}$$

Then, a base permutation, β, has the form

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 \text{ or } 5 & 0 & 1 \text{ or } 2 & 1 \text{ or } 2 & 3 & 4 \text{ or } 5 \end{pmatrix}.$$

The possible base permutations are then $\beta_1 = \psi^5$ (which corresponds to the conjugacy class listed in Table 4), $\beta_2 = (0, 5, 4, 3, 1)$, $\beta_3 = (0, 4, 3, 2, 1)$, and $\beta_4 = (0, 4, 3, 1)$. As we shall presently see, unlike Examples 3 and 4, each candidate base permutation yields a non-empty conjugacy class.

We now determine the conjugacy class for $\beta_2 = (0, 5, 4, 3, 1)$. In order to do this, we must first find an example from this class. We shall see that the techniques we used for Examples 3 and 4 will not work in this case. Therefore, we must appeal to more rudimentary techniques. In particular, Theorem 1 will prove useful. We need permutations τ_2 and π yielding a $(3, 12, \tau_2)$-permutiple such that $\pi\tau_2\pi^{-1} = (0, 5, 4, 3, 1)$. Rearranging, we have $\tau_2 = (\pi^{-1}(0), \pi^{-1}(5), \pi^{-1}(4), \pi^{-1}(3), \pi^{-1}(1))$. In order to reduce the number of candidate permutations, we attempt to find a value of π which fixes 0. Then $\tau_2 = (0, \pi^{-1}(5), \pi^{-1}(4), \pi^{-1}(3), \pi^{-1}(1))$. Applying Theorem 12, the new permutiple $(d_{\pi(5)}, d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_{12}$ has the carry vector $(c_{\pi(5)}, c_{\pi(4)}, c_{\pi(3)}, c_{\pi(2)}, c_{\pi(1)}, c_{\pi(0)})$. Applying Theorem 1 to the jth digit, we have that $3d_{\pi\tau_2(j)} - d_{\pi(j)} = 12c_{\pi\psi(j)} - c_{\pi(j)}$. We know by the above that $\pi(0) = 0$ and $\tau_2(0) = \pi^{-1}(5)$. Therefore, for $j = 0$, we have $3d_5 - d_0 = 3 \cdot 5 - 3 = 12c_{\pi(1)}$. Thus, $c_{\pi(1)} = 1 = c_1$ or c_4. So $\pi(1)$ must be either 1 or 4. We shall attempt to find a solution with $\pi(1) = 1$. Then $\tau_2 = (0, \pi^{-1}(5), \pi^{-1}(4), \pi^{-1}(3), 1)$. Another application of Theorem 1 for $j = 1$ yields $3d_{\pi\tau_2(1)} - d_{\pi(1)} = 12c_{\pi\psi(1)} - c_{\pi(1)}$, or $-1 = 3 \cdot 3 - 10 = 3d_0 - d_1 = 12c_{\pi(2)} - c_1 = 12c_{\pi(2)} - 1$. Thus, $c_{\pi(2)} = 0$, so $\pi(2)$ equals either 0 or 2, but since $\pi(0) = 0$, it must be that $\pi(2) = 2$. Applying Theorem 1 for $j = 2$, we obtain $3d_{\pi\tau_2(2)} - d_{\pi(2)} = 12c_{\pi\psi(2)} - c_{\pi(2)}$, which, using the above information, reduces to $c_{\pi(3)} = 1$. That is, $\pi(3)$ is either 1 or 3. But since $\pi(1) = 1$, it follows that $\pi(3) = 4$. Thus, $\tau_2 = (0, \pi^{-1}(5), 3, \pi^{-1}(3), 1)$. In the above fashion, Theorem 1 for $j = 3$ then gives that $c_{\pi(4)} = 2$. Then $\pi(4)$ must equal either 3 or 5. Letting $\pi(4) = 3$, we have $\tau_2 = (0, 5, 3, 4, 1)$.

