A METHOD FOR FINDING ALL PERMUTIPLES WITH A FIXED SET OF DIGITS FROM A SINGLE KNOWN EXAMPLE

Benjamin V. Holt

Department of Mathematics, Southwestern Oregon Community College, Oregon, USA

`benjamin.holt@socc.edu`

Abstract

A permutiple is a natural number that is a nontrivial multiple of a permutation of its digits in some base. Special cases of permutiples include cyclic numbers (multiples of cyclic permutations of their digits) and palintiple numbers (multiples of their digit reversals). While cyclic numbers have a fairly straightforward description, palintiple numbers admit many varieties and cases. A previous paper attempts to get a better handle on the general case by constructing new examples of permutiples with the same set of digits, multiplier, and length as a known example. However, the results are not sufficient for finding all possible examples except when the multiplier divides the base. Using an approach based on the methods of this previous paper, we develop a new method which enables us to find all examples under any conditions.

1. Introduction

A *permutiple*, as the name suggests, is a number which is an integer multiple of some permutation of its digits in some natural number base, b, greater than one [4]. Well-studied cases include cyclic numbers [1, 5], that is, numbers which are multiples of cyclic permutations of their digits. A base-10 example of a cyclic number is $714285 = 5 \cdot 142857$. A richer, but much less well-understood, case is palintiple numbers [2, 3], also known as reverse multiples [6, 7, 8, 9], which are multiples of their digit reversals. The large variety of palintiple types can be organized using a graph-theoretical construction by Sloane [7] called *Young graphs* which are a modification of the work of Young [8, 9]. The most widely known examples of palintiples, also in base 10, include $87912 = 4 \cdot 21978$ and $98901 = 9 \cdot 10989$.

The work of Holt [4] establishes methods for finding new examples of general permutiples from old examples. For instance, using these methods, we are able to find new examples, such as $79128 = 4 \cdot 19782$, from the example with the same digits mentioned above. Although these methods shed some light on the problem, they are not able to account for all the desired examples under more general conditions. In

particular, the results are not sufficient for finding the permutiple $78912 = 4 \cdot 19728$ from the known example already mentioned. In this paper, we fill this gap by providing a simpler, yet more general, method for finding all permutiples with the same set of digits, multiplier, and length from single known example.

2. Basic Notation, Definitions, and Results

We shall use $(d_k, d_{k-1}, \ldots, d_0)_b$ to denote the natural number $\sum_{j=0}^{k} d_j b^j$ where $0 \le d_j < b$ for all $0 \le j \le k$. The following is a definition of permutiple numbers.

Definition 1 ([4]). Let n be a natural number and σ be a permutation on $\{0, 1, 2, \ldots, k\}$. We say that $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-*permutiple* provided

$$(d_k, d_{k-1}, \ldots, d_1, d_0)_b = n(d_{\sigma(k)}, d_{\sigma(k-1)}, \ldots, d_{\sigma(1)}, d_{\sigma(0)})_b.$$

This definition, along with basic facts about single-digit multiplication, gives the following result which relates a permutiple's digits and carries.

Theorem 1 ([4]). *Let $(d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b, σ)-permutiple and let c_j be the jth carry. Then*

$$bc_{j+1} - c_j = nd_{\sigma(j)} - d_j$$

for all $0 \le j \le k$.

Letting ψ denote the $(k+1)$-cycle $(0, 1, 2, \ldots, k)$, we may write the above relations more conveniently in matrix form as

$$(bP_\psi - I)\mathbf{c} = (nP_\sigma - I)\mathbf{d},$$

where I is the identity matrix, P_ψ and P_σ are permutation matrices, $\mathbf{c}$ is a column vector containing the carries, and $\mathbf{d}$ is a column vector containing the digits. We also note that indexing is from 0 to k rather than from 1 to k.

The problem posed by Holt [4] is the following: given an (n, b, σ)-permutiple, $(d_k, d_{k-1}, \ldots, d_0)_b$, find all permutations, π, such that $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$, is also a permutiple.

