BOOK: DIFFERENTIAL EQUATIONS FOR ENGINEERS

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Differential Equations for Engineers

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CHAPTER OVERVIEW

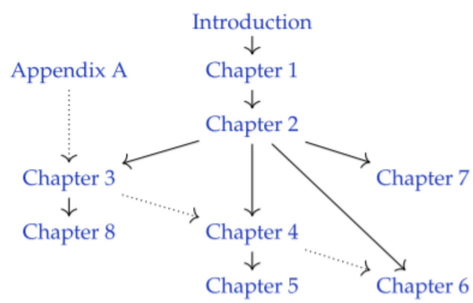
1: Introduction

1.1: Notes About These Notes

This book originated from my class notes for Math 286 at the University of Illinois at Urbana-Champaign (UIUC) in Fall 2008 and Spring 2009. It is a first course on differential equations for engineers. Using this book, I also taught Math 285 at UIUC, Math 20D at University of California, San Diego (UCSD), and Math 4233 at Oklahoma State University (OSU). Normally these courses are taught with Edwards and Penney, *Differential Equations and Boundary Value Problems: Computing and Modeling* [EP], or Boyce and DiPrima's *Elementary Differential Equations and Boundary Value Problems* [BD], and this book aims to be more or less a drop-in replacement. Other books I used as sources of information and inspiration are E.L. Ince's classic (and inexpensive) *Ordinary Differential Equations* [I], Stanley Farlow's *Differential Equations and Their Applications* [F], now available from Dover, Berg and McGregor's *Elementary Partial Differential Equations* [BM], and William Trench's free book *Elementary Differential Equations with Boundary Value Problems* [T]. See the Further Reading chapter at the end of the book.

1.1.1: Organization

The organization of this book to some degree requires chapters be done in order. Later chapters can be dropped. The dependence of the material covered is roughly:



There are a few references in chapters 4 and 5 to chapter 3 (some linear algebra), but these references are not essential and can be skimmed over, so chapter 3 can safely be dropped, while still covering chapters 4 and 5. Chapter 6 does not depend on chapter 4 except that the PDE section 6.5 makes a few references to chapter 4, although it could, in theory, be covered separately. The more in-depth appendix A on linear algebra can replace the short review Section 3.2 for a course that combines linear algebra and ODE.

1.1.2: Typical Types of Courses

Several typical types of courses can be run with the book. There are the two original courses at UIUC, both cover ODE as well some PDE. Either, there is the 4 hours-a-week for a semester (Math 286 at UIUC):

Introduction (0.2), chapter 1 (1.1-1.7), chapter 2, chapter 3, chapter 4 (4.1-4.9), chapter 5 (or 6 or 7 or 8).

Or, the second course at UIUC is at 3 hours-a-week (Math 285 at UIUC):

Introduction (0.2), chapter 1 (1.1-1.7), chapter 2, chapter 4 (4.1-4.9), (and maybe chapter 5, 6 or 7).

A semester-long course at 3 hours a week that doesn't cover either systems or PDE will cover, beyond the introduction, chapter 1, chapter 2, chapter 6, and chapter 7, (with sections skipped as above). On the other hand, a typical course that covers systems will probably need to skip Laplace and power series and cover chapter 1, chapter 2, chapter 3, and chapter 8.

If sections need to be skipped in the beginning, a good core of the sections on single ODE is: 0.2, 1.1-1.4, 1.6, 2.1, 2.2, 2.4-2.6.

The complete book can be covered at a reasonably fast pace at approximately 76 lectures (without appendix A) or 86 lectures (with appendix A replacing Section 3.2). This is not accounting for exams, review, or time spent in a computer lab. A two-quarter or a two-semester course can be easily run with the material. For example (with some sections perhaps strategically skipped):

Semester 1: Introduction, chapter 1, chapter 2, chapter 6, chapter 7.
Semester 2: Chapter 3, chapter 8, chapter 4, chapter 5.

A combined course on ODE with linear algebra can run as:

Introduction, chapter 1 (1.1-1.7), chapter 2, appendix A, chapter 3 (w/o Section 3.2), (possibly chapter 8).

The chapter on the Laplace transform (chapter 6), the chapter on Sturm–Liouville (chapter 5), the chapter on power series (chapter 7), and the chapter on nonlinear systems (chapter 8), are more or less interchangeable and can be treated as "topics". If chapter 8 is covered, it may be best to place it right after chapter 3, and chapter 5 is best covered right after chapter 4. If time is short, the first two sections of chapter 7 make a reasonable self-contained unit.

1.1.3: Computer Resources

The book's website https://www.jirka.org/diffyqs/ contains the following resources:

1. Interactive SAGE demos.
2. Online WeBWorK homeworks (using either your own WeBWorK installation or Edfinity) for most sections, customized for this book.
3. The PDFs of the figures used in this book.

I taught the UIUC courses using IODE (https://faculty.math.illinois.edu/iode/). IODE is a free software package that works with Matlab (proprietary) or Octave (free software). The graphs in the book were made with the Genius software (see https://www.jirka.org/genius.html). I use Genius in class to show these (and other) graphs.

The LaTeX source of the book is also available for possible modification and customization at github (https://github.com/jirilebl/diffyqs).

1.1.4: Acknowledgments

Firstly, I would like to acknowledge Rick Laugesen. I used his handwritten class notes the first time I taught Math 286. My organization of this book through chapter 5, and the choice of material covered, is heavily influenced by his notes. Many examples and computations are taken from his notes. I am also heavily indebted to Rick for all the advice he has given me, not just on teaching Math 286. For spotting errors and other suggestions, I would also like to acknowledge (in no particular order): John P. D'Angelo, Sean Raleigh, Jessica Robinson, Michael Angelini, Leonardo Gomes, Jeff Winegar, Ian Simon, Thomas Wicklund, Eliot Brenner, Sean Robinson, Jannett Susberry, Dana Al-Quadi, Cesar Alvarez, Cem Bagdatlioglu, Nathan Wong, Alison Shive, Shawn White, Wing Yip Ho, Joanne Shin, Gladys Cruz, Jonathan Gomez, Janelle Louie, Navid Froutan, Grace Victorine, Paul Pearson, Jared Teague, Ziad Adwan, Martin Weilandt, Sönmez Şahutoğlu, Pete Peterson, Thomas Gresham, Prentiss Hyde, Jai Welch, Simon Tse, Andrew Browning, James Choi, Dusty Grundmeier, John Marriott, Jim Kruidenier, Barry Conrad, Wesley Snider, Colton Koop, Sarah Morse, Erik Boczko, Asif Shakeel, Chris Peterson, Nicholas Hu, Paul Seeburger, Jonathan McCormick, David Leep, William Meisel, Shishir Agrawal, Tom Wan, Andres Valloud, and probably others I have forgotten. Finally, I would like to acknowledge NSF grants DMS-0900885 and DMS-1362337.

1.2: Introduction to Differential Equations

1.2.1: Differential Equations

The laws of physics are generally written down as differential equations. Therefore, all of science and engineering use differential equations to some degree. Understanding differential equations is essential to understanding almost anything you will study in your science and engineering classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science and engineering are concerned. As an analogy, suppose all your classes from now on were given in Swahili. It would be important to first learn Swahili, or you would have a very tough time getting a good grade in your classes.

You saw many differential equations already without perhaps knowing about it. And you even solved simple differential equations when you took calculus. Let us see an example you may not have seen:

$$\frac{dx}{dt} + x = 2\cos t. \tag{1.2.1}$$

Here x is the *dependent variable* and t is the *independent variable*. Equation (1.2.1) is a basic example of a *differential equation*. It is an example of a *first order differential equation*, since it involves only the first derivative of the dependent variable. This equation arises from Newton's law of cooling where the ambient temperature oscillates with time.

1.2.2: Solutions of Differential Equations

Solving the differential equation means finding x in terms of t. That is, we want to find a function of t, which we call x, such that when we plug x, t, and $\frac{dx}{dt}$ into (1.2.1), the equation holds; that is, the left hand side equals the right hand side. It is the same idea as it would be for a normal (algebraic) equation of just x and t. We claim that

$$x = x(t) = \cos t + \sin t$$

is a *solution*. How do we check? We simply plug x into equation (1.2.1)! First we need to compute $\frac{dx}{dt}$. We find that $\frac{dx}{dt} = -\sin t + \cos t$. Now let us compute the left-hand side of (1.2.1).

$$\frac{dx}{dt} + x = \underbrace{(-\sin t + \cos t)}_{\frac{dx}{dt}} + \underbrace{(\cos t + \sin t)}_{x} = 2\cos t.$$

Yay! We got precisely the right-hand side. But there is more! We claim $x = \cos t + \sin t + e^{-t}$ is also a solution. Let us try,

$$\frac{dx}{dt} = -\sin t + \cos t - e^{-t}.$$

We plug into the left-hand side of (1.2.1)

$$\frac{dx}{dt} + x = \underbrace{(-\sin t + \cos t - e^{-t})}_{\frac{dx}{dt}} + \underbrace{(\cos t + \sin t + e^{-t})}_{x} = 2\cos t.$$

And it works yet again!

So there can be many different solutions. For this equation all solutions can be written in the form

$$x = \cos t + \sin t + Ce^{-t},$$

for some constant C. Different constants C will give different solutions, so there are really infinitely many possible solutions. See Figure 1.2.1 for the graph of a few of these solutions. We will see how we find these solutions a few lectures from now.

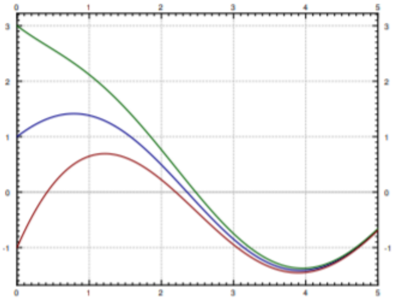


Figure 1.2.1: Few solutions of $\frac{dx}{dt} + x = 2\cos t$.

Solving differential equations can be quite hard. There is no general method that solves every differential equation. We will generally focus on how to get exact formulas for solutions of certain differential equations, but we will also spend a little bit of time on getting approximate solutions. And we will spend some time on understanding the equations without solving them.

Most of this book is dedicated to *ordinary differential equations* or ODEs, that is, equations with only one independent variable, where derivatives are only with respect to this one variable. If there are several independent variables, we get *partial differential equations* or PDEs.

Even for ODEs, which are very well understood, it is not a simple question of turning a crank to get answers. When you can find exact solutions, they are usually preferable to approximate solutions. It is important to understand how such solutions are found. Although in real applications you will leave much of the actual calculations to computers, you need to understand what they are doing. It is often necessary to simplify or transform your equations into something that a computer can understand and solve. You may even need to make certain assumptions and changes in your model to achieve this.

To be a successful engineer or scientist, you will be required to solve problems in your job that you never saw before. It is important to learn problem solving techniques, so that you may apply those techniques to new problems. A common mistake is to expect to learn some prescription for solving all the problems you will encounter in your later career. This course is no exception.

Below is a video on verifying a solution to a differential equation.

1.2.3: Differential Equations in Practice

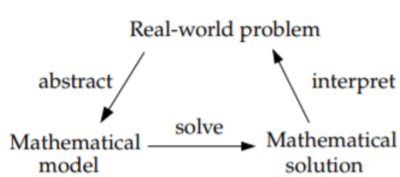


Figure 1.2.2

So how do we use differential equations in science and engineering? First, we have some *real-world problem* we wish to understand. We make some simplifying assumptions and create a *mathematical model*. That is, we translate the real-world situation into a set of differential equations. Then we apply mathematics to get some sort of a *mathematical solution*. There is still something left to do. We have to interpret the results. We have to figure out what the mathematical solution says about the real-world problem we started with.

Learning how to formulate the mathematical model and how to interpret the results is what your physics and engineering classes do. In this course, we will focus mostly on the mathematical analysis. Sometimes we will work with simple real-world examples so that we have some intuition and motivation about what we are doing.

Let us look at an example of this process. One of the most basic differential equations is the standard *exponential growth model*. Let P denote the population of some bacteria on a Petri dish. We assume that there is enough food and enough space. Then the rate of growth of bacteria is proportional to the population—a large population grows quicker. Let t denote time (say in seconds) and P the population. Our model is

$$\frac{dP}{dt} = kP,$$

for some positive constant $k > 0$.

Example 1.2.1

Suppose there are 100 bacteria at time 0 and 200 bacteria 10 seconds later. How many bacteria will there be 1 minute from time 0 (in 60 seconds)?

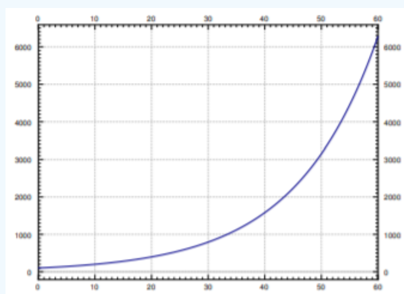


Figure $1.2.2$: Bacteria growth in the first 60 seconds.

Solution

First we need to solve the equation. We claim that a solution is given by

$$P(t) = Ce^{kt},$$

where C is a constant. Let us try:

$$\frac{dP}{dt} = Cke^{kt} = kP.$$

And it really is a solution.

OK, now what? We do not know C, and we do not know k. But we know something. We know $P(0) = 100$, and we know $P(10) = 200$. Let us plug these conditions in and see what happens.

$$\begin{aligned} 100 &= P(0) = Ce^{k0} = C, \\ 200 &= P(10) = 100\, e^{k10}. \end{aligned} \tag{1.2.2}$$

Therefore, $2 = e^{10k}$ or $\frac{\ln 2}{10} = k \approx 0.069$. So

$$P(t) = 100\, e^{(\ln 2)t/10} \approx 100\, e^{0.069t}.$$

At one minute, $t = 60$, the population is $P(60) = 6400$. See Figure $1.2.2$.

Let us talk about the interpretation of the results. Does our solution mean that there must be exactly 6400 bacteria on the plate at 60s? No! We made assumptions that might not be true exactly, just approximately. If our assumptions are reasonable, then there will be approximately 6400 bacteria. Also, in real life P is a discrete quantity, not a real number. However, our model has no problem saying that for example at 61 seconds, $P(61) \approx 6859.35$.

Normally, the k in $P' = kP$ is known, and we want to solve the equation for different *initial conditions*. What does that mean? Take $k = 1$ for simplicity. Suppose we want to solve the equation $\frac{dP}{dt} = P$ subject to $P(0) = 1000$ (the initial condition). Then the solution turns out to be (exercise)

$$P(t) = 1000\, e^t.$$

We call $P(t) = Ce^t$ *the general solution*, as every solution of the equation can be written in this form for some constant C. We need an initial condition to find out what C is, in order to find the *particular solution* we are looking for. Generally, when we say "particular solution," we just mean some solution.

Below is a video on verifying a solution to a differential equation and finding a particular solution.

1.2.4: Fundamental Equations

A few equations appear often and it is useful to just memorize what their solutions are. Let us call them the four fundamental equations. Their solutions are reasonably easy to guess by recalling properties of exponentials, sines, and cosines. They are also simple to check, which is something that you should always do. No need to wonder if you remembered the solution correctly.

First such equation is

$$\frac{dy}{dx} = ky,$$

for some constant $k > 0$. Here y is the dependent and x the independent variable. The general solution for this equation is

$$y(x) = Ce^{kx}.$$

We saw above that this function is a solution, although we used different variable names.

Next,

$$\frac{dy}{dx} = -ky,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = Ce^{-kx}.$$

Exercise 1.2.1

Check that the y given is really a solution to the equation.

Next, take the *second order differential equation*

$$\frac{d^2y}{dx^2} = -k^2y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx).$$

Since the equation is a second order differential equation, we have two constants in our general solution.

? Exercise 1.2.2

Check that the y given is really a solution to the equation.

Finally, consider the second order differential equation

$$\frac{d^2y}{dx^2} = k^2 y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 e^{kx} + C_2 e^{-kx},$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(kx).$$

For those that do not know, cosh and sinh are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

They are called the *hyperbolic cosine* and *hyperbolic sine*. These functions are sometimes easier to work with than exponentials. They have some nice familiar properties such as $\cosh 0 = 1$, $\sinh 0 = 0$, and $\frac{d}{dx}\cosh x = \sinh x$ (no that is not a typo) and $\frac{d}{dx}\sinh x = \cosh x$.

? Exercise 1.2.3

Check that both forms of the y given are really solutions to the equation.

✓ Example 1.2.2

In equations of higher order, you get more constants you must solve for to get a particular solution. The equation $\frac{d^2y}{dx^2} = 0$ has the general solution $y = C_1 x + C_2$; simply integrate twice and don't forget about the constant of integration. Consider the initial conditions $y(0) = 2$ and $y'(0) = 3$. We plug in our general solution and solve for the constants:

$$2 = y(0) = C_1 \cdot 0 + C_2 = C_2, \qquad 3 = y'(0) = C_1.$$

In other words, $y = 3x + 2$ is the particular solution we seek.

An interesting note about cosh: The graph of cosh is the exact shape of a hanging chain. This shape is called a *catenary*. Contrary to popular belief this is not a parabola. If you invert the graph of cosh, it is also the ideal arch for supporting its weight. For example, the gateway arch in Saint Louis is an inverted graph of cosh—if it were just a parabola it might fall. The formula used in the design is inscribed inside the arch:

$$y = -127.7 \text{ ft} \cdot \cosh(x/127.7 \text{ ft}) + 757.7 \text{ ft}.$$

1.3: Classification of Differential Equations

There are many types of differential equations, and we classify them into different categories based on their properties. Let us quickly go over the most basic classification. We already saw the distinction between ordinary and partial differential equations:

- *Ordinary differential equations* or (ODE) are equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.
- *Partial differential equations* or (PDE) are equations that depend on partial derivatives of several variables. That is, there are several independent variables.

Let us see some examples of ordinary differential equations:

$$\begin{aligned} &\frac{dy}{dt} = ky, && \text{(Exponential growth)} \\ &\frac{dy}{dt} = k(A-y), && \text{(Newton's law of cooling)} \\ &m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t). && \text{(Mechanical vibrations)} \end{aligned} \tag{1.3.1}$$

And of partial differential equations:

$$\begin{aligned} &\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0, && \text{(Transport equation)} \\ &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, && \text{(Heat equation)} \\ &\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. && \text{(Wave equation in 2 dimensions)} \end{aligned} \tag{1.3.2}$$

If there are several equations working together, we have a so-called *system of differential equations*. For example,

$$y' = x, \qquad x' = y$$

is a simple system of ordinary differential equations. Maxwell's equations for electromagnetics,

$$\begin{aligned} &\nabla \cdot \vec{D} = \rho, && \nabla \cdot \vec{B} = 0, \\ &\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, && \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \end{aligned} \tag{1.3.3}$$

are a system of partial differential equations. The divergence operator $\nabla \cdot$ and the curl operator $\nabla \times$ can be written out in partial derivatives of the functions involved in the x, y, and z variables.

The next bit of information is the *order* of the equation (or system). The order is simply the order of the largest derivative that appears. If the highest derivative that appears is the first derivative, the equation is of first order. If the highest derivative that appears is the second derivative, then the equation is of second order. For example, Newton's law of cooling above is a first order equation, while the mechanical vibrations equation is a second order equation. The equation governing transversal vibrations in a beam,

$$a^4 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0,$$

is a fourth order partial differential equation. It is fourth order as at least one derivative is the fourth derivative. It does not matter that the derivative in t is only of second order.

In the first chapter, we will start attacking first order ordinary differential equations, that is, equations of the form $\frac{dy}{dx} = f(x, y)$. In general, lower order equations are easier to work with and have simpler behavior, which is why we start with them.

We also distinguish how the dependent variables appear in the equation (or system). In particular, we say an equation is *linear* if the dependent variable (or variables) and their derivatives appear linearly, that is only as first powers, they are not multiplied together, and no other functions of the dependent variables appear. In other words, the equation is a sum of terms, where each term is some function of the independent variables or some function of the independent variables multiplied by a dependent variable or its derivative. Otherwise, the equation is called *nonlinear*. For example, an ordinary differential equation is linear if it can be put into the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=b(x). \tag{1.3.4}$$

The functions a_0, a_1, ..., a_n are called the *coefficients*. The equation is allowed to depend arbitrarily on the independent variable. So

$$e^x\frac{d^2 y}{dx^2}+\sin(x)\frac{dy}{dx}+x^2 y=\frac{1}{x} \tag{1.3.5}$$

is still a linear equation as y and its derivatives only appear linearly.

All the equations and systems above as examples are linear. It may not be immediately obvious for Maxwell's equations unless you write out the divergence and curl in terms of partial derivatives. Let us see some nonlinear equations. For example ,

$$\frac{\partial y}{\partial t}+y\frac{\partial y}{\partial x}=\nu\frac{\partial^2 y}{\partial x^2},$$

is a nonlinear second order partial differential equation. It is nonlinear because y and $\frac{\partial y}{\partial x}$ are multiplied together. The equation

$$\frac{dx}{dt}=x^2 \tag{1.3.6}$$

is a nonlinear first order differential equation as there is a second power of the dependent variable x.

A linear equation may further be called *homogenous* if all terms depend on the dependent variable. That is, if no term is a function of the independent variables alone. Otherwise, the equation is called *homogeneous* or *inhomogeneous*. For example, the exponential growth equation, the wave equation, or the transport equation above are homogeneous. The mechanical vibrations equation above is nonhomogeneous as long as $f(t)$ is not the zero function. Similarly, if the ambient temperature A is nonzero, Newton's law of cooling is nonhomogeneous. A homogeneous linear ODE can be put into the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Compare to (1.3.4) and notice there is no function $b(x)$.

If the coefficients of a linear equation are actually constant functions, then the equation is said to have *constant coefficients*. The coefficients are the functions multiplying the dependent variable(s) or one of its derivatives, not the function $b(x)$ standing alone. A constant coefficient nonhomogeneous ODE is an equation of the form

$$a_n\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1} y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0 y=b(x),$$

where $a_0, a_1, \ldots, a_n$ are all constants, but b may depend on the independent variable x. The mechanical vibrations equation above is a constant coefficient nonhomogeneous second order ODE. The same nomenclature applies to PDEs, so the transport equation, heat equation and wave equation are all examples of constant coefficient linear PDEs.

Finally, an equation (or system) is called *autonomous* if the equation does not depend on the independent variable. For autonomous ordinary differential equations, the independent variable is then thought of as time. Autonomous equation means an equation that does not change with time. For example, Newton's law of cooling is autonomous, so is equation (1.3.6). On the other hand, mechanical vibrations or (1.3.5) are not autonomous.

Below is a video on defining and classifying differential equations.