From the above, $\pi = (4, 3)$ and $\tau_2 = (0, 5, 3, 4, 1)$ give us a solution from the conjugacy class corresponding to $\beta_2 = (0, 5, 4, 3, 1)$, namely, the $(3, 12, \tau_2)$-permutiple,

$$(d_{\pi(5)}, d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_{12} = (5, 8, 1, 6, 10, 3)_{12} = 3 \cdot (1, 10, 8, 6, 3, 5)_{12},$$

with carries $(\hat{c}_5, \hat{c}_4, \hat{c}_3, \hat{c}_2, \hat{c}_1, \hat{c}_0) = (2, 2, 1, 0, 1, 0)$. Taking

$$\hat{p} = (\hat{d}_5, \hat{d}_4, \hat{d}_3, \hat{d}_2, \hat{d}_1, \hat{d}_0)_{12} = (5, 8, 1, 6, 10, 3)_{12}$$

to be our reference example from this class, we now use Corollary 3 to find the remaining elements of this class:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = P_{\pi\psi\pi^{-1}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}.$$

We express the above in component form as

$$\begin{aligned}
\hat{c}_{\pi\psi\pi^{-1}(0)} &= 1 = \hat{c}_1 \text{ or } \hat{c}_3,\\
\hat{c}_{\pi\psi\pi^{-1}(1)} &= 0 = \hat{c}_0 \text{ or } \hat{c}_2,\\
\hat{c}_{\pi\psi\pi^{-1}(2)} &= 1 = \hat{c}_1 \text{ or } \hat{c}_3,\\
\hat{c}_{\pi\psi\pi^{-1}(3)} &= 2 = \hat{c}_4 \text{ or } \hat{c}_5,\\
\hat{c}_{\pi\psi\pi^{-1}(4)} &= 2 = \hat{c}_4 \text{ or } \hat{c}_5,\\
\hat{c}_{\pi\psi\pi^{-1}(5)} &= 0 = \hat{c}_0 \text{ or } \hat{c}_2.
\end{aligned}$$

Therefore, $\pi\psi\pi^{-1}$ can be expressed as

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 \text{ or } 3 & 0 \text{ or } 2 & 1 \text{ or } 3 & 5 & 4 & 0 \text{ or } 2 \end{pmatrix}.$$

In particular, we have either $(0,1,2,3,4,5)$, or $(0,3,4,5,2,1)$, and since $\hat{c}_0 = \hat{c}_2 = 0$, the four permutations induced by the digits of $\hat{p}$ are the identity ε (corresponding to $\hat{p}$), $\pi_7 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 0 & 1 \end{pmatrix}$, $\pi_8 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 4 & 5 & 2 & 1 \end{pmatrix}$, and $\pi_9 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 0 & 3 & 4 & 5 \end{pmatrix}$. The conjugacy class corresponding to $\beta_2 = (0,5,4,3,1)$ is given by the table below.

$\left(\hat{d}_{\pi(5)}, \hat{d}_{\pi(4)}, \ldots, \hat{d}_{\pi(0)}\right)_{12}$	π	τ	$\left(\hat{c}_{\pi(5)}, \hat{c}_{\pi(4)}, \ldots, \hat{c}_{\pi(0)}\right)$
$(5,8,1,6,10,3)_{12}$	ε	$\tau_2 = (0,5,3,4,1)$	$(2,2,1,0,1,0)$
$(10,3,5,8,1,6)_{12}$	π_7	$\pi_7^{-1}\tau_2\pi_7$	$(1,0,2,2,1,0)$
$(10,6,5,8,1,3)_{12}$	π_8	$\pi_8^{-1}\tau_2\pi_8$	$(1,0,2,2,1,0)$
$(5,8,1,3,10,6)_{12}$	π_9	$\pi_9^{-1}\tau_2\pi_9$	$(2,2,1,0,1,0)$

Table 5: The conjugacy class corresponding to $\beta_2 = (0,5,4,3,1)$.