To sort through the types of new examples that arise, Holt [4] defines the notion permutiple *conjugacy.* For completeness, we state this definition here.

Definition 2 ([4]). Suppose $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-permutiple. Then, an (n, b, τ_1)-permutiple, $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \ldots, d_{\pi_1(0)})_b$, and an (n, b, τ_2)-permutiple, $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \ldots, d_{\pi_2(0)})_b$, are said to be *conjugate* if $\pi_1 \tau_1 \pi_1^{-1} = \pi_2 \tau_2 \pi_2^{-1}$.

Conjugacy defines an equivalence relation on the collection of permutiples having digits d_k, d_{k-1}, $\ldots$, d_0. Holt [4] refers to these equivalence classes as *conjugacy*

classes. The common permutation of a conjugacy class, $\beta = \pi_1\tau_1\pi_1^{-1} = \pi_2\tau_2\pi_2^{-1}$, is referred to its *base permutation*. The methods of Holt [4] are sufficient for finding all known examples within a conjugacy class, but fall short when trying to find all conjugacy classes outside of that which contains the known example.

3. A Method for Finding All Examples

We consider a generic (n, b, σ)-permutiple, $(d_k, d_{k-1}, \ldots, d_0)_b$, with carries c_k, c_{k-1}, $\ldots$, c_0, and an (n, b, τ)-permutiple with the same digits, $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$, but not necessarily the same carries, $\hat{c}_k$, $\hat{c}_{k-1}$, $\ldots$, $\hat{c}_0$. Then, in the notation established above,

$$(nP_\tau - I)P_\pi \mathbf{d} = (bP_\psi - I)\hat{\mathbf{c}}. \tag{1}$$

Reducing modulo b, we have

$$(nP_\tau - I)P_\pi \mathbf{d} \equiv -\hat{\mathbf{c}} \mod b.$$

Multiplying the above by $P_{\pi^{-1}}$ and rearranging, we obtain

$$\mathbf{d} + (b - n)P_{\pi\tau\pi^{-1}}\mathbf{d} \equiv P_{\pi^{-1}}\hat{\mathbf{c}} \mod b. \tag{2}$$

Equation (2) in component form, along with the fact from Holt [4] that the carries of any permutiple must be less than or equal to $n - 1$, gives us our main result.

Theorem 2. *Suppose $(d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b, σ)-permutiple. Then, in order for the number $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ to be an (n, b, τ)-permutiple, it must be that*

$$\lambda\left(d_j + (b-n)d_{\pi\tau\pi^{-1}(j)}\right) \leq n - 1$$

for all $0 \leq j \leq k$, where λ is the least non-negative residue modulo b.

The above enables us to find all possible base permutations, $\beta = \pi\tau\pi^{-1}$, for each conjugacy class by imposing necessary conditions on what $\pi\tau\pi^{-1}$ can be. A big advantage of the result is that it does not require any prior knowledge of what the carry sequence should be. In fact, once we narrow down the possible candidates for β, we may then determine the values of the candidate set of carries by substituting in the known digits into Equation (2); the permuted carries are contained in the column vector $\mathbf{v} = P_{\pi^{-1}}\hat{\mathbf{c}}$.

With the base permutations in hand, we then rewrite Equation (1) as

$$(nP_\tau - I)P_\pi \mathbf{d} = (bP_\psi - I)P_\pi \mathbf{v}$$

since $\hat{\mathbf{c}} = P_\pi \mathbf{v}$. Multiplying both sides by $P_{\pi^{-1}}$ we have

$$(nP_{\pi\tau\pi^{-1}} - I)\mathbf{d} = (bP_{\pi\psi\pi^{-1}} - I)\mathbf{v},$$

or

$$(nP_\beta - I)\mathbf{d} = (bP_{\pi\psi\pi^{-1}} - I)\mathbf{v}.$$

Rearranging, we obtain

$$bP_{\pi\psi\pi^{-1}}\mathbf{v} = (nP_\beta - I)\mathbf{d} + \mathbf{v}. \tag{3}$$