1.E: Introduction (Exercises)

1.E.1: title="0.2: Introduction to Differential Equations" href="/Bookshelves/Differential_Equations/Book:_Differential_Equations_for_Engineers_(Lebl)/0:_Introduction/0.2:_Introduction_to_Differential_Equations">Introduction to Differential Equations

Exercise $1.E.0.2.1$

Show that $x = e^{4t}$ is a solution to $x''' - 12x'' + 48x' - 64x = 0$.

Exercise $1.E.0.2.2$

Show that $x = e^{t}$ is not a solution to $x''' - 12x'' + 48x' - 64x = 0$.

Exercise $1.E.0.2.3$

Is $y = \sin t$ a solution to $\left(\frac{dy}{dt}\right)^2 = 1 - y^2$? Justify.

Exercise $1.E.0.2.4$

Let $y'' + 2y' - 8y = 0$. Now try a solution of the form $y = e^{rx}$ for some (unknown) constant r. Is this a solution for some r? If so, find all such r.

Exercise $1.E.0.2.5$

Verify that $x = Ce^{-2t}$ is a solution to $x' = -2x$. Find C to solve for the initial condition $x(0) = 100$.

Exercise $1.E.0.2.6$

Verify that $x = C_1e^{-t} + C_2e^{2t}$ is a solution to $x'' - x' - 2x = 0$. Find C_1 and C_2 to solve for the initial conditions $x(0) = 10$ and $x'(0) = 0$.

Exercise $1.E.0.2.7$

Find a solution to $(x')^2 + x^2 = 4$ using your knowledge of derivatives of functions that you know from basic calculus.

Exercise $1.E.0.2.8$

Solve:

a. $\dfrac{dA}{dt} = -10A, \quad A(0) = 5$
b. $\dfrac{dH}{dx} = 3H, \quad H(0) = 1$
c. $\dfrac{d^2y}{dx^2} = 4y, \quad y(0) = 0, \quad y'(0) = 1$
d. $\dfrac{d^2x}{dy^2} = -9x, \quad x(0) = 1, \quad x'(0) = 0$

Exercise $1.E.0.2.9$

Is there a solution to $y' = y$, such that $y(0) = y(1)$?

Exercise $1.E.0.2.10$

The population of city X was 100 thousand 20 years ago, and the population of city X was 120 thousand 10 years ago. Assuming constant growth, you can use the exponential population model (like for the bacteria). What do you estimate the population is now?

Exercise $1.E.0.2.11$

Suppose that a football coach gets a salary of one million dollars now, and a raise of 10% every year (so exponential model, like population of bacteria). Let s be the salary in millions of dollars, and t is time in years.

a. What is $s(0)$ and $s(1)$.
b. Approximately how many years will it take for the salary to be 10 million.
c. Approximately how many years will it take for the salary to be 20 million.
d. Approximately how many years will it take for the salary to be 30 million.

Exercise $1.E.0.2.12$

Show that $x = e^{-2t}$ is a solution to $x'' + 4x' + 4x = 0$.

Answer

Compute $x' = -2e^{-2t}$ and $x'' = 4e^{-2t}$. Then $(4e^{-2t}) + 4(-2e^{-2t}) + 4(e^{-2t}) = 0$.

Exercise $1.E.0.2.13$

Is $y = x^2$ a solution to $x^2y'' - 2y = 0$? Justify.

Answer

Yes.

Exercise $1.E.0.2.14$

Let $xy'' - y' = 0$. Try a solution of the form $y = x^r$. Is this a solution for some r? If so, find all such r.

Answer

$y = x^r$ is a solution for $r = 0$ and $r = 2$.

Exercise $1.E.0.2.15$

Verify that $x = C_1e^t + C_2$ is a solution to $x'' - x' = 0$. Find C_1 and C_2 so that x satisfies $x(0) = 10$ and $x'(0) = 100$.

Answer

$C_1 = 100, C_2 = -90$

Exercise $1.E.0.2.16$

Solve $\frac{d\varphi}{ds} = 8\varphi$ and $\varphi(0) = -9$.

Answer

$\varphi = -9e^{8s}$

Exercise $1.E.0.2.17$

Solve:

a. $\dfrac{dx}{dt} = -4x, \quad x(0) = 9$
b. $\dfrac{d^2x}{dt^2} = -4x, \quad x(0) = 1, \quad x'(0) = 2$
c. $\dfrac{dp}{dq} = 3p, \quad p(0) = 4$
d. $\dfrac{d^2T}{dx^2} = 4T, \quad T(0) = 0, \quad T'(0) = 6$

Answer

a. $x = 9e^{-4t}$
b. $x = \cos(2t) + \sin(2t)$
c. $p = 4e^{3q}$
d. $T = 3\sinh(2x)$

1.E.2: title="0.3: Classification of Differential Equations" href="/Bookshelves/Differential_Equations/Book:_Differential_Equations_for_Engineers_(Lebl)/0:_Introduction/0.3:_Classification_of_Differential_Equations">Classification of Differential Equations

Exercise $1.E.0.3.1$

Classify the following equations. Are they ODE or PDE? Is it an equation or a system? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous, constant coefficient? If it is an ODE, is it autonomous?

a. $\sin(t)\dfrac{d^2x}{dt^2} + \cos(t)x = t^2$
b. $\dfrac{\partial u}{\partial x} + 3\dfrac{\partial u}{\partial y} = xy$
c. $y'' + 3y + 5x = 0, \quad x'' + x - y = 0$
d. $\dfrac{\partial^2 u}{\partial t^2} + u\dfrac{\partial^2 u}{\partial s^2} = 0$
e. $x'' + tx^2 = t$
f. $\dfrac{d^4x}{dt^4} = 0$

Exercise $1.E.0.3.2$

If $\vec{u} = (u_1, u_2, u_3)$ is a vector, we have the divergence $\nabla \cdot \vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$ and curl $\nabla \times \vec{u} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)$. Notice that curl of a vector is still a vector. Write out Maxwell's equations in terms of partial derivatives and classify the system.

Exercise $1.E.0.3.3$

Suppose F is a linear function, that is, $F(x, y) = ax + by$ for constants a and b. What is the classification of equations of the form $F(y', y) = 0$.

Exercise $1.E.0.3.4$

Write down an explicit example of a third order, linear, nonconstant coefficient, nonautonomous, nonhomogeneous system of two ODE such that every derivative that could appear, does appear.

Exercise $1.E.0.3.5$

Classify the following equations. Are they ODE or PDE? Is it an equation or a system? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous, constant coefficient? If it is an ODE, is it autonomous?

a. $\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial^2 v}{\partial y^2} = \sin(x)$
b. $\frac{dx}{dt} + \cos(t)x = t^2 + t + 1$
c. $\frac{d^7 F}{dx^7} = 3F(x)$
d. $y'' + 8y' = 1$
e. $x'' + tyx' = 0, \quad y'' + txy = 0$
f. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial s^2} + u^2$

Answer

a. PDE, equation, second order, linear, nonhomogeneous, constant coefficient.
b. ODE, equation, first order, linear, nonhomogeneous, not constant coefficient, not autonomous.
c. ODE, equation, seventh order, linear, homogeneous, constant coefficient, autonomous.
d. ODE, equation, second order, linear, nonhomogeneous, constant coefficient, autonomous.
e. ODE, system, second order, nonlinear.
f. PDE, equation, second order, nonlinear.

Exercise $1.E.0.3.6$

Write down the general *zero*th order linear ordinary differential equation. Write down the general solution.

Answer

equation: $a(x)y = b(x)$, solution: $y = \frac{b(x)}{a(x)}$.

Exercise $1.E.0.3.7$

For which k is $\frac{dx}{dt} + x^k = t^{k+2}$ linear. Hint: there are two answers.

Answer

$k = 0$ or $k = 1$

CHAPTER OVERVIEW

2: First order ODEs

2.1: Integrals as solutions

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y)$$

or just

$$y' = f(x, y)$$

In general, there is no simple formula or procedure one can follow to find solutions. In the next few lectures we will look at special cases where solutions are not difficult to obtain. In this section, let us assume that f is a function of x alone, that is, the equation is

$$y' = f(x) \tag{2.1.1}$$

We could just integrate (antidifferentiate) both sides with respect to x.

$$\int y'(x)dx = \int f(x)dx + C$$

that is

$$y(x) = \int f(x)dx + C$$

This $y(x)$ is actually the general solution. So to solve Equation 2.1.1, we find some antiderivative of $f(x)$ and then we add an arbitrary constant to get the general solution.

Now is a good time to discuss a point about calculus notation and terminology. Calculus textbooks muddy the waters by talking about the integral as primarily the so-called **indefinite integral**. The indefinite integral is really the **antiderivative** (in fact the whole one-parameter family of antiderivatives). There really exists only one integral and that is the definite integral. The only reason for the indefinite integral notation is that we can always write an antiderivative as a (definite) integral. That is, by the fundamental theorem of calculus we can always write $\int f(x)dx + C$ as

$$\int_{x_0}^{x} f(t)dt + C$$

Hence the terminology to integrate when we may really mean to antidifferentiate. Integration is just one way to compute the antiderivative (and it is a way that always works, see the following examples). Integration is defined as the area under the graph, it only happens to also compute antiderivatives. For sake of consistency, we will keep using the indefinite integral notation when we want an antiderivative, and you should always think of the definite integral.

Example 2.1.1

Find the general solution of $y' = 3x^2$.

Solution

Elementary calculus tells us that the general solution must be $y = x^3 + C$. Let us check: $y' = 3x^2$. We have gotten precisely our equation back.

Normally, we also have an initial condition such as $y(x_0) = y_0$ for some two numbersx_0 and y_0 x_0 is usually 0, but not always). We can then write the solution as a definite integral in a nice way. Suppose our problem is $y' = f(x)$, $y(x_0) = y_0$. Then the solution is

$$y(x) = \int_{x_0}^{x} f(s)ds + y_0 \tag{2.1.2}$$

Let us check! We compute $y' = f(x)$, via the fundamental theorem of calculus, and by Jupiter, y is a solution. Is it the one satisfying the initial condition? Well, $y(x_0) = \int_{x_0}^{x_0} f(x)dx + y_0 = y_0$. It is!

Do note that the definite integral and the indefinite integral (antidifferentiation) are completely different beasts. The definite integral always evaluates to a number. Therefore, Equation 2.1.2 is a formula we can plug into the calculator or a computer, and it will be happy to calculate specific values for us. We will easily be able to plot the solution and work with it just like with any other function. It is not so crucial to always find a closed form for the antiderivative.

Below is a video on using integration to solve a differential equation.

Example 2.1.2

Solve

$$y' = e^{-x^2}, \;\; y(0) = 1.$$

By the preceding discussion, the solution must be

$$y(x) = \int_0^x e^{-s^2}\, ds + 1.$$

Solution

Here is a good way to make fun of your friends taking second semester calculus. Tell them to find the closed form solution. Ha ha ha (bad math joke). It is not possible (in closed form). There is absolutely nothing wrong with writing the solution as a definite integral. This particular integral is in fact very important in statistics.

Using this method, we can also solve equations of the form

$$y' = f(y)$$

Let us write the equation in Leibniz notation.

$$\frac{dy}{dx} = f(y)$$

Now we use the inverse function theorem from calculus to switch the roles of x and y to obtain

$$\frac{dy}{dx} = \frac{1}{f(y)}$$

What we are doing seems like algebra with dx and dy. It is tempting to just do algebra with dx and dy as if they were numbers. And in this case it does work. Be careful, however, as this sort of hand-waving calculation can lead to trouble, especially when more than one independent variable is involved. At this point we can simply integrate,

$$x(y) = \int \frac{1}{f(y)} dy + C$$

Finally, we try to solve for y.

Example 2.1.3

Previously, we guessed $y' = ky$ (for some $k > 0$) has the solution $y = Ce^{kx}$. We can now find the solution without guessing. First we note that $y = 0$ is a solution. Henceforth, we assume $y \neq 0$. We write

$$\frac{dx}{dy} = \frac{1}{ky}$$

We integrate to obtain

$$x(y) = x = \frac{1}{k}\ln|y| + D$$

where D is an arbitrary constant. Now we solve for$y$ (actually for $|y|$).

$$|y| = e^{kx-kD} = e^{-kD}e^{kx}$$

If we replace e^{-kD} with an arbitrary constant C we can get rid of the absolute value bars (which we can do as D was arbitrary). In this way, we also incorporate the solution $y = 0$. We get the same general solution as we guessed before, $y = Ce^{kx}$.

Example 2.1.4

Find the general solution of $y' = y^2$.

Solution

First we note that $y = 0$ is a solution. We can now assume that $y \neq 0$. Write

$$\frac{dx}{dy} = \frac{1}{y^2}$$

We integrate to get

$$x = \frac{-1}{y} + C$$

We solve for $y = \dfrac{1}{C - x}$. So the general solution is

$$y = \frac{1}{C - x} \quad or \quad y = 0$$

Note the singularities of the solution. If for example $C = 1$, then the solution as we approach $x = 1$. See Figure 2.1.1. Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The equation $y' = y^2$ is very nice and defined everywhere, but the solution is only defined on some interval $(-\infty, C)$ or (C, ∞). Usually when this happens we only consider one of these the solution. For example if we impose a condition $y(0) = 1$, then the solution is $y = \frac{1}{1-x}$, and we would consider this solution only for x on the interval $(-\infty, 1)$. In the figure, it is the left side of the graph.

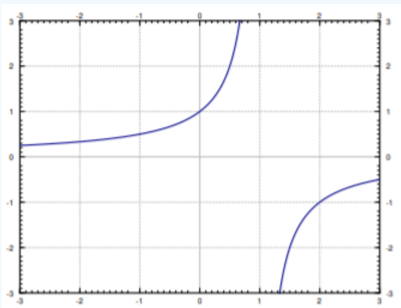


Figure 2.1.1: Plot of $y = \frac{1}{1-x}$.

Below is a video on using integration to solve an inital value problem.

Classical problems leading to differential equations solvable by integration are problems dealing with velocity, acceleration and distance. You have surely seen these problems before in your calculus class.

Example 2.1.5

Suppose a car drives at a speed $e^{t/2}$ meters per second, where t is time in seconds. How far did the car get in 2 seconds (starting at $t=0$)? How far in 10 seconds?

Solution

Let x denote the distance the car traveled. The equation is

$$x' = e^{t/2}$$

We can just integrate this equation to get that

$$x(t) = 2e^{t/2} + C$$

We still need to figure out C. We know that when $t=0$, then $x=0$. That is, $x(0)=0$. So

$$0 = x(0) = 2e^{0/2} + C = 2 + C$$

Thus $C=-2$ and

$$x(t) = 2e^{t/2} - 2$$

Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain

$$x(2) = 2e^{2/2} - 2 \approx 3.44 \text{ meters}, \quad x(10) = 2e^{10/2} - 2 \approx 294 \text{ meters}$$

Example 2.1.6

Suppose that the car accelerates at a rate of $t^2 \frac{m}{s^2}$. At time $t=0$ the car is at the 1 meter mark and is traveling at 10 $^{m}/_{s}$. Where is the car at time $t=10$.

Solution

Well this is actually a second order problem. If x is the distance traveled, then x' is the velocity, and x'' is the acceleration. The equation with initial conditions is

$$x'' = t^2, \quad x(0) = 1, \quad x'(0) = 10$$

What if we say $x' = v$. Then we have the problem

$$v' = t^2, \quad v(0) = 10$$

Once we solve for v, we can integrate and find x.

2.2: Slope fields

The general first order equation we are studying looks like

$$y' = f(x, y)$$

In general, we cannot simply solve these kinds of equations explicitly. It would be nice if we could at least figure out the shape and behavior of the solutions, or if we could find approximate solutions. At this point it may be good to first try the Lab I and/or Project I from the IODE website.

2.2.1: Slope fields

The equation $y' = f(x, y)$ gives you a slope at each point in the (x, y)-plane. And this is the slope a solution $y(x)$ would have at x if its value was y. In other words, $f(x, y)$ is the slope of a solution whose graph runs through the point (x, y). At a point (x, y), we plot a short line with the slope $f(x, y)$. For example, if $f(x, y) = xy$, then at point $(2, 1.5)$ we draw a short line of slope $xy = 2 \times 1.5 = 3$. So, if $y(x)$ is a solution and $y(2) = 1.5$, then the equation mandates that $y'(2) = 3$. See Figure $2.2.1$.

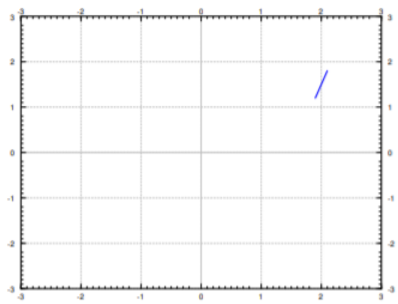


Figure 2.2.1: The slope $y' = xy$ at $(2, 1.5)$.

To get an idea of how solutions behave, we draw such lines at lots of points in the plane, not just the point $(2, 1.5)$. We would ideally want to see the slope at every point, but that is just not possible. Usually we pick a grid of points fine enough so that it shows the behavior, but not too fine so that we can still recognize the individual lines. We call this picture the of the equation. See Figure $2.2.2$ for the slope field of the equation $y' = xy$. Usually in practice, one does not do this by hand, but has a computer do the drawing.

Below is a video on slope fields.

Suppose we are given a specific initial condition $y(x_0) = y_0$. A solution, that is, the graph of the solution, would be a curve that follows the slopes we drew. For a few sample solutions, see Figure $2.2.3$. It is easy to roughly sketch (or at least imagine) possible solutions in the slope field, just from looking at the slope field itself. You simply sketch a line that roughly fits the little line segments and goes through your initial condition.

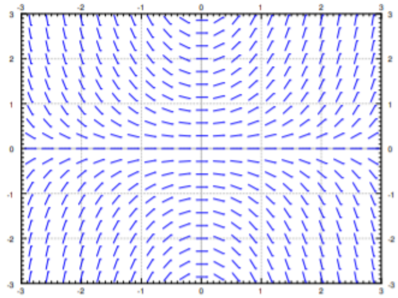


Figure $2.2.3$: Slope field of $y' = xy$.

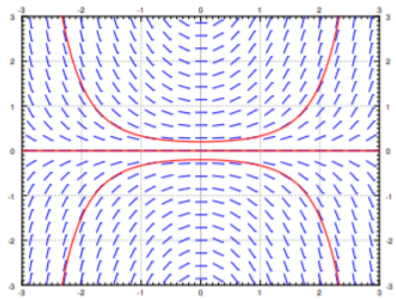


Figure $2.2.3$: Slope field of $y' = xy$ with a graph of solutions satisfying $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.

By looking at the slope field we get a lot of information about the behavior of solutions without having to solve the equation. For example, in Figure $2.2.3$ we see what the solutions do when the initial conditions are $y(0) > 0$, $y(0) = 0$ and $y(0) < 0$. A small change in the initial condition causes quite different behavior. We see this behavior just from the slope field and imagining what solutions ought to do.

We see a different behavior for the equation $y' = -y$. The slope field and a few solutions is in see Figure $2.2.4$. If we think of moving from left to right (perhaps x is time and time is usually increasing), then we see that no matter what $y(0)$ is, all solutions tend to zero as x tends to infinity. Again that behavior is clear from simply looking at the slope field itself.

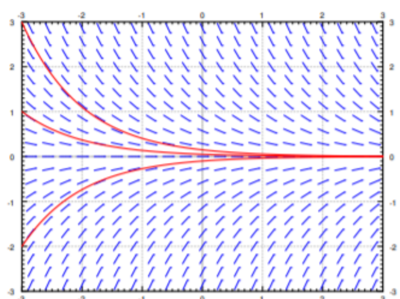


Figure $2.2.4$: Slope field of $y' = -y$ with a graph of a few solutions.

Below is a video on choosing which differential equation corresponds to the given slope field.

2.2.2: Existence and uniqueness

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

i. Does a solution exist?
ii. Is the solution unique (if it exists)?

What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

Example 2.2.1

Attempt to solve

$$y' = \frac{1}{x}, \quad y(0) = 0.$$

Solution

Integrate to find the general solution $y = \ln|x| + C$. Note that the solution does not exist at $x = 0$. See Figure $2.2.5$ on the following page. The equation may have been written as the seemingly harmless $xy' = 1$.

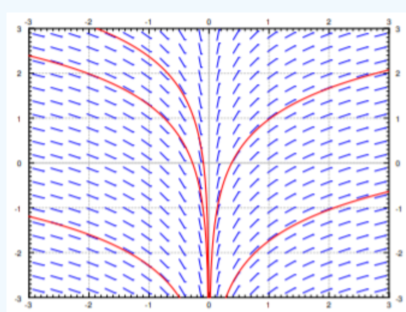


Figure $2.2.5$: Slope field of $y' = \frac{1}{x}$.

Example 2.2.2

Solve:

$$y' = 2\sqrt{|y|}, \quad y(0) = 0.$$

Solution

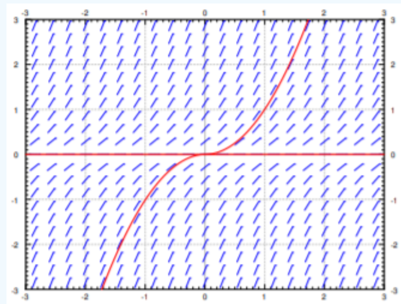


Figure 2.2.6: Slope field of $y' = 2\sqrt{|y|}$ with two solutions satisfying $y(0) = 0$.

Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as **Picard's theorem**.[1]

Theorem 2.2.1

Picard's theorem on existence and uniqueness

If $f(x,y)$ is *continuous (as a function of two variables) and* $\frac{\partial f}{\partial y}$ *exists and is continuous near some* (x_0, y_0), then a solution to

$$y' = f(x,y), \quad y(x_0) = y_0$$

exists (at least for x *in some small interval) and is unique.*

Note that the problems $y' = \frac{1}{x}$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

Example 2.2.3

For some constant A, solve:

$$y' = y^2 \quad y(0) = A$$

.

Solution

We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = \frac{1}{y^2}$, so $x = -\frac{1}{y} + C$, so $y = \frac{1}{C - x}$. If $y(0) = A$, then $C = \frac{1}{A}$ so

$$y = \frac{1}{\frac{1}{A} - x}$$

If $A = 0$, then $y = 0$ is a solution.

For example, when $A = 1$ the solution “blows up” at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice.

For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds “globally” unlike for the equation $y' = y^2$.

2.2.3: Footnotes

[1] Named after the French mathematician Charles Émile Picard (1856 – 1941)

2.3: Separable Equations

When a differential equation is of the form $y' = f(x)$, we can just integrate: $y = \int f(x)dx + C$. Unfortunately this method no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields

$$y = \int f(x, y)dx + C$$

Notice the dependence on y in the integral.

2.3.0.1: Separable equations

Let us suppose that the equation is *separable*. That is, let us consider

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the Leibniz notation

$$\frac{dy}{dx} = f(x)g(y)$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x)dx$$

Now both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y.