Finding an example from the conjugacy class corresponding to $\beta_3 = (0,4,3,2,1)$, repeated use of Theorem 1 as above yields the permutations $\pi = (1,2,3,4,5)$ and $\tau_3 = (5,0,3,2,1)$. From these permutations, we obtain the reference $(3,12,\tau_3)$-permutiple,

$$(\hat{d}_5, \hat{d}_4, \hat{d}_3, \hat{d}_2, \hat{d}_1, \hat{d}_0)_{12} = (d_{\pi(5)}, d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_{12} = (10,5,1,8,6,3)_{12},$$

with carries $(\hat{c}_5, \hat{c}_4, \hat{c}_3, \hat{c}_2, \hat{c}_1, \hat{c}_0) = (1,2,1,2,0,0)$.

In similar fashion to calculations for the above conjugacy classes, the reader may verify that this example via Corollary 3 gives the following four permutations: the identity ε, $\pi_{10} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{pmatrix}$, $\pi_{11} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 3 & 2 & 5 \end{pmatrix}$, and $\pi_{12} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 & 0 \end{pmatrix}$, and thus, the conjugacy class corresponding to $\beta_3 = (0,4,3,2,1)$ is given by the following table.

$\left(\hat{d}_{\pi(5)}, \hat{d}_{\pi(4)}, \ldots, \hat{d}_{\pi(0)}\right)_{12}$	π	τ	$\left(\hat{c}_{\pi(5)}, \hat{c}_{\pi(4)}, \ldots, \hat{c}_{\pi(0)}\right)$
$(10,5,1,8,6,3)_{12}$	ε	$\tau_3 = (5,0,3,2,1)$	$(1,2,1,2,0,0)$
$(3,10,5,1,8,6)_{12}$	π_{10}	$\pi_{10}^{-1}\tau_3\pi_{10}$	$(0,1,2,1,2,0)$
$(10,8,1,5,6,3)_{12}$	π_{11}	$\pi_{11}^{-1}\tau_3\pi_{11}$	$(1,2,1,2,0,0)$
$(3,10,8,1,5,6)_{12}$	π_{12}	$\pi_{12}^{-1}\tau_3\pi_{12}$	$(0,1,2,1,2,0)$

Table 6: The conjugacy class corresponding to $\beta_3 = (0,4,3,2,1)$.

To find an example from our final conjugacy class corresponding to $\beta_4 = (0,4,3,1)$, we use Theorem 1 as above to find the permutations $\pi = (5,1,2,4)$ and $\tau_4 = (0,2,3,5)$. From these permutations, we obtain the reference $(3,12,\tau_4)$-permutiple,

$$(\hat{d}_5,\hat{d}_4,\hat{d}_3,\hat{d}_2,\hat{d}_1,\hat{d}_0)_{12} = (d_{\pi(5)},d_{\pi(4)},d_{\pi(3)},d_{\pi(2)},d_{\pi(1)},d_{\pi(0)})_{12} = (10,5,8,1,6,3)_{12},$$

with carries $(\hat{c}_5,\hat{c}_4,\hat{c}_3,\hat{c}_2,\hat{c}_1,\hat{c}_0) = (1,2,2,1,0,0)$. Again, we shall leave the details of the calculations involving Corollary 3 to the reader who may verify that this permutiple yields the following two permutations: the identity ε and $\pi_{13} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{pmatrix}$. The conjugacy class for $\beta_4 = (0,4,3,1)$ is given by the table below.

$\left(\hat{d}_{\pi(5)}, \hat{d}_{\pi(4)}, \ldots, \hat{d}_{\pi(0)}\right)_{12}$	π	τ	$\left(\hat{c}_{\pi(5)}, \hat{c}_{\pi(4)}, \ldots, \hat{c}_{\pi(0)}\right)$
$(10,5,8,1,6,3)_{12}$	ε	$\tau_4 = (0,2,3,5)$	$(1,2,2,1,0,0)$
$(3,10,5,8,1,6)_{12}$	π_{13}	$\pi_{13}^{-1}\tau_4\pi_{13}$	$(0,1,2,2,1,0)$

Table 7: The conjugacy class corresponding to $\beta_4 = (0,4,3,1)$.