Now, since $\mathbf{d}$, $\mathbf{v}$, and $\beta = \pi\tau\pi^{-1}$ are known, the only unknown in Equation (3) is π. This is to say that Equation (3) gives us a list of candidate permutations, π, for which $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ is an (n, b, τ)-permutiple. Moreover, we note that Equation (3) is equivalent to Equation (1). So, Theorem 3 of Holt [4] guarantees that every permutation, π, satisfying Equation (3) yields a permutiple so long as $\hat{c}_0 = 0$. From there, determining τ itself is a matter of either of computing $\tau = \pi^{-1}\beta\pi$ or dividing $(d_{\pi(k)}, d_{\pi(k-1)}, \ldots, d_{\pi(0)})_b$ by n.

We now illustrate the above method by resolving a case for which the techniques of Holt [4] were insufficient for recovering all permutiples from a known example.

Example 1. We shall find all 5-digit, $(4, 10, \tau)$-permutiples with the same digits as the base-10 example $87912 = 4 \cdot 21978$. In more general notation, we state our known example as $(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$ so that

$$\mathbf{d} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 9 \\ 7 \\ 8 \end{bmatrix}.$$

Theorem 2 tells us that for all $0 \leq j \leq 4$ we must have

$$\lambda\left(d_j + 6d_{\pi\tau\pi^{-1}(j)}\right) \leq 3.$$

That is,

$$\lambda\left(2 + 6d_{\pi\tau\pi^{-1}(0)}\right) \leq 3,$$
$$\lambda\left(1 + 6d_{\pi\tau\pi^{-1}(1)}\right) \leq 3,$$
$$\lambda\left(9 + 6d_{\pi\tau\pi^{-1}(2)}\right) \leq 3,$$
$$\lambda\left(7 + 6d_{\pi\tau\pi^{-1}(3)}\right) \leq 3,$$
$$\lambda\left(8 + 6d_{\pi\tau\pi^{-1}(4)}\right) \leq 3.$$

The above inequalities yield the possibilities

$$\pi\tau\pi^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 \text{ or } 3 & 0, 2, \text{ or } 3 & 1 \text{ or } 2 & 0, 2, \text{ or } 3 \end{pmatrix}.$$

We note that $\pi\tau\pi^{-1}(3) = 2$ would give us a relation that is not a permutation, so we are left with

$$\pi\tau\pi^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 \text{ or } 3 & 0, 2, \text{ or } 3 & 1 & 0, 2, \text{ or } 3 \end{pmatrix}.$$

The possible base permutations are then $\beta_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$, $\beta_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 3 & 1 & 2 \end{pmatrix}$, $\beta_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 2 & 1 & 3 \end{pmatrix}$, and $\beta_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 0 & 1 & 2 \end{pmatrix}$.

The reader will notice that β_1 is the reversal permutation, ρ, and is the base permutation of our known example. Also, $\beta_1 = \rho$ is the digit permutation appearing in our known example. It is here that we underscore, as in Holt [4], that a base permutation need not be a digit permutation itself in conjugacy classes outside the one which contains the known example.

From here, we substitute $\mathbf{d}$ and each possible $\beta_j = \pi\tau\pi^{-1}$ into Equation (2) to determine $\mathbf{v} = P_{\pi^{-1}}\hat{\mathbf{c}}$, which gives a possible set of carries. To these, we then apply Equation (3) to recover π.

Applying Equation (2) to $\beta_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$ gives

$$\mathbf{v} \equiv P_{\pi^{-1}}\hat{\mathbf{c}} \equiv \begin{bmatrix} 2 \\ 1 \\ 9 \\ 7 \\ 8 \end{bmatrix} + 6P_{\beta_1} \begin{bmatrix} 2 \\ 1 \\ 9 \\ 7 \\ 8 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix} \quad \mathrm{mod}\ 10.$$

Since $0 \le \hat{c}_j \le 3$ for each $0 \le j \le 4$, we conclude that

$$\mathbf{v} = P_{\pi^{-1}}\hat{\mathbf{c}} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix},$$

which is no surprise since these are the carries, c_j, of our known example.