Example 2.3.1

Take the equation

$$y' = xy$$

First note that $y = 0$ is a solution, so assume $y \neq 0$ from now on, so that we can divide by y. Write the equation as $\frac{dy}{dx} = xy$, then

$$\int \frac{dy}{y} = \int x dx + C.$$

We compute the antiderivatives to get

$$\ln|y| = \frac{x^2}{2} + C$$

Or

$$|y| = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}} e^C = De^{\frac{x^2}{2}}$$

where $D > 0$ is some constant. Because $y = 0$ is a solution and because of the absolute value we actually can write:

$$y = De^{\frac{x^2}{2}}$$

for any number D (including zero or negative).

We check:

$$y' = Dxe^{\frac{x^2}{2}} = x\left(De^{\frac{x^2}{2}}\right) = xy$$

Yay!

We should be a little bit more careful with this method. You may be worried that we were integrating in two different variables. We seemingly did a different operation to each side. Let us work through this method more rigorously. Take

$$\frac{dy}{dx} = f(x)g(y)$$

We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$!

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

We integrate both sides with respect to x.

$$\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx + C$$

We use the change of variables formula (substitution) on the left hand side:

$$\int \frac{1}{g(y)}dy = \int f(x)dx + C$$

And we are done.

2.3.1: Implicit solutions

It is clear that we might sometimes get stuck even if we can do the integration. For example, take the separable equation

$$y' = \frac{xy}{y^2+1}$$

We separate variables,

$$\frac{y^2+1}{y}dy = \left(y + \frac{1}{y}\right)dy = xdx$$

We integrate to get

$$\frac{y^2}{2} + ln|y| = \frac{x^2}{2} + C$$

or perhaps the easier looking expression (where $D = 2C$)

$$y^2 + 2ln|y| = x^2 + D$$

It is not easy to find the solution explicitly as it is hard to solve for y. We, therefore, leave the solution in this form and call it an *implicit solution*. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x, and remember that y is a function of x, to get

$$y'\left(2y + \frac{2}{y}\right) = 2x$$

Multiply both sides by y and divide by $2(y^2+1)$ and you will get exactly the differential equation. We leave this computation to the reader.

If you have an implicit solution, and you want to compute values for y, you might have to be tricky. You might get multiple solutions y for each x, so you have to pick one. Sometimes you can graph x as a function of y, and then flip your paper. Sometimes you have to do more.

Computers are also good at some of these tricks. More advanced mathematical software usually has some way of plotting solutions to implicit equations. For example, for $C = 0$ if you plot all the points (x, y) that are solutions to $y^2 + 2\ln|y| = x^2$, you find the two curves in Figure 2.3.1. This is not quite a graph of a function. For each x there are two choices of y. To find a function you would have to pick one of these two curves. You pick the one that satisfies your initial condition if you have one. For example, the top curve satisfies the condition $y(1) = 1$. So for each C we really got two solutions. As you can see, computing values from an implicit solution can be somewhat tricky. But sometimes, an implicit solution is the best we can do.

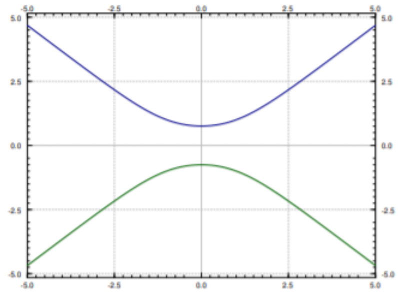


Figure 2.3.1: The implicit solution $y^2 + 2\ln|y| = x^2$ to $y' = \frac{xy}{y^2+1}$.

The equation above also has the solution $y = 0$. So the general solution is

$$y^2 + 2\ln|y| = x^2 + C, \qquad \text{and} \qquad y = 0.$$

These outlying solutions such as $y = 0$ are sometimes called *singular solutions*.

Below is a video on solving a separable differential equation.

Example 2.3.2

Solve $x^2 y' = 1 - x^2 + y^2 - x^2 y^2$, $y(1) = 0$.

Solution

First factor the right hand side to obtain

$$x^2 y' = (1 - x^2)(1 + y^2)$$

Separate variables, integrate, and solve for y

$$\begin{aligned} \frac{y'}{1+y^2} &= \frac{1-x^2}{x^2}, \\ \frac{y'}{1+y^2} &= \frac{1}{x^2} - 1, \\ \arctan(y) &= -\frac{1}{x^2} - x + C, \\ y &= \tan\left(-\frac{1}{x} - x + C\right) \end{aligned} \tag{2.3.1}$$

Solve for the initial condition, $0 = \tan(-2 + C)$ to get $C = 2$ (or $C = 2 + \pi$, or $C = 2 + 2\pi$, etc.). The particular solution we seek is, therefore,

$$y = \tan\left(\frac{-1}{x} - x + 2\right).$$

Example 2.3.3

Juan made a cup of coffee, and Juan likes to drink coffee only once reaches 60 degrees Celsius and will not burn him. Initially at time $t = 0$ minutes, Juan measured the temperature and the coffee was 89 degrees Celsius. One minute later, Juan measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Juan start drinking?

Solution

Let T be the temperature of the coffee in degrees Celsius, and let A be the ambient (room) temperature, also in degrees Celsius. states that the rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

for some constant k. For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants)

$$\begin{aligned} \frac{1}{T-A}\frac{dT}{dt} &= -k, \\ \ln(T-A) &= -kt + C, \quad (\text{note that} T - A > 0) \\ T - A &= De^{-kt}, \\ T &= A + De^{-kt} \end{aligned} \tag{2.3.2}$$

That is, $T = 22 + D\,e^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67\,e^{-kt}$. The second condition says $85 = T(1) = 22 + 67\,e^{-k}$. Solving for k we get $k = -\ln\frac{85-22}{67} \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. Namely, we solve

$$60 = 22 + 67e^{-0.0616t}$$

to get $t = -\frac{\ln\frac{60-22}{67}}{0.0616} \approx 9.21$ minutes. So Juan can begin to drink the coffee at just over 9 minutes from the time Juan made it. That is probably about the amount of time it took us to calculate how long it would take. See Figure 2.3.2.

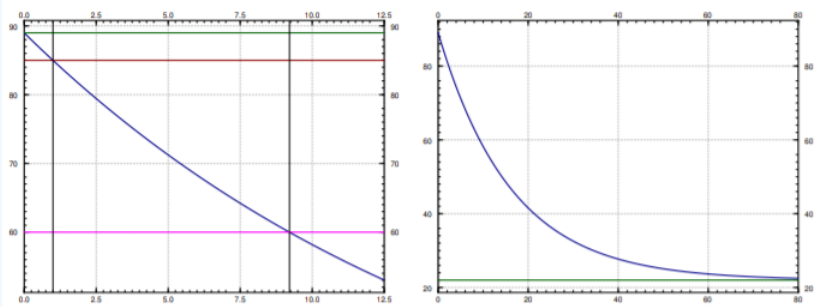


Figure 2.3.2: Graphs of the coffee temperature function $T(t)$. On the left, horizontal lines are drawn at temperatures 60, 85, and 89. Vertical lines are drawn at $t = 1$ and $t = 9.21$. Notice that the temperature of the coffee hits 85 at $t = 1$, and 60 at $t \approx 9.21$. On the right, the graph is over a longer period of time, with a horizontal line at the ambient temperature 22.

Example 2.3.4

Find the general solution to $y' = \frac{-xy^2}{3}$ (including singular solutions).

Solution

First note that $y = 0$ is a solution (a singular solution). Now assume that $y \neq 0$.

$$\begin{aligned} -\frac{3}{y^2} y' &= x, \\ \frac{3}{y} &= \frac{x^2}{2} + C, \\ y &= \frac{3}{\frac{x^2}{2} + C} = \frac{6}{x^2 + 2C}. \end{aligned} \tag{2.3.3}$$

So the general solution is,

$$y = \frac{6}{x^2 + 2C}, \quad \text{and} \quad y = 0.$$

2.4: Linear equations and the integrating factor

One of the most important types of equations we will learn how to solve are the so-called linear equations. In fact, the majority of the course is about linear equations. In this lecture we focus on the first order linear equation. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x). \tag{2.4.1}$$

Here the word "linear" means linear in y and y'; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations. The trick is to rewrite the left hand side of (2.4.1) as a derivative of a product of y with another function. To this end we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}[r(x)y]$$

This is the left hand side of (2.4.1) multiplied by $r(x)$. So if we multiply (2.4.1) by $r(x)$, we obtain

$$\frac{d}{dx}[r(x)y] = r(x)f(x)$$

Now we integrate both sides. The right hand side does not depend on y and the left hand side is written as a derivative of a function. Afterwards, we solve for y. The function $r(x)$ is called the integrating factor and the method is called the integrating factor method.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)dx}$$

We compute:

$$\begin{aligned} y' + p(x)y &= f(x), \\ e^{\int p(x)dx} y' + e^{\int p(x)dx} p(x)y &= e^{\int p(x)dx} f(x), \\ \frac{d}{dx}\left[e^{\int p(x)dx} y\right] &= e^{\int p(x)dx} f(x), \\ e^{\int p(x)dx} y &= \int e^{\int p(x)dx} f(x)dx + C, \\ y &= e^{-\int p(x)dx} \left(\int e^{\int p(x)dx} f(x)dx + C\right). \end{aligned} \tag{2.4.2}$$

Of course, to get a closed form formula for y, we need to be able to find a closed form formula for the integrals appearing above.

Below is a video on solving a differential equation using an integrating factor.

Example 2.4.1

Solve

$$y' + 2xy = e^{x-x^2}, \quad y(0) = -1$$

Solution

First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$ to get

$$\begin{aligned} e^{x^2} y' + 2xe^{x^2} y &= e^{x-x^2} e^{x^2}, \\ \frac{d}{dx}\left[e^{x^2} y\right] &= e^x. \end{aligned} \tag{2.4.3}$$

We integrate

$$\begin{aligned} e^{x^2} y &= e^x + C, \\ y &= e^{x-x^2} + Ce^{-x^2}. \end{aligned} \tag{2.4.4}$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

Exercise 2.4.1

Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above. An advice: Do not try to remember the formula itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

. Look at the solution and write the integrals as definite integrals.

$$y(x) = e^{-\int_{x_0}^{x} p(s)ds} \left(\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)ds} f(t)dt + y_0 \right) \tag{2.4.5}$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

Exercise 2.4.2

Check that $y(x_0) = y_0$ in formula (2.4.5).

Exercise 2.4.3

Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.

$$y' + y = e^{x^2 - x}, \quad y(0) = 10$$

Note

Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is always an explicit formula (2.4.5) for the solution. Second, it follows from the formula (2.4.5) that if $p(x)$ and $f(x)$ are continuous on some interval (a, b), then the solution $y(x)$ exists and is differentiable on (a, b). Compare with the simple nonlinear example we have seen previously, $y' = y^2$, and compare to Theorem 1.2.1.

Example 2.4.2

Let us discuss a common simple application of linear equations. This type of problem is used often in real life. For example, linear equations are used in figuring out the concentration of chemicals in bodies of water (rivers and lakes).

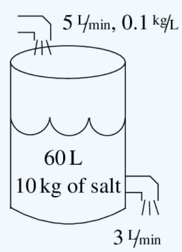


Figure 2.4.1

A 100 liter tank contains 10 kilograms of salt dissolved in 60 liters of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per liter is flowing in at the rate of 5 liters a minute. The solution in the tank is well stirred and flows out at a rate of 3 liters a minute. How much salt is in the tank when the tank is full?

Solution

Let us come up with the equation. Let x denote the kilograms of salt in the tank, let t denote the time in minutes. For a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in x concentration in})\Delta t - (\text{rate out x concentration out})\Delta t.$$

Dividing through by Δt and taking the limit $\Delta t \to 0$ we see that

$$\frac{dx}{dt} = (\text{rate in x concentration in}) - (\text{rate out x concentration out})$$

In our example, we have

$$\begin{aligned} \text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5-3)t}. \end{aligned} \tag{2.4.6}$$

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \text{ x } 0.1) - \left(3 \frac{x}{60+2t}\right)$$

Or in the form (2.4.1)

$$\frac{dx}{dt} + \frac{3}{60+2t} x = 0.5$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60+2t} dt\right) = \exp\left(\frac{3}{2} \ln(60+2t)\right) = (60+2t)^{3/2}$$

We multiply both sides of the equation to get

$$\begin{aligned}
(60+2t)^{3/2} \frac{dx}{dt} + (60+2t)^{3/2} \frac{3}{60+2t} x &= 0.5(60+2t)^{3/2}, \\
\frac{d}{dt}\left[(60+2t)^{3/2} x\right] &= 0.5(60+2t)^{3/2}, \\
(60+2t)^{3/2} x &= \int 0.5(60+2t)^{3/2} dt + C, \\
x &= (60+2t)^{-3/2} \int \frac{(60+2t)^{3/2}}{2} dt + C(60+2t)^{-3/2}, \\
x &= (60+2t)^{-3/2} \frac{1}{10} (60+2t)^{5/2} + C(60+2t)^{-3/2}, \\
x &= \frac{(60+2t)}{10} + C(60+2t)^{-3/2}.
\end{aligned} \tag{2.4.7}$$

We need to find C. We know that at $t=0$, $x=10$. So

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2}$$

or

$$C = 4(60^{3/2}) \approx 1859.03.$$

We are interested in x when the tank is full. So we note that the tank is full when $60+2t=100$, or when $t=20$. So

$$\begin{aligned}
x(20) &= \frac{60+40}{10} + C(60+40)^{-3/2} \\
&\approx 10 + 1859.03(100)^{-3/2} \approx 11.86.
\end{aligned} \tag{2.4.8}$$

See Figure $2.4.2$ for the graph of x over t.

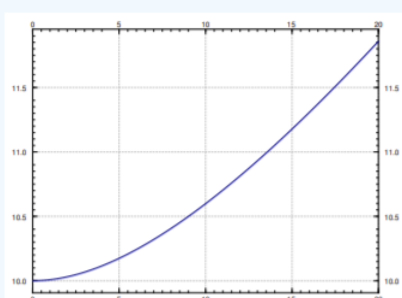


Figure $2.4.2$: Graph of the solution x kilograms of salt in the tank at time t.

The concentration at the end is approximately $0.1186\,{}^{\text{kg}}/_{\text{liter}}$ and we started with $\frac{1}{6}$ or $0.167\,{}^{\text{kg}}/_{\text{liter}}$.

Playlist of Videos on Integrating Factor

2.5: Existence and Uniqueness of Solutions of Nonlinear Equations

Although there are methods for solving some nonlinear equations, it is impossible to find useful formulas for the solutions of most. Whether we are looking for exact solutions or numerical approximations, it is useful to know conditions that imply the existence and uniqueness of solutions of initial value problems for nonlinear equations. In this section we state such a condition and illustrate it with examples.

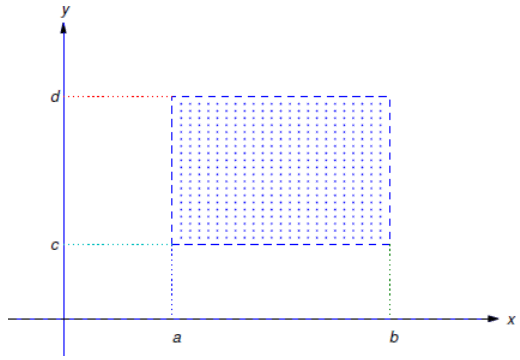


Figure 2.5.1 : An open rectangle

Some terminology: an *open rectangle* R is a set of points (x, y) such that

$$a < x < b \quad \text{and} \quad c < y < d$$

(Figure 2.5.1). We'll denote this set by $R : \{a < x < b, c < y < d\}$. "Open" means that the boundary rectangle (indicated by the dashed lines in Figure 2.5.1) is not included in R.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first order nonlinear differential equations. We omit the proof, which is beyond the scope of this book.

Theorem 2.5.1 : existence and uniqueness

a. If f is continuous on an open rectangle

$$R : \{a < x < b, c < y < d\}$$

that contains (x_0, y_0) then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.5.1}$$

has at least one solution on some open subinterval of (a, b) that contains x_0.

b. If both f and f_y are continuous on R then Equation 2.5.1 has a unique solution on some open subinterval of (a, b) that contains x_0

It's important to understand exactly what Theorem 2.5.1 says.

- (a) is an *existence theorem*. It guarantees that a solution exists on some open interval that contains x_0, but provides no information on how to find the solution, or to determine the open interval on which it exists. Moreover, (a) provides no information on the number of solutions that Equation 2.5.1 may have. It leaves open the possibility that Equation 2.5.1 may have two or more solutions that differ for values of x arbitrarily close to x_0. We will see in Example 2.5.6 that this can happen.
- (b) is a *uniqueness theorem*. It guarantees that Equation 2.5.1 has a unique solution on some open interval (a,b) that contains x_0. However, if $(a, b) \neq (-\infty, \infty)$, Equation 2.5.1 may have more than one solution on a larger interval that contains (a, b). For example, it may happen that $b < \infty$ and all solutions have the same values on (a, b), but two solutions y_1 and y_2 are defined on some interval (a, b_1) with $b_1 > b$, and have different values for $b < x < b_1$; thus, the graphs of the y_1 and y_2 "branch off" in different directions at $x = b$. (See Example 2.5.7 and Figure 2.5.3). In this case, continuity implies that $y_1(b) = y_2(b)$ (call their common value y), and y_1 and y_2 are both solutions of the initial value problem

$$y = f(x, y), \quad y(b) = \bar{y} \tag{2.5.2}$$

that differ on every open interval that contains b. Therefore f or f_y must have a discontinuity at some point in each open rectangle that contains (b, y), since if this were not so, 2.5.2 would have a unique solution on some open interval that contains b. We leave it to you to give a similar analysis of the case where $a > -\infty$.

Example 2.5.1

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0. \tag{2.5.3}$$

Since

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}$$

are continuous for all (x, y), Theorem 2.5.1 implies that if (x_0, y_0) is arbitrary, then Equation 2.5.3 has a unique solution on some open interval that contains x_0.

Example 2.5.2

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0. \tag{2.5.4}$$

Here

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{4x^2 y}{(x^2 + y^2)^2}$$

are continuous everywhere except at $(0, 0)$. If $(x_0, y_0) \neq (0, 0)$, there's an open rectangle R that contains (x_0, y_0) that does not contain $(0, 0)$. Since f and f_y are continuous on R, Theorem 2.5.1 implies that if $(x_0, y_0) \neq (0, 0)$ then Equation 2.5.4 has a unique solution on some open interval that contains x_0.

Example 2.5.3

Consider the initial value problem

$$y' = \frac{x + y}{x - y}, \quad y(x_0) = y_0. \tag{2.5.5}$$

Here

$$f(x, y) = \frac{x + y}{x - y} \quad \text{and} \quad f_y(x, y) = \frac{2x}{(x - y)^2}$$

are continuous everywhere except on the line $y = x$. If $y_0 \neq x_0$, there's an open rectangle R that contains (x_0, y_0) that does not intersect the line $y = x$. Since f and f_y are continuous on R, Theorem 2.5.1 implies that if $y_0 \neq x_0$, Equation 2.5.5 has a unique solution on some open interval that contains x_0.

Example 2.5.4

In Example 2.2.4, we saw that the solutions of

$$y' = 2xy^2 \tag{2.5.6}$$

are

$$y \equiv 0 \quad \text{and} \quad y = -\frac{1}{x^2+c},$$

where c is an arbitrary constant. In particular, this implies that no solution of Equation 2.5.6 other than $y \equiv 0$ can equal zero for any value of x. Show that Theorem $2.5.1b$ implies this.

We'll obtain a contradiction by assuming that Equation 2.5.6 has a solution y_1 that equals zero for some value of x, but is not identically zero. If y_1 has this property, there's a point x_0 such that $y_1(x_0) = 0$, but $y_1(x) \neq 0$ for some value of x in every open interval that contains x_0. This means that the initial value problem

$$y' = 2xy^2, \quad y(x_0) = 0 \tag{2.5.7}$$

has two solutions $y \equiv 0$ and $y = y_1$ that differ for some value of x on every open interval that contains x_0. This contradicts Theorem 2.5.1 (b), since in Equation 2.5.6 the functions

$$f(x,y) = 2xy^2 \quad \text{and} \quad f_y(x,y) = 4xy.$$

are both continuous for all (x, y), which implies that Equation 2.5.7 has a unique solution on some open interval that contains x_0.

Below is a video on finding values where there is no guarantee of existence and uniqueness of a solution to a differential equation.

Example 2.5.5

Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \tag{2.5.8}$$

a. For what points (x_0, y_0) does Theorem $2.5.1a$ imply that Equation 2.5.8 has a solution?
b. For what points (x_0, y_0) does Theorem $2.5.1b$ imply that Equation 2.5.8 has a unique solution on some open interval that contains x_0?

Solution a

Since

$$f(x,y) = \frac{10}{3}xy^{2/5}$$

is continuous for all (x, y), Theorem 2.5.1 implies that Equation 2.5.8 has a solution for every (x_0, y_0).

Solution b

Here

$$f_y(x,y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all (x,y) with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both f and f_y are continuous, and Theorem 2.5.1 implies that Equation 2.5.8 has a unique solution on some open interval that contains x_0.

If $y = 0$ then $f_y(x,y)$ is undefined, and therefore discontinuous; hence, Theorem 2.5.1 does not apply to Equation 2.5.8 if $y_0 = 0$.

Example 2.5.6

Example 2.5.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = 0 \tag{2.5.9}$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

Solution

By inspection, $y \equiv 0$ is a solution of the differential equation

$$y' = \frac{10}{3} x y^{2/5}. \tag{2.5.10}$$

Since $y \equiv 0$ satisfies the initial condition $y(0) = 0$, it is a solution of Equation 2.5.9.

Now suppose y is a solution of Equation 2.5.10 that is not identically zero. Separating variables in Equation 2.5.10 yields

$$y^{-2/5} y' = \frac{10}{3} x$$

on any open interval where y has no zeros. Integrating this and rewriting the arbitrary constant as $5c/3$ yields

$$\frac{5}{3} y^{3/5} = \frac{5}{3}(x^2 + c).$$

Therefore

$$y = (x^2 + c)^{5/3}. \tag{2.5.11}$$

Since we divided by y to separate variables in Equation 2.5.10, our derivation of Equation 2.5.11 is legitimate only on open intervals where y has no zeros. However, Equation 2.5.11 actually defines y for all x, and differentiating Equation 2.5.11 shows that

$$y' = \frac{10}{3} x (x^2 + c)^{2/3} = \frac{10}{3} x y^{2/5}, \; -\infty < x < \infty$$

Therefore Equation 2.5.11 satisfies Equation 2.5.10 on $(-\infty, \infty)$ even if $c \leq 0$, so that $y(\sqrt{|c|}) = y(-\sqrt{|c|}) = 0$. In particular, taking $c = 0$ in Equation 2.5.11 yields

$$y = x^{10/3}$$

as a second solution of Equation 2.5.9. Both solutions are defined on $(-\infty, \infty)$, and they differ on every open interval that contains $x_0 = 0$ (Figure 2.5.2). In fact, there are *four* distinct solutions of Equation 2.5.9 defined on $(-\infty, \infty)$ that differ from each other on every open interval that contains $x_0 = 0$. Can you identify the other two?