4. Future Directions and Concluding Remarks

While we have developed methods for finding all permutiples having the same digits and carries as a known example (which, as we have seen, allow us to find all desired examples when n divides b), the next step forward is to find methods which give us all the desired examples.

Aside from the above problem, the work we have done here leaves many questions. Are there other conditions or divisibility criteria which, using the methods developed here, or some slight variation thereof, allow us to find all permutiples having the same digits as a single permutiple example already in hand? How do the orders of base permutations of each conjugacy class relate to the order of σ? Are there any restrictions on the size of the conjugacy classes? Regarding the general permutiple problem, other questions certainly abound. What kinds of permutations can σ be? What sort of restrictions might there be on the order of σ?

As we have already seen, repeated application of Theorem 1, Corollary 3, and Corollary 4 give us methods for finding permutations of the digits of a starting permutiple which yield other permutiples. However, in fairness, we should add that some examples require substantially more calculation than others in order to find every example. For instance, considering the base-10 cyclic example given in the introductory paragraph, $714285 = 5 \cdot 142857$, the reader may verify that Corollary 4 requires us to examine 36 possible base permutations, among which, only four actually yield non-empty conjugacy classes. This is all to say that future efforts should look for criteria for determining when base permutations yield non-empty conjugacy classes.

Another avenue of investigation is finding a way to generalize Sloane's [9] Young graph representation of palintiple structure in order to visualize permutiple structure. We imagine that such a construction would allow us to classify, and better understand, permutiple structure. We also imagine that it would be substantially more complex. Finally, as Young graphs themselves are an interesting area of study in their own right, we conjecture that its generalization would also justify future study.

It is also worth mentioning that the matrices $nP_\sigma - I$ and $bP_\psi - I$, as well as their inverses (given by $\frac{1}{n^{|\sigma|}-1}\sum_{\ell=0}^{|\sigma|-1}(nP_\sigma)^\ell$ and $\frac{1}{b^{k+1}-1}\sum_{\ell=0}^{k}(bP_\psi)^\ell$, respectively), all have interesting properties. For instance, both $bP_\psi - I$ and its inverse are circulant matrices. Moreover, both $nP_\sigma - I$ and $bP_\psi - I$ have the property that every column and row sum to $n-1$ and $b-1$, respectively. In addition to these properties, we ask if these matrices are endowed with other special properties when (n, b, σ)-permutiples exist.

With the palintiple problem firmly in mind, the difficulty of particular digit permutation problems might make the general permutiple problem seem intractable. However, the methods developed here seem to prove otherwise; there is certainly structure that one can take advantage of. Moreover, as noted by Holt [4], studying the general problem may very well offer insight into particular problems which study only one kind of permutation. In particular, it might be possible to derive (in the manner described by Holt [4]) entire palintiple classes from certain permutiple types such as those mentioned by Holt [4]. What is more, it seems that general permutiples can also be derived from other permutiples. As a noteworthy example, we present the cyclic $(6, 12, \psi^3)$-permutiple, $(10, 3, 5, 1, 8, 6)_{12} = 6\cdot(1, 8, 6, 10, 3, 5)_{12}$, whose non-zero carries are the digits of the $(2, 6)$-palintiple, $(4, 3, 5, 1, 2)_6 = 2 \cdot (2, 1, 5, 3, 4)_6$, which served as our initial base-6 example. Another example is the starting permutiple of Example 5, $(5, 1, 8, 6, 10, 3)_{12} = 3 \cdot (1, 8, 6, 10, 3, 5)_{12}$, with carries $(c_5, c_4, c_3, c_2, c_1, c_0) = (2, 1, 2, 0, 1, 0)$. The carries (excluding $c_0 = 0$) are the digits of a $(2, 3)$-palintiple, which again, as shown by Holt [4], gives rise to an entire family of derived palintiples. When choosing these examples, we did not intentionally seek out such occurrences; we only later noticed that such examples of "derived permutiples" seem to naturally abound. We suspect that there are deep and interesting connections between permutiples of differing bases which have yet to be explored.

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