Applying Equation (3),

$$10P_{\pi\psi\pi^{-1}} \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix} = (4P_{\beta_1} - I) \begin{bmatrix} 2 \\ 1 \\ 9 \\ 7 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix} = 10 \begin{bmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

The possibilities are then expressed as

$$\pi\psi\pi^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1,2,\ \text{or}\ 3 & 1,2,\ \text{or}\ 3 & 1,2,\ \text{or}\ 3 & 0\ \text{or}\ 4 & 0\ \text{or}\ 4 \end{pmatrix}.$$

Since the above must be a 5-cycle, our possibilities are reduced to

$$\pi\psi\pi^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1\ \text{or}\ 2 & 2\ \text{or}\ 3 & 1\ \text{or}\ 3 & 4 & 0 \end{pmatrix}.$$

That is, $\pi\psi\pi^{-1} = (\pi(0), \pi(1), \pi(2), \pi(3), \pi(4))$ can be either $\psi = (0, 1, 2, 3, 4)$ or $(0, 2, 1, 3, 4) = (1, 2)\psi(1, 2)$. Thus, $\pi = \varepsilon$, the identity permutation, and $\pi = (1, 2)$ both solve Equation (3). Moreover, since the first carry, $\hat{c}_0 = v_{\pi(0)}$, must always be zero, we have that $\pi(0)$ can be either 0 or 4. The possibility $\pi(0) = 4$ gives us two more permutations: $\pi = \psi^4$ and $\pi = (1, 2)\psi^4$.

The entire conjugacy class for $\beta_1 = \rho$ is listed below.

$(d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_{10}$	π	τ	$(\hat{c}_4, \hat{c}_3, \hat{c}_2, \hat{c}_1, \hat{c}_0)$
$(8, 7, 9, 1, 2)_{10}$	ε	ρ	$(0, 3, 3, 3, 0)$
$(8, 7, 1, 9, 2)_{10}$	$(1, 2)$	$(1, 2)\rho(1, 2)$	$(0, 3, 3, 3, 0)$
$(7, 9, 1, 2, 8)_{10}$	ψ^4	$\psi^{-4}\rho\psi^4$	$(3, 3, 3, 0, 0)$
$(7, 1, 9, 2, 8)_{10}$	$(1, 2)\psi^4$	$\psi^{-4}(1, 2)\rho(1, 2)\psi^4$	$(3, 3, 3, 0, 0)$

Remark 1. The reader will notice that Equation (3) does the same work as Corollary 2 in Holt [4]. In fact, the above analysis is identical in form to that found in Example 3 of Holt [4].

We now find the conjugacy class for $\beta_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 3 & 1 & 2 \end{pmatrix}$. Again, by Equation (2),

$$\mathbf{v} \equiv P_{\pi^{-1}}\hat{\mathbf{c}} \equiv \begin{bmatrix} 2 \\ 1 \\ 9 \\ 7 \\ 8 \end{bmatrix} + 6P_{\beta_2}\begin{bmatrix} 2 \\ 1 \\ 9 \\ 7 \\ 8 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} \quad \text{mod } 10.$$

In similar fashion to the above case, we may argue that

$$\mathbf{v} = P_{\pi^{-1}}\hat{\mathbf{c}} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 3 \\ 2 \end{bmatrix}.$$

Using Equation (3), we have

$$10P_{\pi\psi\pi^{-1}}\begin{bmatrix} 0 \\ 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} = (4P_{\beta_2} - I)\begin{bmatrix} 2 \\ 1 \\ 9 \\ 7 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} = 10\begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \\ 3 \end{bmatrix},$$

which allows for

$$\pi\psi\pi^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 \text{ or } 3 & 2 & 4 & 0 & 3 \text{ or } 1 \end{pmatrix}.$$

Again, since the above must be a 5-cycle, we are left with a single possibility:

$$\pi\psi\pi^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \end{pmatrix}.$$

It follows that $\pi\psi\pi^{-1} = (\pi(0), \pi(1), \pi(2), \pi(3), \pi(4)) = (0, 1, 2, 4, 3)$. Thus, the only possible permutation is $\pi = (3, 4)$.