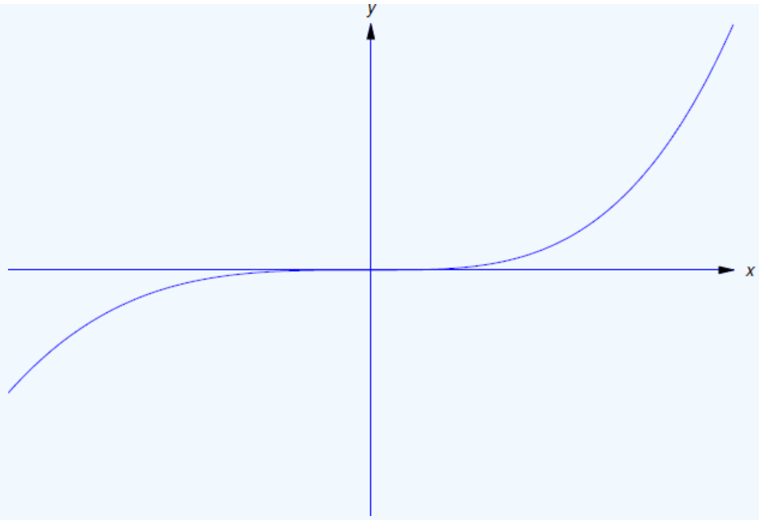


Figure 2.5.2 : Two solutions ($y = 0$ and $y = x^{1/2}$) of Equation 2.5.9 that differ on every interval containing $x_0 = 0$

Example 2.5.7

From Example 2.5.5 , the initial value problem

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = -1 \tag{2.5.12}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval (a, b) on which it is unique.

Solution

Let y be any solution of Equation 2.5.12. Because of the initial condition $y(0) = -1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation 2.5.11. Setting $x = 0$ and $y = -1$ in Equation 2.5.11 yields $c = -1$, so

$$y = (x^2 - 1)^{5/3} \tag{2.5.13}$$

for x in I. Therefore every solution of Equation 2.5.12 differs from zero and is given by Equation 2.5.13 on $(-1, 1)$; that is, Equation 2.5.13 is the unique solution of Equation 2.5.12 on $(-1, 1)$. This is the largest open interval on which Equation 2.5.12 has a unique solution. To see this, note that Equation 2.5.13 is a solution of Equation 2.5.12 on $(-\infty, \infty)$. From *Exercise 2.2.15*, there are infinitely many other solutions of Equation 2.5.12 that differ from Equation 2.5.13 on every open interval larger than $(-1, 1)$. One such solution is

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 \le x \le 1, \\ 0, & |x| > 1. \end{cases}$$

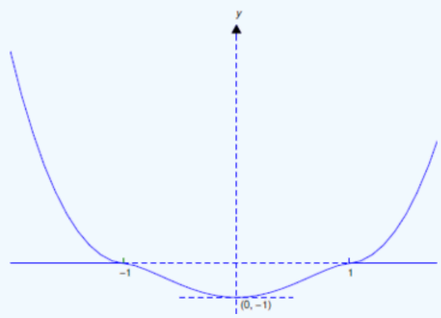


Figure 2.5.3 : Two solutions of Equation 2.5.12 on (−1,1) that coincide on (−1, 1), but on no larger open interval. (right)

Example 2.5.8

From Example 2.5.5), the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 1 \tag{2.5.14}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

Solution

Let y be any solution of Equation 2.5.14. Because of the initial condition $y(0) = 1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation 2.5.11. Setting $x = 0$ and $y = 1$ in Equation 2.5.11 yields $c = 1$, so

$$y = (x^2 + 1)^{5/3} \tag{2.5.15}$$

for x in I. Therefore every solution of Equation 2.5.14 differs from zero and is given by Equation 2.5.15 on $(-\infty, \infty)$; that is, Equation 2.5.15 is the unique solution of Equation 2.5.14 on $(-\infty, \infty)$. Figure 2.5.4) shows the graph of this solution.

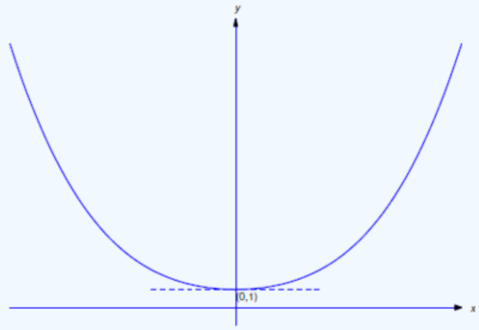


Figure 2.5.4 : The unique solution of Equation 2.5.14.

2.5E: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises)

Q2.3.1

In *Exercises 2.3.1-2.3.13*, find all (x_0, y_0) for which Theorem 2.3.1 implies that the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ has (a) a solution and (b) a unique solution on some open interval that contains x_0.

1. $y' = \frac{x^2+y^2}{\sin x}$
2. $y' = \frac{e^x+y}{x^2+y^2}$
3. $y' = \tan xy$
4. $y' = \frac{x^2+y^2}{\ln xy}$
5. $y' = (x^2+y^2)y^{1/3}$
6. $y' = 2xy$
7. $y' = \ln(1+x^2+y^2)$
8. $y' = \frac{2x+3y}{x-4y}$
9. $y' = (x^2+y^2)^{1/2}$
10. $y' = x(y^2-1)^{2/3}$
11. $y' = (x^2+y^2)^2$
12. $y' = (x+y)^{1/2}$
13. $y' = \frac{\tan y}{x-1}$

Q2.3.2

14. Apply Theorem 2.3.1 to the initial value problem

$$y' + p(x)y = q(x), \quad y(x_0) = y_0 \tag{2.5E.1}$$

for a linear equation, and compare the conclusions that can be drawn from it to those that follow from Theorem 2.1.2.

15.

a. Verify that the function

$$y = \begin{cases} (x^2-1)^{5/3}, & -1 < x < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{2.5E.2}$$

is a solution of the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1 \tag{2.5E.3}$$

on $(-\infty, \infty)$. HINT: *You'll need the definition*

$$y'(\bar{x}) = \lim_{x \to \bar{x}} \frac{y(x) - y(\bar{x})}{x - \bar{x}} \tag{2.5E.4}$$

to verify that y satisfies the differential equation at $\bar{x} = \pm 1$.

b. Verify that if $\epsilon_i = 0$ or 1 for $i = 1, 2$ and $a, b > 1$, then the function

$$y = \begin{cases} \epsilon_1(x^2-a^2)^{5/3}, & -\infty < x < -a, \\ 0, & -a \le x \le -1, \\ (x^2-1)^{5/3}, & -1 < x < 1, \\ 0, & 1 \le x \le b, \\ \epsilon_2(x^2-b^2)^{5/3}, & b < x < \infty, \end{cases} \tag{2.5E.5}$$

is a solution of the initial value problem of a on $(-\infty, \infty)$.

16. Use the ideas developed in *Exercise 2.3.15* to find infinitely many solutions of the initial value problem

$$y' = y^{2/5}, \quad y(0) = 1 \tag{2.5E.6}$$

on $(-\infty, \infty)$.

17. Consider the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(x_0) = y_0. \tag{A}$$

a. For what points (x_0, y_0) does Theorem 2.3.1 imply that (A) has a solution?
b. For what points (x_0, y_0) does Theorem 2.3.1 imply that (A) has a unique solution on some open interval that contains x_0?

18. Find nine solutions of the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(0) = 1 \tag{2.5E.7}$$

that are all defined on $(-\infty, \infty)$ and differ from each other for values of x in every open interval that contains $x_0 = 0$.

19. From Theorem 2.3.1, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(0) = 9 \tag{2.5E.8}$$

has a unique solution on an open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

20.

a. From Theorem 2.3.1, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(3) = -7 \tag{A}$$

has a unique solution on some open interval that contains $x_0 = 3$. Determine the largest such open interval, and find the solution on this interval.
b. Find infinitely many solutions of (A), all defined on $(-\infty, \infty)$.

21. Prove:

a. If

$$f(x, y_0) = 0, \quad a < x < b, \tag{A}$$

andx_0 is in (a, b), then $y \equiv y_0$ is a solution of

$$y' = f(x, y), \quad y(x_0) = y_0$$

on (a, b).
b. If f and f_y are continuous on an open rectangle that contains (x_0, y_0) and (A) holds, no solution of $y' = f(x, y)$ other than $y \equiv y_0$ can equal y_0 at any point in (a, b).

2.6: Substitution

Just as when solving integrals, one method to try is to change variables to end up with a simpler equation to solve.

2.6.1: Substitution

The equation

$$y' = (x - y + 1)^2 \tag{2.6.1}$$

is neither separable nor linear. What can we do? How about trying to change variables, so that in the new variables the equation is simpler. We use another variable v, which we treat as a function of x. Let us try

$$v = x - y + 1. \tag{2.6.2}$$

We need to figure out y' in terms of v', v and x. We differentiate (in x) to obtain $v' = 1 - y'$. So $y' = 1 - v'$. We plug this into the equation to get

$$1 - v' = v^2 \tag{2.6.3}$$

In other words, $v' = 1 - v^2$. Such an equation we know how to solve by separating variables:

$$\frac{1}{1 - v^2} dv = dx \tag{2.6.4}$$

So

$$\frac{1}{2} \ln \left| \frac{v+1}{v-1} \right| = x + C, \quad \text{or} \quad \left| \frac{v+1}{v-1} \right| = e^{2x+2C}, \quad \text{or} \quad \frac{v+1}{v-1} = De^{2x},$$

for some constant D. Note that $v = 1$ and $v = -1$ are also solutions.

Now we need to "unsubstitute" to obtain

$$\frac{x - y + 2}{x - y} = De^{2x} \tag{2.6.5}$$

and also the two solutions $x - y + 1 = 1$ or $y = x$, and $x - y + 1 = -1$ or $y = x + 2$. We solve the first equation for y.

$$\begin{aligned} x - y + 2 &= (x - y)\, De^{2x}, \\ x - y + 2 &= Dxe^{2x} - yDe^{2x}, \\ -y + yDe^{2x} &= Dxe^{2x} - x - 2, \\ y(-1 + De^{2x}) &= Dxe^{2x} - x - 2, \\ y &= \frac{Dxe^{2x} - x - 2}{De^{2x} - 1}. \end{aligned} \tag{2.6.6}$$

Note that $D = 0$ gives $y = x + 2$, but no value of D gives the solution $y = x$.

Substitution in differential equations is applied in much the same way that it is applied in calculus. You guess. Several different substitutions might work. There are some general things to look for. We summarize a few of these in a table.

When you see	Try substituting
yy'	$v = y^2$
$y^2 y'$	$v = y^3$
$(\cos y) y'$	$v = \sin y$
$(\sin y) y'$	$v = \cos y$
$y' e^y$	$v = e^y$

Usually you try to substitute in the "most complicated" part of the equation with the hopes of simplifying it. The above table is just a rule of thumb. You might have to modify your guesses. If a substitution does not work (it does not make the equation any simpler), try a different one.

2.6.2: Bernoulli Equations

There are some forms of equations where there is a general rule for substitution that always works. One such example is the so-called Bernoulli equation.[1]

$$y' + p(x)y = q(x)y^n \tag{2.6.7}$$

$$y' + p(x)\,y = q(x)\,y^n \tag{2.6.8}$$

This equation looks a lot like a linear equation except for the y^n. If $n = 0$ or $n = 1$, then the equation is linear and we can solve it. Otherwise, the substitution $v = y^{1-n}$ transforms the Bernoulli equation into a linear equation. Note that n need not be an integer.

Example 1.5.1: Bernoulli Equation

Solve

$$xy' + y(x+1) + xy^5 = 0, \;\; y(1) = 1$$

Solution

First, the equation is Bernoulli $p(x) = \dfrac{x+1}{x}$ (and $q(x) = -1$). We substitute

$$v = y^{1-5} = y^{-4}, \quad v' = -4y^{-5}y'$$

In other words, $\left(\dfrac{-1}{4}\right) y^5 v' = y'$. So

$$\begin{aligned} xy' + y(x+1) + xy^5 &= 0, \\ \frac{-xy^5}{4}v' + y(x+1) + xy^5 &= 0, \\ \frac{-x}{4}v' + y^{-4}(x+1) + x &= 0, \\ \frac{-x}{4}v' + v(x+1) + x &= 0, \end{aligned} \tag{2.6.9}$$

and finally

$$v' - \frac{4(x+1)}{x}v = 4$$

Now the equation is linear. We can use the integrating factor method. In particular, we use formula (1.4.17). Let us assume that $x > 0$ so $|x| = x$. This assumption is OK, as our initial condition is $x = 1$. Let us compute the integrating factor. Here $p(s)$ from formula (1.4.17) is $\dfrac{-4(s+1)}{s}$.

$$\begin{aligned} e^{\int_1^x p(s)ds} &= \exp\left(\int_1^x \frac{-4(s+1)}{s} ds\right) = e^{-4x - 4ln(x) + 4} = e^{-4x+4}x^{-4} = \frac{e^{-4x+4}}{x^4}, \\ e^{-\int_1^x p(s)ds} &= e^{4x + 4ln(x) - 4} = e^{4x-4}x^4 \end{aligned} \tag{2.6.10}$$

We now plug in to (1.4.17)

$$\begin{aligned} v(x) &= e^{-\int_1^x p(s)ds}\left(\int_1^x e^{\int_1^t p(s)ds} 4dt + 1\right), \\ &= e^{4x-4}x^4\left(\int_1^x 4\frac{e^{-4t+4}}{t^4} dt + 1\right) \end{aligned} \tag{2.6.11}$$

Note that the integral in this expression is not possible to find in closed form. As we said before, it is perfectly fine to have a definite integral in our solution. Now “unsubstitute”

$$\begin{aligned} y^{-4} &= e^{4x-4}x^4\left(4\int_1^x \frac{e^{-4t+4}}{t^4}dt+1\right), \\ y &= \frac{e^{-x+1}}{x\left(4\int_1^x \frac{e^{-4t+4}}{t^4}dt+1\right)^{1/4}} \end{aligned} \tag{2.6.12}$$

2.6.3: Homogeneous Equations

Another type of equations we can solve by substitution are the so-called homogeneous equations. Suppose that we can write the differential equation as

$$y' = F\left(\frac{y}{x}\right) \tag{2.6.13}$$

Here we try the substitutions

$$v = \frac{y}{x} \quad \text{and therefore} \quad y' = v + xv' \tag{2.6.14}$$

We note that the equation is transformed into

$$v + xv' = F(v) \quad \text{or} \quad xv' = F(v) - v \quad \text{or} \quad \frac{v'}{F(v)-v} = \frac{1}{x} \tag{2.6.15}$$

Hence an implicit solution is

$$\int \frac{1}{F(v)-v}dv = \ln|x| + C \tag{2.6.16}$$

Example 1.5.2

Solve

$$x^2y' = y^2 + xy, \quad y(1) = 1$$

Solution

We put the equation into the form $y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$. We substitute $v = \dfrac{y}{x}$ to get the separable equation

$$xv' = v^2 + v - v = v^2$$

which has a solution

$$\begin{aligned} \int \frac{1}{v^2}dv &= ln\,|x| + C, \\ \frac{-1}{v} &= ln\,|x| + C, \\ v &= \frac{-1}{ln\,|x| + C}. \end{aligned} \tag{2.6.17}$$

We unsubstitute

$$\begin{aligned} \frac{y}{x} &= \frac{-1}{ln\,|x| + C}, \\ y &= \frac{-x}{ln\,|x| + C} \end{aligned} \tag{2.6.18}$$

We want $y(1) = 1$, so

$$1 = y(1) = \frac{-1}{ln\,|1| + C} = \frac{-1}{C}$$

Thus $C = -1$ and the solution we are looking for is

$$y = \frac{-x}{ln\,|x| - 1}$$

2.6.4: Footnotes

[1] There are several things called Bernoulli equations, this is just one of them. The Bernoullis were a prominent Swiss family of mathematicians. These particular equations are named for Jacob Bernoulli (1654–1705).

2.6.5: Contributors and Attributions

- ◦ Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

2.7: Autonomous equations

Let us consider general differential equation problems of the form

$$\frac{dx}{dt} = f(x)$$

where the derivative of solutions depends only on x (the dependent variable). Such equations are called **autonomous** equations. If we think of t as time, the naming comes from the fact that the equation is independent of time.

Let us come back to the cooling coffee problem (see Example 1.3.3). Newton's law of cooling says that

$$\frac{dx}{dt} = -k(x - A)$$

where x is the temperature, t is time, k is some constant, and A is the ambient temperature. See Figure $2.7.1$ for an example with $k = 0.3$ and $A = 5$.

Note the solution $x = A$ (in the figure $x = 5$). We call these constant solutions the equilibrium solutions. The points on the x axis where $f(X) = 0$ are called critical points. The point $x = A$ is a critical point. In fact, each critical point corresponds to an equilibrium solution. Note also, by looking at the graph, that the solution $x = A$ is "stable" in that small perturbations in x do not lead to substantially different solutions as t grows. If we change the initial condition a little bit, then as $t \to \infty$ we get $x \to A$. We call such critical points stable. In this simple example it turns out that all solutions in fact go to A as $t \to \infty$. If a critical point is not stable we would say it is unstable.

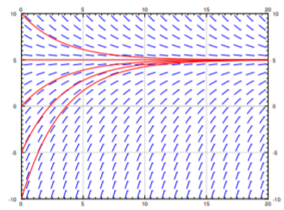


Figure $2.7.1$: The slope field and some solutions of $x' = 0.3(5 - x)$.

Let us consider the logistic equation

$$\frac{dx}{dt} = kx(M - x)$$

for some positive k and M. This equation is commonly used to model population if we know the limiting population M, that is the maximum sustainable population. The logistic equation leads to less catastrophic predictions on world population than $x' = kx$. In the real world there is no such thing as negative population, but we will still consider negative x for the purposes of the math (see Figure $2.7.2$ for an example).

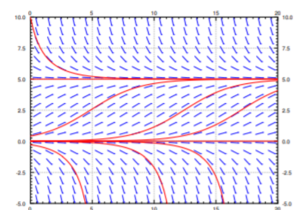


Figure $2.7.2$: The slope field and some solutions of $x' = 0.1x(5 - x)$.

Note two critical points, $x = 0$ and $x = 5$. The critical point at $x = 5$. is stable. On the other hand the critical point at $x = 0$. is unstable.

It is not really necessary to find the exact solutions to talk about the long term behavior of the solutions. For example, from the above slope field plot, we can easily see that

$$\lim_{t\to\infty} x(t) = \begin{cases} 5 & \text{if } x(0) > 0, \\ 0 & \text{if } x(0) = 0, \\ \text{DNE or } -\infty & \text{if } x(0) < 0. \end{cases}$$

Where DNE means "does not exist." From just looking at the slope field we cannot quite decide what happens if $x(0) < 0$. It could be that the solution does not exist for t all the way to ∞. Think of the equation $x' = x^2$, we have seen that it only exists for some finite period of time. Same can happen here. In our example equation above it will actually turn out that the solution does not exist for all time, but to see that we would have to solve the equation. In any case, the solution does go to $-\infty$, but it may get there rather quickly.

If we are interested only in the long term behavior of the solution, we would be doing unnecessary work if we solved the equation exactly. We could draw the slope field, but it is easier to just look at the or , which is a simple way to visualize the behavior of autonomous equations. In this case there is one dependent variable x. We draw the x-axis, we mark all the critical points, and then we draw arrows in between. Since x is the dependent variable we draw the axis vertically, as it appears in the slope field diagrams above. If $f(x) > 0$, we draw an up arrow. If $f(x) < 0$, we draw a down arrow. To figure this out, we could just plug in some x between the critical points, $f(x)$ will have the same sign at all x between two critical points as long $f(x)$ is continuous. For example, $f(6) = -0.6 < 0$, so $f(x) < 0$ for $x > 5$, and the arrow above $x = 5$ is a down arrow. Next, $f(1) = 0.4 > 0$, so $f(x) > 0$ whenever $0 < x < 5$, and the arrow points up. Finally, $f(-1) = -0.6 < 0$ so $f(x) < 0$ when $x < 0$, and the arrow points down.

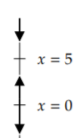


Figure $2.7.3$

Armed with the phase diagram, it is easy to sketch the solutions approximately: As time t moves from left to right, the graph of a solution goes up if the arrow is up, and it goes down if the arrow is down.

Below is a video on solving an autonomous differential equation that describes logistic growth.

❓ Exercise 2.7.1

Try sketching a few solutions simply from looking at the phase diagram. Check with the preceding graphs if you are getting the type of curves.

Once we draw the phase diagram, we can easily classify critical points as stable or unstable.[1]

Figure 2.7.4

Since any mathematical model we cook up will only be an approximation to the real world, unstable points are generally bad news.

Let us think about the logistic equation with harvesting. Suppose an alien race really likes to eat humans. They keep a planet with humans on it and harvest the humans at a rate of h million humans per year. Suppose x is the number of humans in millions on the planet and t is time in years. Let M be the limiting population when no harvesting is done and $k > 0$ is some constant depending on how fast humans multiply. Our equation becomes

$$\frac{dx}{dt} = kx(M - x) - h$$

We expand the right hand side and solve for critical points

$$\frac{dx}{dt} = -kx^2 + kMx - h$$

Solving for the critical points A and B from the quadratic equations:

$$A = \frac{kM + \sqrt{(kM)^2 - 4hk}}{2k}, \quad B = \frac{kM - \sqrt{(kM)^2 - 4hk}}{2k}$$

Below is a video on solving an autonomous initial value problem.

❓ Exercise 2.7.2

Sketch a phase diagram for different possibilities. Note that these possibilities are $A > B$, or $A = B$, or A and B both complex (i.e. no real solutions). Hint: Fix some simple k and M and then vary h.

For example, let $M = 8$ and $k = 0.1$. When $h = 1$, then A and B are distinct and positive. The slope field we get is in Figure 2.7.5. As long as the population starts above B, which is approximately 1.55 million, then the population will not die out. It will in fact tend towards $A \approx 6.45$ million. If ever some catastrophe happens and the population drops below B, humans will die out, and the fast food restaurant serving them will go out of business.