The conjugacy class for β_2, therefore, consists of a single element, namely, the example $(7, 8, 9, 1, 2)_{10} = 4 \cdot (1, 9, 7, 2, 8)_{10}$, a $(4, 10, \tau)$-permutiple with carry vector

$$\hat{\mathbf{c}} = P_\pi \mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \\ 3 \end{bmatrix},$$

where $\tau = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 0 & 4 & 2 & 1 \end{pmatrix} = (0, 3, 2, 4, 1) = \pi^{-1}\beta_2\pi$.

Remark 2. The above conjugacy class consists of the example that the results of Holt [4] could not account for.

Considering $\beta_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 2 & 1 & 3 \end{pmatrix}$, another use of Equation (2) yields

$$\mathbf{v} \equiv P_{\pi^{-1}}\hat{\mathbf{c}} \equiv \begin{bmatrix} 2 \\ 1 \\ 9 \\ 7 \\ 8 \end{bmatrix} + 6 P_{\beta_3} \begin{bmatrix} 2 \\ 1 \\ 9 \\ 7 \\ 8 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix} \quad \mod 10.$$

Then, using

$$\mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix},$$

we employ Equation (3) to obtain

$$10 P_{\pi\psi\pi^{-1}} \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix} = (4 P_{\beta_3} - I) \begin{bmatrix} 2 \\ 1 \\ 9 \\ 7 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix} = 10 \begin{bmatrix} 3 \\ 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}.$$

Since there is no permutation, π, which makes the above statement true, we conclude that the conjugacy class corresponding to β_3 is empty. By a similar calculation, β_4 also yields no new examples. With the above, we have found all $(4, 10, \tau)$-permutiples with the same digits as our known example.

4. Summary of Method and Concluding Remarks

To summarize the above method, Theorem 2 provides a list of base permutation candidates. Trying all of these in Equation (2) gives us possibilities for what the carries can be by computing the permuted carry vector $\mathbf{v} = P_{\pi^{-1}}\hat{\mathbf{c}}$. Inserting this information into Equation (3) then allows us to recover permutations, π, which yield new permutiples.

While the above method addresses the main question of Holt [4], we note that there are still plenty of questions that remain from the above considerations. Of particular interest are patterns or restrictions which may exist concerning permutation type and orders of base permutations, as well as the sizes of their corresponding conjugacy classes.

Other tractable lines of inquiry with the goal of finding new permutiples from old include understanding when "derived" permutiples are possible, that is, (n, b, σ)-permutiples whose truncated carry vector, $(c_k, c_{k-1}, \ldots, c_1)$, is also a base-$n$ permutiple. An example of this phenomenon, mentioned by Holt [4], is the cyclic $(6, 12, \psi^3)$-permutiple, $(10, 3, 5, 1, 8, 6)_{12} = 6 \cdot (1, 8, 6, 10, 3, 5)_{12}$, whose nonzero carries are the digits of the $(2, 6)$-palintiple, $(4, 3, 5, 1, 2)_6 = 2 \cdot (2, 1, 5, 3, 4)_6$. In other words, we ask: when is it possible to construct, or "derive," a new permutiple, say $(10, 3, 5, 1, 8, 6)_{12}$, from a known permutiple, such as $(4, 3, 5, 1, 2)_6$, by treating it as a carry vector $(4, 3, 5, 1, 2, 0)$? We note that the less general case of derived palintiples is taken up by Holt [3].

For other research questions and results regarding the general permutiple problem, the reader is directed to Holt [4].

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