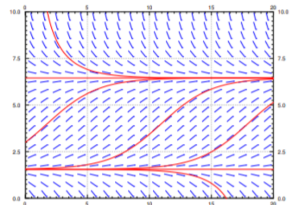


Figure 2.7.5: Slope field and some solutions of $x' = 0.1x(8 - x) - 1$.

When $h = 1.6$, then $A = B = 4$ and there is only **one** critical point and it is **unstable**. When the population starts above 4 million it will tend towards 4 million. If it ever drops below 4 million, humans will die out on the planet. This scenario is not one that we (as the human fast food proprietor) want to be in. A small perturbation of the equilibrium state and we are out of business; there is no room for error (see Figure 2.7.6).

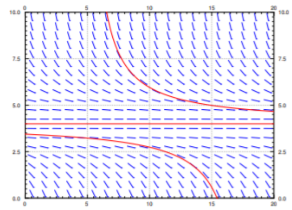


Figure 2.7.6: The slope field and some solutions of $x' = 0.1x(8 - x) - 1.6$.

Finally if we are harvesting at 2 million humans per year, there are no critical points. The population will always plummet towards zero, no matter how well stocked the planet starts (see Figure 2.7.7).

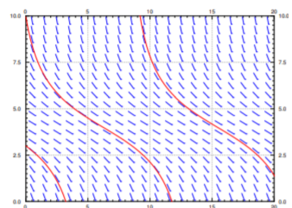


Figure 2.7.7: Slope field and some solutions of $x' = 0.1x(8 - x) - 2$.

2.7.1: Footnotes

[1] Unstable points with one of the arrows pointing towards the critical point are sometimes called semistable.

2.7.2: References

1. Paul W. Berg and James L. McGregor, Elementary Partial Differential Equations, Holden-Day, San Francisco, CA, 1966.
2. William E. Boyce, Richard C. DiPrima, Elementary Differential Equations and Boundary Value Problems, 9th edition, John Wiley & Sons Inc., New York, NY, 2008.
3. C.H. Edwards and D.E. Penney, Differential Equations and Boundary Value Problems: Computing and Modeling, 4th edition, Prentice Hall, 2008.
4. Stanley J. Farlow, An Introduction to Differential Equations and Their Applications, McGraw-Hill, Inc., Princeton, NJ, 1994. (Published also by Dover Publications, 2006.)
5. E.L. Ince, Ordinary Differential Equations, Dover Publications, Inc., New York, NY, 1956.

2.8: Numerical methods- Euler's method

At this point it may be good to first try the Lab II and/or Project II from the IODE website: www.math.uiuc.edu/iode/. As we said before, unless $f(x, y)$ is of a special form, it is generally very hard if not impossible to get a nice formula for the solution of the problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

If the equation can be solved in closed form, we should do that. But what if we have an equation that cannot be solved in closed form? What if we want to find the value of the solution at some particular x? Or perhaps we want to produce a graph of the solution to inspect the behavior. In this section we will learn about the basics of numerical approximation of solutions.

The simplest method for approximating a solution is Euler's Method.[1] It works as follows: Take x_0 and compute the slope $k = f(x_0, y_0)$. The slope is the change in y per unit change in x. Follow the line for an interval of length h on the x-axis. Hence if $y = y_0$ at x_0, then we say that y_1 (the approximate value of y at $x_1 = x_0 + h$) is $y_1 = y_0 + hk$. Rinse, repeat! Let $k = f(x_1, y_1)$, and then compute $x_2 = x_1 + h$, and $y_2 = y_1 + hk$. Now compute x_3 and y_3 using x_2 and y_2, etc. Consider the equation $y' = \frac{y^2}{3}$, $y(0) = 1$, and $h = 1$. Then $x_0 = 0$ and $y_0 = 1$. We compute

$$\begin{aligned} x_1 &= x_0 + h = 0 + 1 = 1, & y_1 &= y_0 + h\, f(x_0, y_0) = 1 + 1 \cdot \frac{1}{3} = \frac{4}{3} \approx 1.333, \\ x_2 &= x_1 + h = 1 + 1 = 2, & y_2 &= y_1 + h\, f(x_1, y_1) = \frac{4}{3} + 1 \cdot \frac{\left(\frac{4}{3}\right)^2}{3} = \frac{52}{27} \approx 1.926. \end{aligned} \tag{2.8.1}$$

We then draw an approximate graph of the solution by connecting the points (x_0, y_0), (x_1, y_1), (x_2, y_2),…. For the first two steps of the method see Figure $2.8.1$.

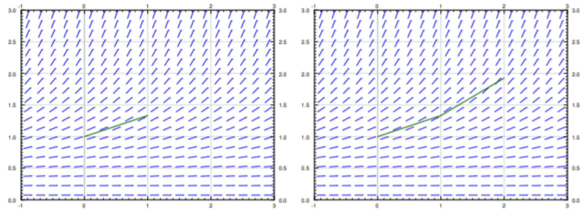


Figure $2.8.1$: First two steps of Euler's method with $h = 1$ for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

More abstractly, for any $i = 0, 1, 2, 3, \ldots$, we compute

$$x_{i+1} = x_i + h, \qquad y_{i+1} = y_i + h\, f(x_i, y_i).$$

The line segments we get are an approximate graph of the solution. Generally it is not exactly the solution. See Figure $2.8.2$ for the plot of the real solution and the approximation.

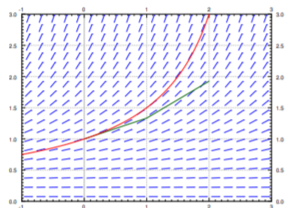


Figure $2.8.2$: Two steps of Euler's method (step size 1) and the exact solution for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

We continue with the equation $y' = \frac{y^2}{3}$, $y(0) = 1$. Let us try to approximate $y(2)$ using Euler's method. In Figures $2.8.1$ and $2.8.2$ we have graphically approximated $y(2)$ with step size 1. With step size 1, we have $y(2) \approx 1.926$. The real answer is 3. We are approximately 1.074 off. Let us halve the step size. Computing y_4 with $h = 0.5$, we find that $y(2) \approx 2.209$, so an error of about 0.791. Table $2.8.1$ gives the values computed for various parameters.

Exercise $2.8.1$

Solve this equation exactly and show that $y(2) = 3$.

The difference between the actual solution and the approximate solution we will call the error. We will usually talk about just the size of the error and we do not care much about its sign. The main point is, that we usually do not know the real solution, so we only have a vague understanding of the error. If we knew the error exactly …what is the point of doing the approximation?

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

Table $2.8.1$: Euler's method approximation of $y(2)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

We notice that except for the first few times, every time we halved the interval the error approximately halved. This halving of the error is a general feature of Euler's method as it is a first order method. In the IODE Project II you are asked to implement a second order method. A second order method reduces the error to approximately one quarter every time we halve the interval (second order as $\frac{1}{4} = \frac{1}{2} \mathrm{x} \frac{1}{2}$).

To get the error to be within 0.1 of the answer we had to already do 64 steps. To get it to within 0.01 we would have to halve another three or four times, meaning doing 512 to 1024 steps. That is quite a bit to do by hand. The improved Euler method from IODE Project II should quarter the error every time we halve the interval, so we would have to approximately do half as many "halvings" to get the same error. This reduction can be a big deal. With 10 halvings (starting at $h = 1$) we have 1024 steps, whereas with 5 halvings we only have to do 32 steps, assuming that the error was comparable to start with. A computer may not care about this difference for a problem this simple, but suppose each step would take a second to compute (the function may be substantially more difficult to compute than $\frac{y^2}{3}$). Then the difference is 32 seconds versus about 17 minutes. Note: We are not being altogether fair, a second order method would probably double the time to do each step. Even so, it is 1 minute versus 17 minutes. Next, suppose that we have to repeat such a calculation for different parameters a thousand times. You get the idea.

Below is a video on using Euler's Method to approximate the solution to a differential equation.

Note that in practice we do not know how large the error is! How do we know what is the right step size? Well, essentially we keep halving the interval, and if we are lucky, we can estimate the error from a few of these calculations and the assumption that the error goes down by a factor of one half each time (if we are using standard Euler).

? Exercise $2.8.2$

In the table above, suppose you do not know the error. Take the approximate values of the function in the last two lines, assume that the error goes down by a factor of 2. Can you estimate the error in the last time from this? Does it (approximately) agree with the table? Now do it for the first two rows. Does this agree with the table?

Let us talk a little bit more about the example $y' = \frac{y^2}{3}$, $y(0) = 1$. Suppose that instead of the value $y(2)$ we wish to find $y(3)$. The results of this effort are listed in Table $2.8.2$ for successive halvings of h. What is going on here? Well, you should solve the equation exactly and you will notice that the solution does not exist at $x = 3$. In fact, the solution goes to infinity when you approach $x = 3$.

h	Approximate $y(3)$
1	3.16232
0.5	4.54329
0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004
0.015625	50.40121
0.0078125	87.75769

Table $2.8.2$: Attempts to use Euler's to approximate $y(3)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

Another case when things can go bad is if the solution oscillates wildly near some point. Such an example is given in IODE Project II. The solution may exist at all points, but even a much better numerical method than Euler would need an insanely small step size to approximate the solution with reasonable precision. And computers might not be able to easily handle such a small step size.

In real applications we would not use a simple method such as Euler's. The simplest method that would probably be used in a real application is the standard Runge-Kutta method (see exercises). That is a fourth order method, meaning that if we halve the interval, the error generally goes down by a factor of 16 (it is fourth order as $\frac{1}{16} = \frac{1}{2} \mathrm{x} \frac{1}{2} \mathrm{x} \frac{1}{2} \mathrm{x} \frac{1}{2}$).

Choosing the right method to use and the right step size can be very tricky. There are several competing factors to consider.

- Computational time: Each step takes computer time. Even if the function f is simple to compute, we do it many times over. Large step size means faster computation, but perhaps not the right precision.
- Roundoff errors: Computers only compute with a certain number of significant digits. Errors introduced by rounding numbers off during our computations become noticeable when the step size becomes too small relative to the quantities we are working with. So reducing step size may in fact make errors worse.
- Stability: Certain equations may be numerically unstable. What may happen is that the numbers never seem to stabilize no matter how many times we halve the interval. We may need a ridiculously small interval size, which may not be practical due to roundoff errors or computational time considerations. Such problems are sometimes called stiff. In the worst case, the numerical computations might be giving us bogus numbers that look like a correct answer. Just because the numbers have stabilized after successive halving, does not mean that we must have the right answer.

Below is a video on using Euler's Method to appoximation a solution to a differential equation.

We have seen just the beginnings of the challenges that appear in real applications. Numerical approximation of solutions to differential equations is an active research area for engineers and mathematicians. For example, the general purpose method used for the ODE solver in Matlab and Octave (as of this writing) is a method that appeared in the literature only in the 1980s.

2.8.1: Footnotes

[1] Named after the Swiss mathematician Leonhard Paul Euler (1707–1783). The correct pronunciation of the name sounds more like “oiler.”

2.9: Exact Equations

Another type of equation that comes up quite often in physics and engineering is an . Suppose $F(x, y)$ is a function of two variables, which we call the . The naming should suggest potential energy, or electric potential. Exact equations and potential functions appear when there is a conservation law at play, such as conservation of energy. Let us make up a simple example. Let

$$F(x, y) = x^2 + y^2.$$

We are interested in the lines of constant energy, that is lines where the energy is conserved; we want curves where $F(x, y) = C$, for some constant C. In our example, the curves $x^2 + y^2 = C$ are circles. See Figure $2.9.1$.

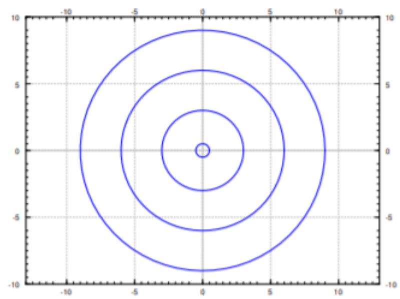


Figure $2.9.1$: Solutions to $F(x, y) = x^2 + y^2 = C$ for various C.

We take the *total derivative* of F:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

For convenience, we will make use of the notation of $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$. In our example,

$$dF = 2x\,dx + 2y\,dy.$$

We apply the total derivative to $F(x, y) = C$, to find the differential equation $dF = 0$. The differential equation we obtain in such a way has the form

$$M\,dx + N\,dy = 0, \qquad \text{or} \qquad M + N\frac{dy}{dx} = 0.$$

An equation of this form is called *exact* if it was obtained as $dF = 0$ for some potential function F. In our simple example, we obtain the equation

$$2x\,dx + 2y\,dy = 0, \qquad \text{or} \qquad 2x + 2y\frac{dy}{dx} = 0.$$

Since we obtained this equation by differentiating $x^2 + y^2 = C$, the equation is exact. We often wish to solve for y in terms of x. In our example,

$$y = \pm\sqrt{C^2 - x^2}.$$

An interpretation of the setup is that at each point $\vec{v} = (M, N)$ is a vector in the plane, that is, a direction and a magnitude. As M and N are functions of (x, y), we have a *vector field*. The particular vector field $\vec{v}$ that comes from an exact equation is a so-called *conservative vector field*, that is, a vector field that comes with a potential function $F(x, y)$, such that

$$\vec{v} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right).$$

Let γ be a path in the plane starting at (x_1, y_1) and ending at (x_2, y_2). If we think of $\vec{v}$ as force, then the work required to move along γ is

$$\int_\gamma \vec{v}(\vec{r}) \cdot d\vec{r} = \int_\gamma M\,dx + N\,dy = F(x_2, y_2) - F(x_1, y_1).$$

That is, the work done only depends on endpoints, that is where we start and where we end. For example, suppose F is gravitational potential. The derivative of F given by $\vec{v}$ is the gravitational force. What we are saying is that the work required to move a heavy box from the ground floor to the roof, only depends on the change in potential energy. That is, the work done is the same no matter what path we took; if we took the stairs or the elevator. Although if we took the elevator, the elevator is doing the work for us. The curves $F(x, y) = C$ are those where no work need be done, such as the heavy box sliding along without accelerating or breaking on a perfectly flat roof, on a cart with incredibly well oiled wheels.

An exact equation is a conservative vector field, and the implicit solution of this equation is the potential function.

2.9.1: Solving exact equations

Now you, the reader, should ask: Where did we solve a differential equation? Well, in applications we generally know M and N, but we do not know F. That is, we may have just started with $2x + 2y\frac{dy}{dx} = 0$, or perhaps even

$$x + y\frac{dy}{dx} = 0.$$

It is up to us to find some potential F that works. Many different F will work; adding a constant to F does not change the equation. Once we have a potential function F, the equation $F\big(x, y(x)\big) = C$ gives an implicit solution of the ODE.

Below is a video on solving an exact first order differential equation.

Below is another video on solving an exact first order differential equation.

✔ Example 2.9.1

Let us find the general solution to $2x + 2y\frac{dy}{dx} = 0$. Forget we knew what F was.

Solution

If we know that this is an exact equation, we start looking for a potential function F. We have $M = 2x$ and $N = 2y$. If F exists, it must be such that $F_x(x, y) = 2x$. Integrate in the x variable to find

$$F(x, y) = x^2 + A(y), \tag{2.9.1}$$

for some function $A(y)$. The function A is the , though it is only constant as far as x is concerned, and may still depend on y. Now differentiate (2.9.1) in y and set it equal to N, which is what F_y is supposed to be:

$$2y = F_y(x, y) = A'(y).$$

Integrating, we find $A(y) = y^2$. We could add a constant of integration if we wanted to, but there is no need. We found $F(x, y) = x^2 + y^2$. Next for a constant C, we solve

$$F\big(x, y(x)\big) = C.$$

for y in terms of x. In this case, we obtain $y = \pm\sqrt{C^2 - x^2}$ as we did before.

Exercise 2.9.1

Why did we not need to add a constant of integration when integrating $A'(y) = 2y$? Add a constant of integration, say 3, and see what F you get. What is the difference from what we got above, and why does it not matter?

The procedure, once we know that the equation is exact, is:

i. Integrate $F_x = M$ in x resulting in $F(x, y) = \text{something} + A(y)$.
ii. Differentiate this F in y, and set that equal to N, so that we may find $A(y)$ by integration.

The procedure can also be done by first integrating in y and then differentiating in x. Pretty easy huh? Let's try this again.

Example 2.9.2

Consider now $2x + y + xy\frac{dy}{dx} = 0$.

OK, so $M = 2x + y$ and $N = xy$. We try to proceed as before. Suppose F exists. Then $F_x(x, y) = 2x + y$. We integrate:

$$F(x, y) = x^2 + xy + A(y)$$

for some function $A(y)$. Differentiate in y and set equal to N:

$$N = xy = F_y(x, y) = x + A'(y).$$

But there is no way to satisfy this requirement! The function xy cannot be written as x plus a function of y. The equation is not exact; no potential function F exists.

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Is there an easier way to check for the existence of F, other than failing in trying to find it? Turns out there is. Suppose $M = F_x$ and $N = F_y$. Then as long as the second derivatives are continuous,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

Let us state it as a theorem. Usually this is called the Poincaré Lemma.[1]

Theorem 2.9.1

Poincaré

If M and N are continuously differentiable functions of (x, y), and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then near any point there is a function $F(x, y)$ such that $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.

The theorem doesn't give us a global F defined everywhere. In general, we can only find the potential locally, near some initial point. By this time, we have come to expect this from differential equations.

Let us return to Example 2.9.2 where $M = 2x + y$ and $N = xy$. Notice $M_y = 1$ and $N_x = y$, which are clearly not equal. The equation is not exact.

Example 2.9.3

Solve

$$\frac{dy}{dx} = \frac{-2x - y}{x - 1}, \qquad y(0) = 1.$$

Solution

We write the equation as

$$(2x + y) + (x - 1)\frac{dy}{dx} = 0,$$

so $M = 2x + y$ and $N = x - 1$. Then

$$M_y = 1 = N_x.$$

The equation is exact. Integrating M in x, we find

$$F(x, y) = x^2 + xy + A(y).$$

Differentiating in y and setting to N, we find

$$x - 1 = x + A'(y).$$

So $A'(y) = -1$, and $A(y) = -y$ will work. Take $F(x,y) = x^2 + xy - y$. We wish to solve $x^2 + xy - y = C$. First let us find C. As $y(0) = 1$ then $F(0,1) = C$. Therefore $0^2 + 0 \times 1 - 1 = C$, so $C = -1$. Now we solve $x^2 + xy - y = -1$ for y to get

$$y = \frac{-x^2 - 1}{x - 1}.$$

Example 2.9.4

Solve

$$-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 0, \qquad y(1) = 2.$$

Solution

We leave to the reader to check that $M_y = N_x$.

This vector field (M,N) is not conservative if considered as a vector field of the entire plane minus the origin. The problem is that if the curve γ is a circle around the origin, say starting at $(1,0)$ and ending at $(1,0)$ going counterclockwise, then if F existed we would expect

$$0 = F(1,0) - F(1,0) = \int_\gamma F_x\, dx + F_y\, dy = \int_\gamma \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy = 2\pi.$$

That is nonsense! We leave the computation of the path integral to the interested reader, or you can consult your multivariable calculus textbook. So there is no potential function F defined everywhere outside the origin $(0,0)$.

If we think back to the theorem, it does not guarantee such a function anyway. It only guarantees a potential function locally, that is only in some region near the initial point. As $y(1) = 2$ we start at the point $(1,2)$. Considering $x > 0$ and integrating M in x or N in y, we find

$$F(x,y) = \arctan\left(\frac{y}{x}\right).$$

The implicit solution is $\arctan\left(\frac{y}{x}\right) = C$. Solving, $y = \tan(C)x$. That is, the solution is a straight line. Solving $y(1) = 2$ gives us that $\tan(C) = 2$, and so $y = 2x$ is the desired solution. See Figure 2.9.1, and note that the solution only exists for $x > 0$.

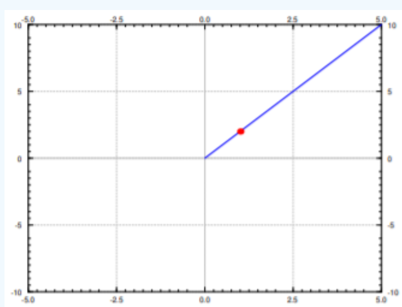


Figure 2.9.1: Solution to $-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 0$, $y(1) = 2$, with initial point marked.

Example 2.9.5

Solve

$$x^2 + y^2 + 2y(x+1)\frac{dy}{dx} = 0.$$

Solution

The reader should check that this equation is exact. Let $M = x^2 + y^2$ and $N = 2y(x+1)$. We follow the procedure for exact equations

$$F(x, y) = \frac{1}{3}x^3 + xy^2 + A(y),$$

and

$$2y(x+1) = 2xy + A'(y).$$

Therefore $A'(y) = 2y$ or $A(y) = y^2$ and $F(x, y) = \frac{1}{3}x^3 + xy^2 + y^2$. We try to solve $F(x, y) = C$. We easily solve for y^2 and then just take the square root:

$$y^2 = \frac{C - (\frac{1}{3})x^3}{x+1}, \qquad \text{so} \qquad y = \pm\sqrt{\frac{C - (\frac{1}{3})x^3}{x+1}}.$$

When $x = -1$, the term in front of $\frac{dy}{dx}$ vanishes. You can also see that our solution is not valid in that case. However, one could in that case try to solve for x in terms of y starting from the implicit solution $\frac{1}{3}x^3 + xy^2 + y^2 = C$. The solution is somewhat messy and we leave it as implicit.

2.9.2: Integrating factors

Sometimes an equation $M\,dx + N\,dy = 0$ is not exact, but it can be made exact by multiplying with a function $u(x, y)$. That is, perhaps for some nonzero function $u(x, y)$,

$$u(x, y)M(x, y)\,dx + u(x, y)N(x, y)\,dy = 0$$

is exact. Any solution to this new equation is also a solution to $M\,dx + N\,dy = 0$.

In fact, a linear equation

$$\frac{dy}{dx} + p(x)y = f(x), \qquad \text{or} \qquad \big(p(x)y - f(x)\big)\,dx + dy = 0$$

is always such an equation. Let $r(x) = e^{\int p(x)\,dx}$ be the integrating factor for a linear equation. Multiply the equation by $r(x)$ and write it in the form of $M + N\frac{dy}{dx} = 0$.

$$r(x)p(x)y - r(x)f(x) + r(x)\frac{dy}{dx} = 0.$$

Then $M = r(x)p(x)y - r(x)f(x)$, so $M_y = r(x)p(x)$, while $N = r(x)$, so $N_x = r'(x) = r(x)p(x)$. In other words, we have an exact equation. Integrating factors for linear functions are just a special case of integrating factors for exact equations.

But how do we find the integrating factor u? Well, given an equation

$$M\,dx + N\,dy = 0,$$

u should be a function such that

$$\frac{\partial}{\partial y}\big[uM\big] = u_y M + uM_y = \frac{\partial}{\partial x}\big[uN\big] = u_x N + uN_x.$$

Therefore,

$$(M_y - N_x)u = u_x N - u_y M.$$

At first it may seem we replaced one differential equation by another. True, but all hope is not lost.

A strategy that often works is to look for a u that is a function of x alone, or a function of y alone. If u is a function of x alone, that is $u(x)$, then we write $u'(x)$ instead of u_x, and u_y is just zero. Then

$$\frac{M_y - N_x}{N}u = u'.$$

In particular, $\frac{M_y - N_x}{N}$ ought to be a function of x alone (not depend on y). If so, then we have a linear equation

$$u' - \frac{M_y - N_x}{N} u = 0.$$

Letting $P(x) = \frac{M_y - N_x}{N}$, we solve using the standard integrating factor method, to find $u(x) = Ce^{\int P(x)\,dx}$. The constant in the solution is not relevant, we need any nonzero solution, so we take $C = 1$. Then $u(x) = e^{\int P(x)\,dx}$ is the integrating factor.

Similarly we could try a function of the form $u(y)$. Then

$$\frac{M_y - N_x}{M} u = -u'.$$

In particular, $\frac{M_y - N_x}{M}$ ought to be a function of y alone. If so, then we have a linear equation

$$u' + \frac{M_y - N_x}{M} u = 0.$$

Letting $Q(y) = \frac{M_y - N_x}{M}$, we find $u(y) = Ce^{-\int Q(y)\,dy}$. We take $C = 1$. So $u(y) = e^{-\int Q(y)\,dy}$ is the integrating factor.

Example 2.9.6

Solve

$$\frac{x^2 + y^2}{x + 1} + 2y \frac{dy}{dx} = 0.$$

Solution

Let $M = \frac{x^2 + y^2}{x + 1}$ and $N = 2y$. Compute

$$M_y - N_x = \frac{2y}{x + 1} - 0 = \frac{2y}{x + 1}.$$

As this is not zero, the equation is not exact. We notice

$$P(x) = \frac{M_y - N_x}{N} = \frac{2y}{x + 1} \frac{1}{2y} = \frac{1}{x + 1}$$

is a function of x alone. We compute the integrating factor

$$e^{\int P(x)\,dx} = e^{\ln(x+1)} = x + 1.$$

We multiply our given equation by $(x + 1)$ to obtain

$$x^2 + y^2 + 2y(x + 1)\frac{dy}{dx} = 0,$$

which is an exact equation that we solved in Example 2.9.5. The solution was

$$y = \pm\sqrt{\frac{C - (\frac{1}{3})x^3}{x + 1}}.$$

Example 2.9.7

Solve

$$y^2 + (xy + 1)\frac{dy}{dx} = 0.$$

Solution

First compute

$$M_y - N_x = 2y - y = y.$$

As this is not zero, the equation is not exact. We observe

$$Q(y) = \frac{M_y - N_x}{M} = \frac{y}{y^2} = \frac{1}{y}$$

is a function of y alone. We compute the integrating factor

$$e^{-\int Q(y)\,dy} = e^{-\ln y} = \frac{1}{y}.$$

Therefore we look at the exact equation

$$y + \frac{xy+1}{y}\frac{dy}{dx} = 0.$$

The reader should double check that this equation is exact. We follow the procedure for exact equations

$$F(x,y) = xy + A(y),$$

and

$$\frac{xy+1}{y} = x + \frac{1}{y} = x + A'(y).$$

Consequently $A'(y) = \frac{1}{y}$ or $A(y) = \ln y$. Thus $F(x,y) = xy + \ln y$. It is not possible to solve $F(x,y) = C$ for y in terms of elementary functions, so let us be content with the implicit solution:

$$xy + \ln y = C.$$

We are looking for the general solution and we divided by y above. We should check what happens when $y = 0$, as the equation itself makes perfect sense in that case. We plug in $y = 0$ to find the equation is satisfied. So $y = 0$ is also a solution.

2.9.3: Footnotes

[1] Named for the French polymath Jules Henri Poincaré (1854–1912).

2.10: First Order Linear PDE

We only considered ODE so far, so let us solve a linear first order PDE. Consider the equation

$$a(x,t)\,u_x + b(x,t)\,u_t + c(x,t)\,u = g(x,t), \qquad u(x,0) = f(x), \qquad -\infty < x < \infty, \quad t > 0,$$

where $u(x,t)$ is a function of x and t. The *initial condition* $u(x,0) = f(x)$ is now a function of x rather than just a number. In these problems, it is useful to think of x as position and t as time. The equation describes the evolution of a function of x as time goes on. Below, the coefficients a, b, c, and the function g are mostly going to be constant or zero. The method we describe works with nonconstant coefficients, although the computations may get difficult quickly.

This method we use is the . The idea is that we find lines along which the equation is an ODE that we solve. We will see this technique again for second order PDE when we encounter the wave equation in Section 4.8.

Example 2.10.1

Consider the equation

$$u_t + \alpha u_x = 0, \qquad u(x,0) = f(x).$$

This particular equation, $u_t + \alpha u_x = 0$, is called the *transport equation*.

The data will propagate along curves called characteristics. The idea is to change to the so-called *characteristic coordinates*. If we change to these coordinates, the equation simplifies. The change of variables for this equation is

$$\xi = x - \alpha t, \qquad s = t.$$

Let's see what the equation becomes. Remember the chain rule in several variables.

$$\begin{aligned} u_t &= u_\xi \xi_t + u_s s_t = -\alpha u_\xi + u_s, \\ u_x &= u_\xi \xi_x + u_s s_x = u_\xi. \end{aligned} \tag{2.10.1}$$

The equation in the coordinates ξ and s becomes

$$\underbrace{(-\alpha u_\xi + u_s)}_{u_t} + \alpha\underbrace{(u_\xi)}_{u_x} = 0,$$

or in other words

$$u_s = 0.$$

That is trivial to solve. Treating ξ as simply a parameter, we have obtained the ODE $\frac{du}{ds} = 0$.

The solution is a function that does not depend on s (but it does depend on ξ). That is, there is some function A such that

$$u = A(\xi) = A(x - \alpha t).$$

The initial condition says that:

$$f(x) = u(x,0) = A(x - \alpha 0) = A(x),$$

so $A = f$. In other words,

$$u(x,t) = f(x - \alpha t).$$

Everything is simply moving right at speed α as t increases. The curve given by the equation

$$\xi = \text{constant}$$

is called the characteristic. See Figure $2.10.1$. In this case, the solution does not change along the characteristic.

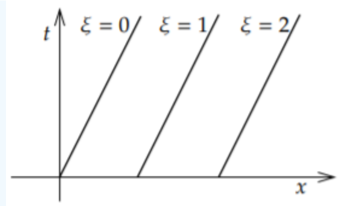


Figure 2.10.1: Characteristic curves.

In the (x, t) coordinates, the characteristic curves satisfy $t = \frac{1}{\alpha}(x - \xi)$, and are in fact lines. The slope of characteristic lines is $\frac{1}{\alpha}$, and for each different ξ we get a different characteristic line.

We see why $u_t + \alpha u_x = 0$ is called the transport equation: everything travels at some constant speed. Sometimes this is called . An example application is material being moved by a river where the material does not diffuse and is simply carried along. In this setup, x is the position along the river, t is the time, and $u(x, t)$ the concentration the material at position x and time t. See Figure 2.10.2 for an example.

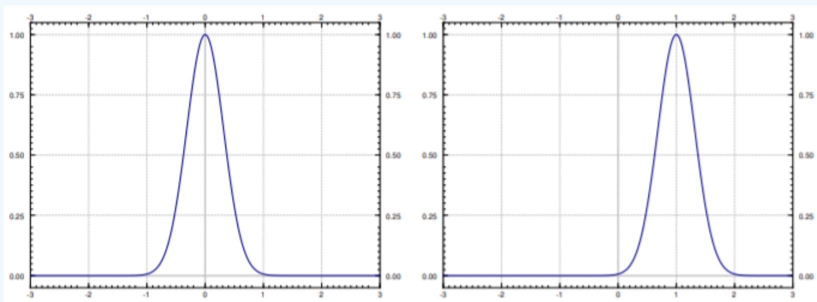


Figure 2.10.2: Example of "transport" in $u_t - u_x = 0$ (that is, $\alpha = 1$) where the initial condition $f(x)$ is a peak at the origin. On the left is a graph of the initial condition $u(x, 0)$. On the right is a graph of the function $u(x, 1)$, that is at time $t = 1$. Notice it is the same graph shifted one unit to the right.

We use similar idea in the more general case:

$$au_x + bu_t + cu = g, \qquad u(x, 0) = f(x).$$

We change coordinates to the characteristic coordinates. Let us call these coordinates (ξ, s). These are coordinates where $au_x + bu_t$ becomes differentiation in the s variable.

Along the characteristic curves (where ξ is constant), we get a new ODE in the s variable. In the transport equation, we got the simple $\frac{du}{ds} = 0$. In general, we get the linear equation

$$\frac{du}{ds} + cu = g. \tag{2.10.2}$$

We think of everything as a function of ξ and s, although we are thinking of ξ as a parameter rather than an independent variable. So the equation is an ODE. It is a linear ODE that we can solve using the integrating factor.

To find the characteristics, think of a curve given parametrically $\big(x(s), t(s)\big)$. We try to have the curve satisfy

$$\frac{dx}{ds} = a, \qquad \frac{dt}{ds} = b.$$

Why? Because when we think of x and t as functions of s we find, using the chain rule,

$$\frac{du}{ds} + cu = \underbrace{\left(u_x \frac{dx}{ds} + u_t \frac{dt}{ds}\right)}_{\frac{du}{ds}} + cu = au_x + bu_t + cu = g.$$

So we get the ODE (2.10.2), which then describes the value of the solution u of the PDE along this characteristic curve. It is also convenient to make sure that $s = 0$ corresponds to $t = 0$, that is $t(0) = 0$. It will be convenient also for $x(0) = \xi$. See Figure 2.10.3.

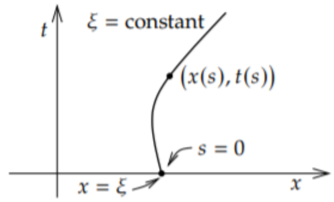


Figure 2.10.3: General characteristic curve.

✔ Example 2.10.2

Consider

$$u_x + u_t + u = x, \qquad u(x,0) = e^{-x^2}.$$

We find the characteristics, that is, the curves given by

$$\frac{dx}{ds} = 1, \qquad \frac{dt}{ds} = 1.$$

So

$$x = s + c_1, \qquad t = s + c_2,$$

for some c_1 and c_2. At $s = 0$ we want $t = 0$, and x should be ξ. So we let $c_1 = \xi$ and $c_2 = 0$:

$$x = s + \xi, \qquad t = s.$$

The ODE is $\frac{du}{ds} + u = x$, and $x = s + \xi$. So, the ODE to solve along the characteristic is

$$\frac{du}{ds} + u = s + \xi.$$

The general solution of this equation, treating ξ as a parameter, is $u = Ce^{-s} + s + \xi - 1$, for some constant C. At $s = 0$, our initial condition is that u is $e^{-\xi^2}$, since at $s = 0$ we have $x = \xi$. Given this initial condition, we find $C = e^{-\xi^2} - \xi + 1$. So,

$$\begin{aligned} u &= \left(e^{-\xi^2} - \xi + 1\right)e^{-s} + s + \xi - 1 \\ &= e^{-\xi^2 - s} + (1 - \xi)e^{-s} + s + \xi - 1. \end{aligned} \tag{2.10.3}$$

Substitute $\xi = x - t$ and $s = t$ to find u in terms of x and t:

$$\begin{aligned} u &= e^{-\xi^2 - s} + (1 - \xi)e^{-s} + s + \xi - 1 \\ &= e^{-(x-t)^2 - t} + (1 - x + t)e^{-t} + x - 1. \end{aligned} \tag{2.10.4}$$

See Figure 2.10.4 for a plot of $u(x,t)$ as a function of two variables.

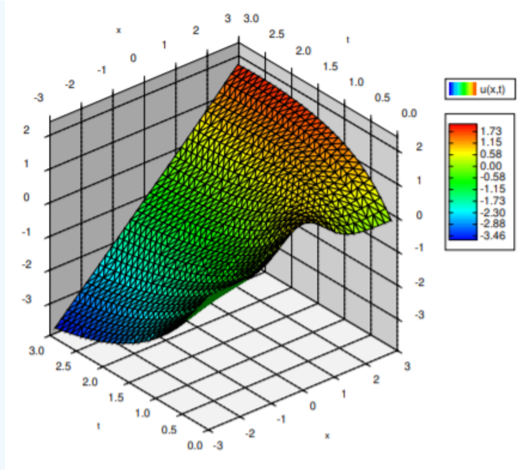


Figure 2.10.4: Plot of the solution $u(x,t)$ to $u_x + u_t + u = x$, $u(x,0) = e^{-x^2}$.

When the coefficients are not constants, the characteristic curves are not going to be straight lines anymore.

Example 2.10.3

Consider the following variable coefficient equation:

$$x u_x + u_t + 2u = 0, \qquad u(x,0) = \cos(x). \tag{2.10.5}$$

We find the characteristics, that is, the curves given by

$$\frac{dx}{ds} = x, \qquad \frac{dt}{ds} = 1.$$

So

$$x = c_1 e^s, \qquad t = s + c_2.$$

At $s = 0$, we wish to get the line $t = 0$, and x should be ξ. So

$$x = \xi e^s, \qquad t = s.$$

OK, the ODE we need to solve is

$$\frac{du}{ds} + 2u = 0.$$

This is for a fixed ξ. At $s = 0$, we should get that u is $\cos(\xi)$, so that is our initial condition. Consequently,

$$u = e^{-2s}\cos(\xi) = e^{-2t}\cos(xe^{-t}).$$

We make a few closing remarks. One thing to keep in mind is that we would get into trouble if the coefficient in front of u_t , that is the b, is ever zero. Let us consider a quick example of what can go wrong:

$$u_x + u = 0, \qquad u(x,0) = \sin(x).$$

This problem has no solution. If we had a solution, it would imply that $u_x(x,0) = \cos(x)$, but $u_x(x,0) + u(x,0) = \cos(x) + \sin(x) \neq 0$. The problem is that the characteristic curve is now the line $t = 0$, and the solution is already provided on that line!

As long as b is nonzero, it is convenient to ensure that b is positive by multiplying by -1 if necessary, so that positive s means positive t.

Another remark is that if a or b in the equation are variable, the computations can quickly get out of hand, as the expressions for the characteristic coordinates become messy and then solving the ODE becomes even messier. In the examples above, b was always 1, meaning we got $s = t$ in the characteristic coordinates. If b is not constant, your expression for s will be more complicated.

Finding the characteristic coordinates is really a system of ODE in general if a depends on t or if b depends on x. In that case, we would need techniques of systems of ODE to solve, see Chapter 3 or Chapter 8. In general, if a and b are not linear functions or constants, finding closed form expressions for the characteristic coordinates may be impossible.

Finally, the method of characteristics applies to nonlinear first order PDE as well. In the nonlinear case, the characteristics depend not only on the differential equation, but also on the initial data. This leads to not only more difficult computations, but also the formation of singularities where the solution breaks down at a certain point in time. An example application where first order nonlinear PDE come up is traffic flow theory, and you have probably experienced the formation of singularities: traffic jams. But we digress.

2.E: First order ODEs (Exercises)

These are homework exercises to accompany Libl's "Differential Equations for Engineering" Textmap. This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Prerequisite for the course is the basic calculus sequence.

2.E.1: 1.1: Integrals as solutions

Exercise $2.E.1.1.1$

Solve for v, and then solve for x. Find $x(10)$ to answer the question.

Exercise $2.E.1.1.2$

Solve $\dfrac{dy}{dx} = x^2 + x$ for $y(1) = 3$.

Exercise $2.E.1.1.3$

Solve $\dfrac{dy}{dx} = \sin(5x)$ for $y(0) = 2$.

Exercise $2.E.1.1.4$

Solve $\dfrac{dy}{dx} = \dfrac{1}{x^2 - 1}$ for $y(0) = 0$.

Exercise $2.E.1.1.5$

Solve $y' = y^3$ for $y(0) = 1$.

Exercise $2.E.1.1.6$: (little harder)

Solve $y' = (y-1)(y+1)$ for $y(0) = 3$.

Exercise $2.E.1.1.7$

Solve $\dfrac{dy}{dx} = \dfrac{1}{y+1}$ for $y(0) = 0$.

Exercise $2.E.1.1.8$: (harder)

Solve $y'' = \sin x$ for $y(0) = 0$, $y'(0) = 2$.

Exercise $2.E.1.1.9$

A spaceship is traveling at the speed $2t^2 + 1$ $^{\text{km}}/_{\text{s}}$ (t is time in seconds). It is pointing directly away from earth and at time $t = 0$ it is 1000 kilometers from earth. How far from earth is it at one minute from time $t = 0$?

Exercise $2.E.1.1.10$

Solve $\dfrac{dx}{dt} = \sin(t^2) + t$, $x(0) = 20$. It is OK to leave your answer as a definite integral.

Exercise $2.E.1.1.11$

A dropped ball accelerates downwards at a constant rate 9.8 meters per second squared. Set up the differential equation for the height above ground h in meters. Then supposing $h(0) = 100$ meters, how long does it take for the ball to hit the ground.

Exercise $2.E.1.1.12$

Find the general solution of $y' = e^x$, and then $y' = e^y$.

Exercise $2.E.1.1.13$

Solve $\frac{dy}{dx} = e^x + x$ and $y(0) = 10$.

Answer

$y = e^x + \frac{x^2}{2} + 9$

Exercise $2.E.1.1.14$

Solve $x' = \frac{1}{x^2}$, $x(1) = 1$.

Answer

$x = (3t - 2)^{1/3}$

Exercise $2.E.1.1.15$

Solve $x' = \frac{1}{\cos(x)}$, $x(0) = \frac{\pi}{2}$.

Answer

$\mathrm{A}x = \sin^{-1}(t+1)$

Exercise $2.E.1.1.16$

Sid is in a car traveling at speed $10t + 70$ miles per hour away from Las Vegas, where t is in hours. At $t = 0$ the Sid is 10 miles away from Vegas. How far from Vegas is Sid 2 hours later?

Answer

170

Exercise $2.E.1.1.17$

Solve $y' = y^n$, $y(0) = 1$, where n is a positive integer. Hint: You have to consider different cases.

Answer

If $n \neq 1$, then $y = ((1-n)x + 1)^{1/(1-n)}$. If $n = 1$, then $y = e^x$.

> **Exercise $2.E.1.1.18$**
>
> The rate of change of the volume of a snowball that is melting is proportional to the surface area of the snowball. Suppose the snowball is perfectly spherical. The volume (in centimeters cubed) of a ball of radius r centimeters is $\frac{4}{3})\pi r^3$. The surface area is $4\pi r^2$. Set up the differential equation for how the radius r is changing. Then, suppose that at time $t=0$ minutes, the radius is 10 centimeters. After 5 minutes, the radius is 8 centimeters. At what time t will the snowball be completely melted?
>
> **Answer**
>
> The equation is $r' = -C$ for some constant C. The snowball will be completely melted in 25 minutes from time $t=0$.

> **Exercise $2.E.1.1.19$**
>
> Find the general solution to $y'''' = 0$. How many distinct constants do you need?
>
> **Answer**
>
> $y = Ax^3 + Bx^2 + Cx + D$, so 4 constants.

2.E.2: 1.2: Slope fields

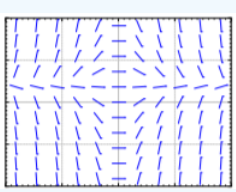
> **Exercise $2.E.1.2.1$**
>
> Sketch slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Can you guess a particular solution by looking at the slope field?

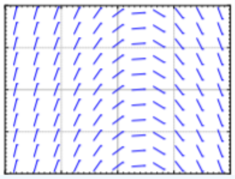
> **Exercise $2.E.1.2.2$**
>
> Sketch slope field for $y' = x^2$.

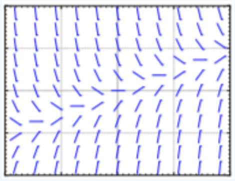
> **Exercise $2.E.1.2.3$**
>
> Sketch slope field for $y' = y^2$.

> **Exercise $2.E.1.2.4$**
>
> Is it possible to solve the equation $y' = \frac{xy}{\cos x}$ for $y(0) = 1$? Justify.

> **Exercise $2.E.1.2.5$**
>
> Is it possible to solve the equation $y' = y\sqrt{|x|}$ for $y(0) = 0$? Is the solution unique? Justify.

> **Exercise $2.E.1.2.6$**
>
> Match equations $y' = 1 - x$, $y' = x - 2y$, $y' = x(1-y)$ to slope fields. Justify.
>
> a. 

b. 

c. 

Exercise $2.E.1.2.7$: (challenging)

Take $y' = f(x, y)$, $y(0) = 0$, where $f(x, y) > 1$ for all x and y. If the solution exists for all x, can you say what happens to $y(x)$ as x goes to positive infinity? Explain.

Exercise $2.E.1.2.8$: (challenging)

Take $(y - x)y' = 0$, $x(0) = 0$.

a. Find two distinct solutions.
b. Explain why this does not violate Picard's theorem.

Exercise $2.E.1.2.9$

Suppose $y' = f(x, y)$. What will the slope field look like, explain and sketch an example, if you know the following about $f(x, y)$:

a. f does not depend on y.
b. f does not depend on x.
c. f, $(t, t) = 0$ for any number t.
d. $f(x, 0) = 0$ and $f(x, 1) = 1$ for all x.

Exercise $2.E.1.2.10$

Find a solution to $y' = |y|$, $y(0) = 0$. Does Picard's theorem apply?

Exercise $2.E.1.2.11$

Take an equation $y' = (y - 2x)g(x, y) + 2$ for some function $g(x, y)$. Can you solve the problem for the initial condition $y(0) = 0$, and if so what is the solution?

Exercise $2.E.1.2.12$: (challenging)

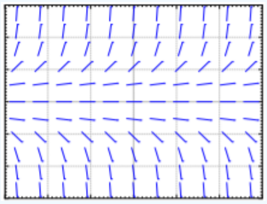
Suppose $y' = f(x, y)$ is such that $f(x, 1) = 0$ for every x, f is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous for every x and y.

a. Guess a solution given the initial condition $y(0) = 1$.
b. Can graphs of two solutions of the equation for different initial conditions ever intersect?
c. Given $y(0) = 0$, what can you say about the solution. In particular, can $y(x) > 1$ for any x? Can $y(x) = 1$ for any x? Why or why not?

Exercise $2.E.1.2.13$

Sketch the slope field of $y' = y^3$. Can you visually find the solution that satisfies $y(0) = 0$?

Answer



$y = 0$ is a solution such that $y(0) = 0$

Exercise $2.E.1.2.14$

Is it possible to solve $y' = xy$ for $y(0) = 0$? Is the solution unique?

Answer

Yes a solution exists. The equation is $y' = f(x, y)$ where $f(x, y) = xy$. The function $f(x, y)$ is continuous and $\frac{\partial f}{\partial y} = x$, which is also continuous near $(0, 0)$. So a solution exists and is unique. (In fact, $y = 0$ is the solution.)

Exercise $2.E.1.2.15$

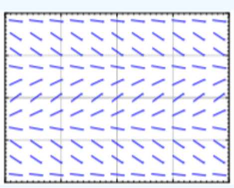
Is it possible to solve $y' = \frac{x}{x^2-1}$ for $y(1) = 0$?

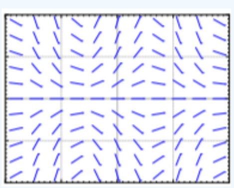
Answer

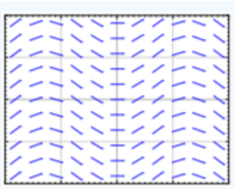
No, the equation is not defined at $(x, y) = (1, 0)$.

Exercise $2.E.1.2.16$

Match equations $y' = \sin x$, $y' = \cos y$, $y' = y\cos(x)$ to slope fields. Justify.

a. 

b. 

c. 

Answer

a. $y' = \cos y$

b. $y' = y\cos(x)$
c. $y' = \sin x$

Justification left to reader.

? Exercise $2.E.1.2.17$: (tricky)

Suppose

$$f(y) = \begin{cases} 0 & if\ y > 0, \\ 1 & if\ y \leq 0. \end{cases}$$

Does $y' = f(y)$, $y(0) = 0$ have a continuously differentiable solution? Does Picard apply? Why, or why not?

Answer

Picard does not apply as f is not continuous at $y = 0$. The equation does not have a continuously differentiable solution. Suppose it did. Notice that $y'(0) = 1$. By the first derivative test, $y(x) > 0$ for small positive x. But then for those x we would have $y'(x) = 0$, so clearly the derivative cannot be continuous.

? Exercise $2.E.1.2.18$

Consider an equation of the form $y' = f(x)$ for some continuous function f, and an initial condition $y(x_0) = y_0$. Does a solution exist for all x? Why or why not?

Answer

The solution is $y(x) = \int_{x_0}^{x} f(s)ds + y_0$, and this does indeed exist for every x.

2.E.3: 1.3: Separable Equations

? Exercise $2.E.1.3.1$

Solve $y' = \frac{x}{y}$.

? Exercise $2.E.1.3.2$

Solve $y' = x^2 y$.

? Exercise $2.E.1.3.3$

Solve $\frac{dx}{dt} = \left(x^2 - 1\right)$, for $x(0) = 0$.

? Exercise $2.E.1.3.4$

Solve $\frac{dx}{dt} = x\sin(t)$, for $x(0) = 1$.

? Exercise $2.E.1.3.5$

Solve $\frac{dy}{dx} = xy + x + y + 1$. Hint: Factor the right hand side.

Exercise $2.E.1.3.6$

Solve $xy' = y + 2x^2y$, where $y(1) = 1$.

Exercise $2.E.1.3.7$

Solve $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$, for $y(0) = 1$.

Exercise $2.E.1.3.8$

Find an implicit solution for $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$, for $y(0) = 1$.

Exercise $2.E.1.3.9$

Find an explicit solution for $y' = xe^{-y}$, $y(0) = 1$.

Exercise $2.E.1.3.10$

Find an explicit solution for $xy' = e^{-y}$, for $y(1) = 1$.

Exercise $2.E.1.3.11$

Find an explicit solution for $y' = ye^{-x^2}$, $y(0) = 1$. It is alright to leave a definite integral in your answer.

Exercise $2.E.1.3.12$

Suppose a cup of coffee is at 100 degrees Celsius at time $t = 0$, it is at 70 degrees at $t = 10$ minutes, and it is at 50 degrees at $t = 20$ minutes. Compute the ambient temperature.

Exercise $2.E.1.3.13$

Solve $y' = 2xy$.

Answer

$$y = Ce^{x^2}$$

Exercise $2.E.1.3.14$

Solve $x' = 3xt^2 - 3t^2$, $x(0) = 2$.

Answer

$$y = e^{t^3} + 1$$

Exercise $2.E.1.3.15$

Find an implicit solution for $x' = \frac{1}{3x^2+1}$ $x(0) = 1$.

Answer

$$x^3 + x = t + 2$$

Exercise $2.E.1.3.16$

Find an explicit solution to $xy' = y^2$, $y(1) = 1$.

Answer

$y = \frac{1}{1-\ln x}$

Exercise $2.E.1.3.17$

Find an implicit solution to $y' = \frac{\sin(x)}{\cos(y)}$.

Answer

$\sin(y) = -\cos(x) + C$

Exercise $2.E.1.3.18$

Take Example 1.3.3 with the same numbers: 89 degrees at $t = 0$, 85 degrees at $t = 1$, and ambient temperature of 22 degrees. Suppose these temperatures were measured with precision of ± 0.5 degrees. Given this imprecision, the time it takes the coffee to cool to (exactly) 60 degrees is also only known in a certain range. Find this range. Hint: Think about what kind of error makes the cooling time longer and what shorter.

Answer

The range is approximately $7.45$to 12.15minutes.

Exercise $2.E.1.3.19$

A population x of rabbits on an island is modeled by $x' = x - (\frac{1}{1000})x^2$, where the independent variable is time in months. At time $t = 0$, there are 40 rabbits on the island.

a. Find the solution to the equation with the initial condition.
b. How many rabbits are on the island in 1 month, 5 months, 10 months, 15 months (round to the nearest integer)

Answer

a. $x = \frac{1000e^t}{e^t+24}$.
b. 102rabbits after one month, 861after 5 months, 999after 10 months, 1000after 15 months.

2.E.4: 1.4: Linear equations and the integrating factor

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

Exercise $2.E.1.4.1$

Solve $y' + xy = x$.

Exercise $2.E.1.4.2$

Solve $y' + 6y = e^x$.

? Exercise $2.E.1.4.3$

Solve $y' + 3x^2 y = \sin(x)\, e^{-x^3}$ with $y(0) = 1$.

? Exercise $2.E.1.4.4$

Solve $y' + \cos(x)\, y = \cos(x)$.

? Exercise $2.E.1.4.5$

Solve $\frac{1}{x^2+1} y' + xy = 3$ with $y(0) = 0$.

? Exercise $2.E.1.4.6$

Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 liters per hour. The first lake contains 100 thousand liters of water and the second lake contains 200 thousand liters of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream.

a. Find the concentration of toxic substance as a function of time in both lakes.
b. When will the concentration in the first lake be below 0.001 kg per liter?
c. When will the concentration in the second lake be maximal?

? Exercise $2.E.1.4.7$

Newton's law of cooling states that $\frac{dx}{dt} = -k(x - A)$ where x is the temperature, t is time, A is the ambient temperature, and $k > 0$ is a constant. Suppose that $A = A_0 \cos(\omega t)$ for some constants A_0 and ω. That is, the ambient temperature oscillates (for example night and day temperatures).

a. Find the general solution.
b. In the long term, will the initial conditions make much of a difference? Why or why not?

? Exercise $2.E.1.4.8$

Initially 5 grams of salt are dissolved in 20 liters of water. Brine with concentration of salt 2 grams of salt per liter is added at a rate of 3 liters a minute. The tank is mixed well and is drained at 3 liters a minute. How long does the process have to continue until there are 20 grams of salt in the tank?

? Exercise $2.E.1.4.9$

Initially a tank contains 10 liters of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 liter per minute. The water is mixed well and drained at 1 liter per minute. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine?

? Exercise $2.E.1.4.10$

Solve $y' + 3x^2 y + x^2$.

Answer

$$y = Ce^{-x^3} + \frac{1}{3}$$

> **Exercise 2.E.1.4.11**
>
> Solve $y' + 2\sin(2x)y = 2\sin(2x)$ with $y(\pi/2) = 3$.
>
> **Answer**
>
> $y = 2e^{\cos(2x)+1} + 1$

> **Exercise 2.E.1.4.12**
>
> Suppose a water tank is being pumped out at $3\ \frac{\text{L}}{\text{min}}$. The water tank starts at $10\ \text{L}$ of clean water. Water with toxic substance is flowing into the tank at $2\ \frac{\text{L}}{\text{min}}$, with concentration $20t\ \frac{\text{g}}{\text{L}}$ at time t. When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?
>
> **Answer**
>
> 250 grams

> **Exercise 2.E.1.4.13**
>
> Suppose we have bacteria on a plate and suppose that we are slowly adding a toxic substance such that the rate of growth is slowing down. That is, suppose that $\frac{dP}{dt} = (2 - 0.1\,t)P$. If $P(0) = 1000$, find the population at $t = 5$.
>
> **Answer**
>
> $P(5) = 1000e^{2\times 5 - 0.05\times 5^2} = 1000e^{8.75} \approx 6.31\times 10^6$

> **Exercise 2.E.1.4.14**
>
> A cylindrical water tank has water flowing in at I cubic meters per second. Let A be the area of the cross section of the tank in meters. Suppose water is flowing from the bottom of the tank at a rate proportional to the height of the water level. Set up the differential equation for h, the height of the water, introducing and naming constants that you need. You should also give the units for your constants.
>
> **Answer**
>
> $Ah' = I - kh$, where k is a constant with units $\frac{\text{m}^2}{\text{s}}$.

2.E.5: 1.5: Substitution

Hint: Answers need not always be in closed form.

> **Exercise 2.E.1.5.1**
>
> Solve $y' + y(x^2 - 1) + xy^6 = 0$, with $y(1) = 1$.

> **Exercise 2.E.1.5.2**
>
> Solve $2yy' + 1 = y^2 + x$, with $y(0) = 1$.

> **Exercise 2.E.1.5.3**
>
> Solve $y' + xy = y^4$, with $y(0) = 1$.

Exercise $2.E.1.5.4$

Solve $yy' + x = \sqrt{x^2 + y^2}$.

Exercise $2.E.1.5.5$

Solve $y' = (x + y - 1)^2$.

Exercise $2.E.1.5.6$

Solve $y' = \dfrac{x^2 - y^2}{xy}$, with $y(1) = 2$.

Exercise $2.E.1.5.7$

Solve $xy' + y + y^2 = 0$, $y(1) = 2$.

Answer

$y = \frac{2}{3x-2}$

Exercise $2.E.1.5.8$

Solve $xy' + y + x = 0$, $y(1) = 1$.

Answer

$y = \frac{3-x^2}{2x}$

Exercise $2.E.1.5.9$

Solve $y^2y' = y^3 - 3x$, $y(0) = 2$.

Answer

$y = \left(7e^{3x} + 3x + 1\right)^{1/3}$

Exercise $2.E.1.5.10$

Solve $2yy' = e^{y^2 - x^2} + 2x$.

Answer

$y = \sqrt{x^2 - \ln(C - x)}$

2.E.6: 1.6: Autonomous equations

Exercise $2.E.1.6.1$

Consider $x' = x^2$.

a. Draw the phase diagram, find the critical points and mark them stable or unstable.
b. Sketch typical solutions of the equation.
c. Find $\lim\limits_{t \to \infty} x(t)$ for the solution with the initial condition $x(0) = -1$.

Exercise $2.E.1.6.2$

Let $x' = \sin x$.

a. Draw the phase diagram for $-4\pi \leq x \leq 4\pi$. On this interval mark the critical points stable or unstable.
b. Sketch typical solutions of the equation.
c. Find $\lim_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = 1$.

Exercise $2.E.1.6.3$

Suppose $f(x)$ is positive for $0 < x < 1$, it is zero when $x = 0$ and $x = 1$, and it is negative for all other x.

a. Draw the phase diagram for $x' = f(x)$, find the critical points and mark them stable or unstable.
b. Sketch typical solutions of the equation.
c. Find $\lim_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = 0.5$.

Exercise $2.E.1.6.4$

Start with the logistic equation $\frac{dx}{dt} = kx(M-x)$. Suppose that we modify our harvesting. That is we will only harvest an amount proportional to current population. In other words we harvest hx per unit of time for some $h > 0$ (Similar to earlier example with h replaced with hx).

a. Construct the differential equation.
b. Show that if $kM > h$, then the equation is still logistic.
c. What happens when $kM < h$?

Exercise $2.E.1.6.5$

A disease is spreading through the country. Let x be the number of people infected. Let the constant S be the number of people susceptible to infection. The infection rate $\frac{dx}{dt}$ is proportional to the product of already infected people, x, and the number of susceptible but uninfected people, $S - x$.

a. Write down the differential equation.
b. Supposing $x(0) > 0$, that is, some people are infected at time $t = 0$, what is $\lim_{t\to\infty} x(t)$.
c. Does the solution to part b) agree with your intuition? Why or why not?

Exercise $2.E.1.6.6$

Let $x' = (x-1)(x-2)x^2$.

a. Sketch the phase diagram and find critical points.
b. Classify the critical points.
c. If $x(0) = 0.5$ then find $\lim_{t\to\infty} x(t)$.

Answer

a. $0,\ 1,\ 2$ are critical points.
b. $x = 0$ is unstable (semistable), $x = 1$ is stable, and $x = 2$ is unstable.
c. 1

Exercise $2.E.1.6.7$

Let $x' = e^{-x}$.

a. Find and classify all critical points.
b. Find $\lim_{t\to\infty} x(t)$ given any initial condition.

Answer

a. There are no critical points.
b. ∞

? Exercise 2.E.1.6.8

Assume that a population of fish in a lake satisfies $\frac{dx}{dt} = kx(M - x)$. Now suppose that fish are continually added at A fish per unit of time.

a. Find the differential equation for x.
b. What is the new limiting population?

Answer

a. $\frac{dx}{dt} = kx(M - x) + A$
b. $\frac{kM + \sqrt{(kM)^2 + 4Ak}}{2k}$

? Exercise 2.E.1.6.9

Suppose $\frac{dx}{dt} = (x - \alpha)(x - \beta)$ for two numbers $\alpha < \beta$.

a. Find the critical points, and classify them.

For b), c), d), find $\lim_{t\to\infty} x(t)$ based on the phase diagram.

b. $x(0) < \alpha$,
c. $\alpha < x(0) < \beta$,
d. $\beta < x(0)$.

Answer

1. α is a stable critical point, β is an unstable one.
2. α
3. α
4. ∞ or DNE.

2.E.7: 1.7: Numerical methods: Euler's method

? Exercise 2.E.1.7.1

Consider $\frac{dx}{dt} = (2t - x)^2$, $x(0) = 2$. Use Euler's method with step size $h = 0.5$ to approximate $x(1)$.

? Exercise 2.E.1.7.2

Consider $\frac{dx}{dt} = t - x$, $x(0) = 1$.

a. Use Euler's method with step sizes $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ to approximate $x(1)$.
b. Solve the equation exactly.
c. Describe what happens to the errors for each h you used. That is, find the factor by which the error changed each time you halved the interval.

? Exercise 2.E.1.7.3

Approximate the value of e by looking at the initial value problem $y' = y$ with $y(0) = 1$ and approximating $y(1)$ using Euler's method with a step size of 0.2.

Exercise $2.E.1.7.4$

Example of numerical instability: Take $y' = -5y$, $y(0) = 1$. We know that the solution should decay to zero as x grows. Using Euler's method, start with $h = 1$ and compute y_1, y_2, y_3, y_4 to try to approximate $y(4)$. What happened? Now halve the interval. Keep halving the interval and approximating $y(4)$ until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.

The simplest method used in practice is the Runge-Kutta method. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ and a step size h. Everything is the same as in Euler's method, except the computation of y_{i+1} and x_{i+1}.

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f(x_i + \frac{h}{2}, y_i + k_1\frac{h}{2}) & x_{i+1} &= x_i + h, \\ k_3 &= f(x_i + \frac{h}{2}, y_i + k_2\frac{h}{2}) & y_{i+1} &= y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}h, \\ k_4 &= f(x_i + h, y_i + k_3 h). \end{aligned} \tag{2.E.1}$$

Exercise $2.E.1.7.5$

Consider $\frac{dy}{dx} = yx^2$, $y(0) = 1$.

a. Use Runge-Kutta (see above) with step sizes $h = 1$ and $h = \frac{1}{2}$ to approximate $y(1)$.
b. Use Euler's method with $h = 1$ and $h = \frac{1}{2}$.
c. Solve exactly, find the exact value of $y(1)$, and compare.

Exercise $2.E.1.7.6$

Let $x' = \sin(xt)$, and $x(0) = 1$. Approximate $x(1)$ using Euler's method with step sizes 1, 0.5, 0.25. Use a calculator and compute up to 4 decimal digits.

Answer

Approximately: $1.0000,\ 1.2397,\ 1.382$

Exercise $2.E.1.7.7$

Let $x' = 2t$, and $x(0) = 0$.

a. Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.
b. Solve exactly, and compute the errors.
c. Compute the factor by which the errors changed.

Answer

a. $0,\ 8,\ 12$
b. $x(4) = 16$, so errors are: $16,\ 8,\ 4$
c. Factors are $0.5,\ 0.5,\ 0.5$

Exercise $2.E.1.7.8$

Let $x' = xe^{xt+1}$, and $x(0) = 0$.

a. Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.
b. Guess an exact solution based on part a) and compute the errors.

Answer

a. $0,\ 0,\ 0$

b. $x = 0$ is a solution so errors are: $0, \ 0, \ 0$.

There is a simple way to improve Euler's method to make it a second order method by doing just one extra step. Consider $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$, and a step size h. What we do is to pretend we compute the next step as in Euler, that is, we start with (x_i, y_i), we compute a slope $k_1 = f(x_i, y_i)$, and then look at the point $(x_i + h, y_i + k_1 h)$. Instead of letting our new point be $(x_i + h, y_i + k_1 h)$, we compute the slope at that point, call it k_2, and then take the average of k_1 and k_2, hoping that the average is going to be closer to the actual slope on the interval from x_i to $x_i + h$. And we are correct, if we halve the step, the error should go down by a factor of $2^2 = 4$. To summarize, the setup is the same as for regular Euler, except the computation of y_{i+1} and x_{i+1}.

$$\begin{aligned} k_1 &= f(x_i, y_i), & x_{i+1} &= x_i + h, \\ k_2 &= f(x_i + h, y_i + k_1 h), & y_{i+1} &= y_i + \frac{k_1 + k_2}{2} h. \end{aligned} \tag{2.E.2}$$

Exercise $2.E.1.7.9$

Consider $\dfrac{dy}{dx} = x + y$, $y(0) = 1$.

a. Use the improved Euler's method (see above) with step sizes $h = \frac{1}{4}$ and $h = \frac{1}{8}$ to approximate $y(1)$.
b. Use Euler's method with $h = \frac{1}{4}$ and $h = \frac{1}{8}$.
c. Solve exactly, find the exact value of $y(1)$.
d. Compute the errors, and the factors by which the errors changed.

Answer

a. Improved Euler: $y(1) \approx 3.3897$ for $h = 1/4$, $y(1) \approx 3.4237$ for $h = 1/8$,
b. Standard Euler: $y(1) \approx 2.8828$ for $h = 1/4$, $y(1) \approx 3.1316$ for $h = 1/8$,
c. $y = 2e^x - x - 1$, so $y(2)$ is approximately 3.4366
d. Approximate errors for improved Euler: 0.046852 for $h = 1/4$, and 0.012881 for $h = 1/8$. For standard Euler: 0.55375 for $h = 1/4$, and 0.30499 for $h = 1/8$. Factor is approximately 0.27 for improved Euler, and 0.55 for standard Euler.

2.E.8: 1.8 Exact Equations

Exercise $2.E.1.8.1$

Solve the following exact equations, implicit general solutions will suffice:

a. $(2xy + x^2)\,dx + (x^2 + y^2 + 1)\,dy = 0$
b. $x^5 + y^5 \frac{dy}{dx} = 0$
c. $e^x + y^3 + 3xy^2 \frac{dy}{dx} = 0$
d. $(x + y)\cos(x) + \sin(x) + \sin(x)y' = 0$

Exercise $2.E.1.8.2$

Find the integrating factor for the following equations making them into exact equations:

a. $e^{xy}\,dx + \frac{y}{x}e^{xy}\,dy = 0$
b. $\frac{e^x + y^3}{y^2}\,dx + 3x\,dy = 0$
c. $4(y^2 + x)\,dx + \frac{2x + 2y^2}{y}\,dy = 0$
d. $2\sin(y)\,dx + x\cos(y)\,dy = 0$

Exercise $2.E.1.8.3$

Suppose you have an equation of the form: $f(x)+g(y)\frac{dy}{dx}=0$.

a. Show it is exact.
b. Find the form of the potential function in terms of f and g.

Exercise $2.E.1.8.4$

Suppose that we have the equation $f(x)\,dx-dy=0$.

a. Is this equation exact?
b. Find the general solution using a definite integral.

Exercise $2.E.1.8.5$

Find the potential function $F(x,y)$ of the exact equation $\frac{1+xy}{x}\,dx+\left(\frac{1}{y}+x\right)dy=0$ in two different ways.

a. Integrate M in terms of x and then differentiate in y and set to N.
b. Integrate N in terms of y and then differentiate in x and set to M.

Exercise $2.E.1.8.6$

A function $u(x,y)$ is said to be a if $u_{xx}+u_{yy}=0$.

a. Show if u is harmonic, $-u_y\,dx+u_x\,dy=0$ is an exact equation. So there exists (at least locally) the so-called function $v(x,y)$ such that $v_x=-u_y$ and $v_y=u_x$.

Verify that the following u are harmonic and find the corresponding harmonic conjugates v:

a. $u=2xy$
b. $u=e^x\cos y$
c. $u=x^3-3xy^2$

Exercise $2.E.1.8.7$

Solve the following exact equations, implicit general solutions will suffice:

a. $\cos(x)+ye^{xy}+xe^{xy}y'=0$
b. $(2x+y)\,dx+(x-4y)\,dy=0$
c. $e^x+e^y\frac{dy}{dx}=0$
d. $(3x^2+3y)\,dx+(3y^2+3x)\,dy=0$

Answer

a. $e^{xy}+\sin(x)=C$
b. $x^2+xy-2y^2=C$
c. $e^x+e^y=C$
d. $x^3+3xy+y^3=C$

Exercise $2.E.1.8.8$

Find the integrating factor for the following equations making them into exact equations:

a. $\frac{1}{y}\,dx+3y\,dy=0$
b. $dx-e^{-x-y}\,dy=0$
c. $\left(\frac{\cos(x)}{y^2}+\frac{1}{y}\right)dx+\frac{x}{y^2}\,dy=0$

d. $\left(2y+\frac{y^2}{x}\right)dx+(2y+x)\,dy=0$

Answer

a. Integrating factor is y, equation becomes $dx+3y^2dy=0$.
b. Integrating factor is e^x, equation becomes $e^xdx-e^{-y}dy=0$.
c. Integrating factor is y^2, equation becomes $(\cos(x)+y)dx+xdy=0$.
d. Integrating factor is x, equation becomes $(2xy+y^2)dx+(x^2+2xy)dy=0$.

Exercise $2.E.1.8.9$

a. Show that every separable equation $y'=f(x)g(y)$ can be written as an exact equation, and verify that it is indeed exact.
b. Using this rewrite $y'=xy$ as an exact equation, solve it and verify that the solution is the same as it was in Example 1.3.1.

Answer

a. The equation is $-f(x)dx+\frac{1}{g(y)}dy$, and this is exact because $M=-f(x)$, $N=\frac{1}{g(y)}$, so $M_y=0=N_x$.
b. $-xdx+\frac{1}{y}dy=0$, leads to potential function $F(x,y)=-\frac{x^2}{2}+\ln|y|$, solving $F(x,y)=C$ leads to the same solution as the example.

2.E.9: 1.9: First Order Linear PDE

Exercise $2.E.1.9.1$

Solve

a. $u_t+9u_x=0$, $u(x,0)=\sin(x)$,
b. $u_t-8u_x=0$, $u(x,0)=\sin(x)$,
c. $u_t+\pi u_x=0$, $u(x,0)=\sin(x)$,
d. $u_t+\pi u_x+u=0$, $u(x,0)=\sin(x)$.

Exercise $2.E.1.9.2$

Solve $u_t+3u_x=1$, $u(x,0)=x^2$.

Exercise $2.E.1.9.3$

Solve $u_t+3u_x=x$, $u(x,0)=e^x$.

Exercise $2.E.1.9.4$

Solve $u_x+u_t+xu=0$, $u(x,0)=\cos(x)$.

Exercise $2.E.1.9.5$

a. Find the characteristic coordinates for the following equations:
$u_x+u_t+u=1$, $u(x,0)=\cos(x)$,) $2u_x+2u_t+2u=2$, $u(x,0)=\cos(x)$.
b. Solve the two equations using the coordinates.
c. Explain why you got the same solution, although the characteristic coordinates you found were different.

Exercise $2.E.1.9.6$

Solve $(1+x^2)u_t+x^2u_x+e^xu=0$, $u(x,0)=0$. Hint: Think a little out of the box.

Exercise $2.E.1.9.7$

Solve

a. $u_t - 5u_x = 0$, $u(x,0) = \frac{1}{1+x^2}$,
b. $u_t + 2u_x = 0$, $u(x,0) = \cos(x)$.

Answer

a. $u = \frac{1}{1+(x+5t)^2}$
b. $u = \cos(x - 2t)$

Exercise $2.E.1.9.8$

Solve $u_x + u_t + tu = 0$, $u(x,0) = \cos(x)$.

Answer

$u = \cos(x - t)e^{-t^2/2}$

Exercise $2.E.1.9.9$

Solve $u_x + u_t = 5$, $u(x,0) = x$.

Answer

$u = x + 4t$

CHAPTER OVERVIEW

3: Applications of First Order Equations

In this chapter, we consider applications of first order differential equations.

Thumbnail: False color time-lapse video of E. coli colony growing on microscope slide. This growth can be model with first order logistic equation. Added approximate scale bar based on the approximate length of 2.0 μm of E. coli bacteria. (CC BY-SA 4.0 International; Stewart EJ, Madden R, Paul G, Taddei F).

3.1: Growth and Decay

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This section begins with a discussion of exponential growth and decay, which you have probably already seen in calculus. We consider applications to radioactive decay, carbon dating, and compound interest. We also consider more complicated problems where the rate of change of a quantity is in part proportional to the magnitude of the quantity, but is also influenced by other other factors for example, a radioactive substance is manufactured at a certain rate, but decays at a rate proportional to its mass, or a saver makes regular deposits in a savings account that draws compound interest.

Since the applications in this section deal with functions of time, we'll denote the independent variable by t. If Q is a function of t, Q' will denote the derivative of Q with respect to t; thus,

$$Q' = \frac{dQ}{dt}.$$

Exponential Growth and Decay

One of the most common mathematical models for a physical process is the *exponential model*, where it is assumed that the rate of change of a quantity Q is proportional to Q; thus

$$Q' = aQ, \tag{3.1.1}$$

where a is the constant of proportionality.

From Example 2.1.3, the general solution of Equation 3.1.1 is

$$Q = ce^{at}$$

and the solution of the initial value problem

$$Q' = aQ, \quad Q(t_0) = Q_0$$

is

$$Q = Q_0 e^{a(t-t_0)}. \tag{3.1.2}$$

Since the solutions of $Q' = aQ$ are exponential functions, we say that a quantity Q that satisfies this equation *grows exponentially* if $a > 0$, or *decays exponentially* if $a < 0$ (Figure 3.1.1).

Figure 3.1.1 : Exponential growth and decay

Radioactive Decay

Experimental evidence shows that radioactive material decays at a rate proportional to the mass of the material present. According to this model the mass $Q(t)$ of a radioactive material present at time t satisfies Equation 3.1.1, where a is a negative constant whose value for any given material must be determined by experimental observation. For simplicity, we'll replace the negative constant a by $-k$, where k is a positive number that we'll call the *decay constant* of the material. Thus, Equation 3.1.1 becomes

$$Q' = -kQ.$$

If the mass of the material present at $t = t_0$ is Q_0, the mass present at time t is the solution of

$$Q' = -kQ, \quad Q(t_0) = Q_0.$$

From Equation 3.1.2 with $a = -k$, the solution of this initial value problem is

$$Q = Q_0 e^{-k(t-t_0)}. \tag{3.1.3}$$

The *half–life* τ of a radioactive material is defined to be the time required for half of its mass to decay; that is, if $Q(t_0) = Q_0$, then

$$Q(\tau + t_0) = \frac{Q_0}{2}. \tag{3.1.4}$$

From Equation 3.1.3 with $t = \tau + t_0$, Equation 3.1.4 is equivalent to

$$Q_0 e^{-k\tau} = \frac{Q_0}{2},$$

so

$$e^{-k\tau} = \frac{1}{2}.$$

Taking logarithms yields

$$-k\tau = \ln \frac{1}{2} = -\ln 2,$$

so the half-life is

$$\tau = \frac{1}{k} \ln 2. \tag{3.1.5}$$

(Figure 3.1.2). The half-life is independent of t_0 and Q_0, since it is determined by the properties of material, not by the amount of the material present at any particular time.

Example 3.1.1

A radioactive substance has a half-life of 1620 years.

a. If its mass is now 4 g (grams), how much will be left 810 years from now?
b. Find the time t_1 when 1.5 g of the substance remain.

Solution a

From Equation 3.1.3 with $t_0 = 0$ and $Q_0 = 4$,

$$Q = 4e^{-kt}, \tag{3.1.6}$$

where we determine k from Equation 3.1.5, with τ= 1620 years:

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{1620}.$$

Substituting this in Equation 3.1.6 yields

$$Q = 4e^{-(t \ln 2)/1620}. \tag{3.1.7}$$

Figure 3.1.2 : Half-life of a radioactive substance

Therefore the mass left after 810 years will be

$$\begin{aligned} Q(810) &= 4e^{-(810\ln 2)/1620} = 4e^{-(\ln 2)/2} \\ &= 2\sqrt{2}\ \text{g}. \end{aligned}$$

Solution b

Setting $t = t_1$ in Equation 3.1.7 and requiring that $Q(t_1) = 1.5$ yields

$$\frac{3}{2} = 4e^{(-t_1 \ln 2)/1620}.$$

Dividing by 4 and taking logarithms yields

$$\ln\frac{3}{8} = -\frac{t_1 \ln 2}{1620}.$$

Since $\ln 3/8 = -\ln 8/3$,

$$t_1 = 1620\frac{\ln 8/3}{\ln 2} \approx 2292.4 \text{ years.}$$

Interest Compounded Continuously

Suppose we deposit an amount of money Q_0 in an interest-bearing account and make no further deposits or withdrawals for t years, during which the account bears interest at a constant annual rate r. To calculate the value of the account at the end of t years, we need one more piece of information: how the interest is added to the account, or—as the bankers say—how it is *compounded*. If the interest is compounded annually, the value of the account is multiplied by $1+r$ at the end of each year. This means that after t years the value of the account is

$$Q(t) = Q_0(1+r)^t.$$

If interest is compounded semiannually, the value of the account is multiplied by $(1+r/2)$ every 6 months. Since this occurs twice annually, the value of the account after t years is

$$Q(t) = Q_0\left(1+\frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded n times per year, the value of the account is multiplied n times per year by $(1+r/n)$; therefore, the value of the account after t years is

$$Q(t) = Q_0\left(1+\frac{r}{n}\right)^{nt}. \tag{3.1.8}$$

Thus, increasing the frequency of compounding increases the value of the account after a fixed period of time. Table 3.1.1 shows the effect of increasing the number of compoundings over $t = 5$ years on an initial deposit of $Q_0 = 100$ (dollars), at an annual interest rate of 6%.

Table 3.1.1 : The effect of compound interest

n **(number of compoundings per year)**	$\$100\left(1+\frac{.06}{n}\right)^{5n}$ **(value in dollars after 5 years)**
1	\$133.82
2	\$134.39
4	\$134.68
8	\$134.83
364	\$134.98

You can see from Table 3.1.1 that the value of the account after 5 years is an increasing function of n. Now suppose the maximum allowable rate of interest on savings accounts is restricted by law, but the time intervals between successive compoundings isn't; then competing banks can attract savers by compounding often. The ultimate step in this direction is to *compound continuously*, by which we mean that $n \to \infty$ in Equation 3.1.8. Since we know from calculus that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r,$$

this yields

$$\begin{aligned} Q(t) &= \lim_{n\to\infty} Q_0\left(1+\frac{r}{n}\right)^{nt} = Q_0\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n\right]^t \\ &= Q_0 e^{rt}. \end{aligned}$$

Observe that $Q = Q_0e^{rt}$ is the solution of the initial value problem

$$Q' = rQ, \quad Q(0) = Q_0;$$

that is, with continuous compounding the value of the account grows exponentially.

Below is a video on exponential growth models and differential equations.

Below is a video on solving a differential equation that models exponential growth.

Example 3.1.2

If \$150 is deposited in a bank that pays $5\frac{1}{2}$% annual interest compounded continuously, the value of the account after t years is

$$Q(t) = 150e^{.055t}$$

dollars. (Note that it is necessary to write the interest rate as a decimal; thus, $r = .055$.) Therefore, after $t = 10$ years the value of the account is

$$Q(10) = 150e^{.55} \approx \$259.99.$$

Example 3.1.3

We wish to accumulate \$10,000 in 10 years by making a single deposit in a savings account bearing $5\frac{1}{2}$% annual interest compounded continuously. How much must we deposit in the account?

Solution

The value of the account at time t is

$$Q(t) = Q_0 e^{.055t}. \tag{3.1.9}$$

Since we want $Q(10)$ to be \$10,000, the initial deposit Q_0 must satisfy the equation

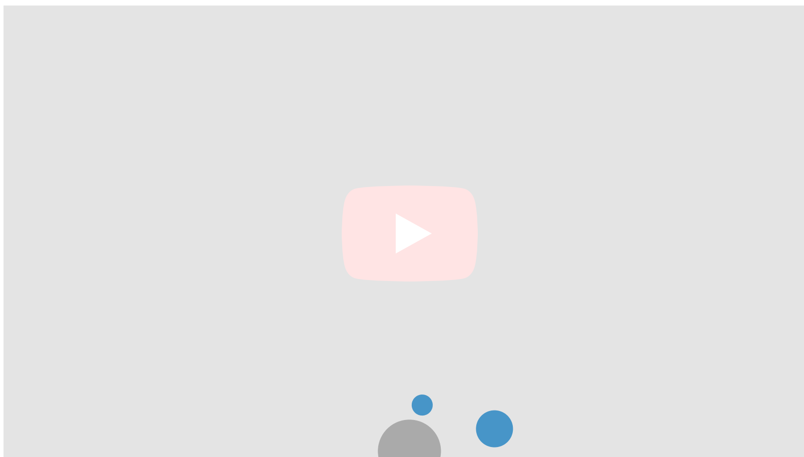
$$10000 = Q_0 e^{.55}, \tag{3.1.10}$$

obtained by setting $t = 10$ and $Q(10) = 10000$ in Equation 3.1.9. Solving Equation 3.1.10 for Q_0 yields

$$Q_0 = 10000e^{-.55} \approx \$5769.50.$$

Below is a video on modeling carbon 14 with a differential equation.

Below is a video on defining and differential equations that model continuous compound interest.



Mixed Growth and Decay

Example 3.1.4

A radioactive substance with decay constant k is produced at a constant rate of a units of mass per unit time.

a. Assuming that $Q(0)=Q_0$, find the mass $Q(t)$ of the substance present at time t.
b. Find $\lim_{t\to\infty} Q(t)$.

Solution a:

Here

$$Q' = \text{ rate of increase of } Q - \text{ rate of decrease of } Q.$$

The rate of increase is the constant a. Since Q is radioactive with decay constant k, the rate of decrease is kQ. Therefore

$$Q' = a - kQ.$$

This is a linear first order differential equation. Rewriting it and imposing the initial condition shows that Q is the solution of the initial value problem

$$Q' + kQ = a, \quad Q(0) = Q_0. \tag{3.1.11}$$

Since e^{-kt} is a solution of the complementary equation, the solutions of Equation 3.1.11 are of the form $Q = ue^{-kt}$, where $u'e^{-kt} = a$, so $u' = ae^{kt}$. Hence,

$$u = \frac{a}{k}e^{kt} + c$$

Figure 3.1.3 : $Q(t)$ approaches the steady state value $\frac{a}{k}$ as $t \to \infty$

and

$$Q = ue^{-kt} = \frac{a}{k} + ce^{-kt}.$$

Since $Q(0) = Q_0$, setting $t = 0$ here yields

$$Q_0 = \frac{a}{k} + c \quad \text{or} \quad c = Q_0 - \frac{a}{k}.$$

Therefore

$$Q = \frac{a}{k} + \left(Q_0 - \frac{a}{k}\right) e^{-kt}. \tag{3.1.12}$$

b. Since $k > 0$, $\lim_{t\to\infty} e^{-kt} = 0$, so from Equation 3.1.12

$$\lim_{t\to\infty} Q(t) = \frac{a}{k}.$$

This limit depends only on a and k, and not on Q_0. We say that a/k is the *steady state* value of Q. From Equation 3.1.12 we also see that Q approaches its steady state value from above if $Q_0 > a/k$, or from below if $Q_0 < a/k$. If $Q_0 = a/k$, then Q remains constant (Figure 3.1.3).

Carbon Dating

The fact that Q approaches a steady state value in the situation discussed in Example 4 underlies the method of *carbon dating*, devised by the American chemist and Nobel Prize Winner *W.S. Libby*.

Carbon 12 is stable, but carbon-14, which is produced by cosmic bombardment of nitrogen in the upper atmosphere, is radioactive with a half-life of about 5570 years. Libby assumed that the quantity of carbon-12 in the atmosphere has been constant throughout time, and that the quantity of radioactive carbon-14 achieved its steady state value long ago as a result of its creation and decomposition over millions of years. These assumptions led Libby to conclude that the ratio of carbon-14 to carbon-12 has been nearly constant for a long time. This constant, which we denote by R, has been determined experimentally.

Living cells absorb both carbon-12 and carbon-14 in the proportion in which they are present in the environment. Therefore the ratio of carbon-14 to carbon-12 in a living cell is always R. However, when the cell dies it ceases to absorb carbon, and the ratio of carbon-14 to carbon-12 decreases exponentially as the radioactive carbon-14 decays. This is the basis for the method of carbon dating, as illustrated in the next example.

Example 3.1.5

An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is between 42 and 44% of the amount present in live individuals. Estimate the age of the village and the length of time for which it survived.

Solution

Let $Q = Q(t)$ be the quantity of carbon-14 in an individual set of remains t years after death, and let Q_0 be the quantity that would be present in live individuals. Since carbon-14 decays exponentially with half-life 5570 years, its decay constant is

$$k = \frac{\ln 2}{5570}.$$

Therefore

$$Q = Q_0 e^{-t(\ln 2)/5570}$$

if we choose our time scale so that $t_0 = 0$ is the time of death. If we know the present value of Q we can solve this equation for t, the number of years since death occurred. This yields

$$t = -5570 \frac{\ln Q/Q_0}{\ln 2}.$$

It is given that $Q = .42Q_0$ in the remains of individuals who died first. Therefore these deaths occurred about

$$t_1 = -5570 \frac{\ln .42}{\ln 2} \approx 6971$$

years ago. For the most recent deaths, $Q = .44Q_0$; hence, these deaths occurred about

$$t_2 = -5570 \frac{\ln .44}{\ln 2} \approx 6597$$

years ago. Therefore it is reasonable to conclude that the village was founded about 7000 years ago, and lasted for about 400 years.

A Savings Program

Example 3.1.6 :

A person opens a savings account with an initial deposit of \$1000 and subsequently deposits \$50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest compounded continuously.

Solution

Observe that Q isn't continuous, since there are 52 discrete deposits per year of \$50 each. To construct a mathematical model for this problem in the form of a differential equation, we make the simplifying assumption that the deposits are made continuously at a rate of \$2600 per year. This is essential, since solutions of differential equations are continuous functions. With this assumption, Q increases continuously at the rate

$$Q' = 2600 + 0.06Q$$

and therefore Q satisfies the differential equation

$$Q' - .06Q = 2600. \tag{3.1.13}$$

(Of course, we must recognize that the solution of this equation is an approximation to the true value of Q at any given time. We'll discuss this further below.) Since $e^{.06t}$ is a solution of the complementary equation, the solutions of Equation 3.1.13 are of the form $Q = ue^{.06t}$, where $u'e^{.06t} = 2600$. Hence, $u' = 2600e^{-.06t}$,

$$u = -\frac{2600}{.06} e^{-0.06t} + c$$

and

$$Q = ue^{.06t} = -\frac{2600}{.06} + ce^{.06t}. \tag{3.1.14}$$

Setting $t=0$ and $Q=1000$ here yields

$$c=1000+\frac{2600}{0.06},$$

and substituting this into Equation 3.1.14 yields

$$Q=1000e^{.06t}+\frac{2600}{.06}(e^{.06t}-1) \tag{3.1.15}$$

where the first term is the value due to the initial deposit and the second is due to the subsequent weekly deposits.

Mathematical models must be tested for validity by comparing predictions based on them with the actual outcome of experiments. Example 6 is unusual in that we can compute the exact value of the account at any specified time and compare it with the approximate value predicted by Equation 3.1.15 (See *Exercise 4.1.21*). The following table gives a comparison for a ten year period. Each exact answer corresponds to the time of the year-end deposit, and each year is assumed to have exactly 52 weeks.

Table 3.1.3

Year	Approximate Value of Q (Example 3.1.6)	Exact Value of P (*Exercise 4.1.21*)	Error $Q-P$	Percentage Error $(Q-P)/P$
1	\$3741.42	\$3739.87	\$1.55	.0413
2	6652.36	6649.17	3.19	.0479
3	9743.30	9738.37	4.93	.0506
4	13,025.38	13,018.60	6.78	.0521
5	16,510.41	16,501.66	8.75	.0530
6	20,210.94	20,200.11	10.83	.0536
7	24,140.30	24,127.25	13.05	.0541
8	28,312.63	28,297.23	15.40	.0544
9	32,742.97	32[illegible]07	17.90	.0547
10	37,447.27	37,426.72	20.55	.0549

3.1E: Growth and Decay (Exercises)

Q4.1.1

1. The half-life of a radioactive substance is 3200 years. Find the quantity $Q(t)$ of the substance left at time $t>0$ if $Q(0)=20$ g.

2. The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to 1/10 of its original mass.

3. A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?

4. A tree contains a known percentage p_0 of a radioactive substance with half-life τ. When the tree dies the substance decays and isn't replaced. If the percentage of the substance in the fossilized remains of such a tree is found to be p_1, how long has the tree been dead?

5. If t_p and t_q are the times required for a radioactive material to decay to $1/p$ and $1/q$ times its original mass (respectively), how are t_p and t_q related?

6. Find the decay constant k for a radioactive substance, given that the mass of the substance is Q_1 at time t_1 and Q_2 at time t_2.

7. A process creates a radioactive substance at the rate of 2 g/hr and the substance decays at a rate proportional to its mass, with constant of proportionality $k=.1(\mathrm{hr})^{-1}$. If $Q(t)$ is the mass of the substance at time t, find $\lim_{t\to\infty} Q(t)$.

8. A bank pays interest continuously at the rate of 6%. How long does it take for a deposit of Q_0 to grow in value to $2Q_0$?

9. At what rate of interest, compounded continuously, will a bank deposit double in value in 8 years?

10. A savings account pays 5% per annum interest compounded continuously. The initial deposit is Q_0 dollars. Assume that there are no subsequent withdrawals or deposits.

a. How long will it take for the value of the account to triple?
b. What is Q_0 if the value of the account after 10 years is $100,000 dollars?

11. A candymaker makes 500 pounds of candy per week, while his large family eats the candy at a rate equal to $Q(t)/10$ pounds per week, where $Q(t)$ is the amount of candy present at time t.

a. Find $Q(t)$ for $t>0$ if the candymaker has 250 pounds of candy at $t=0$.
b. Find $\lim_{t\to\infty} Q(t)$.

12. Suppose a substance decays at a yearly rate equal to half the square of the mass of the substance present. If we start with 50 g of the substance, how long will it be until only 25 g remain?

13. A super bread dough increases in volume at a rate proportional to the volume V present. If V increases by a factor of 10 in 2 hours and $V(0)=V_0$, find V at any time t. How long will it take for V to increase to $100V_0$?

14. A radioactive substance decays at a rate proportional to the amount present, and half the original quantity Q_0 is left after 1500 years. In how many years would the original amount be reduced to $3Q_0/4$? How much will be left after 2000 years?

15. A wizard creates gold continuously at the rate of 1 ounce per hour, but an assistant steals it continuously at the rate of 5% of however much is there per hour. Let $W(t)$ be the number of ounces that the wizard has at time t. Find $W(t)$ and $\lim_{t\to\infty} W(t)$ if $W(0)=1$.

16. A process creates a radioactive substance at the rate of 1 g/hr, and the substance decays at an hourly rate equal to 1/10 of the mass present (expressed in grams). Assuming that there are initially 20 g, find the mass $S(t)$ of the substance present at time t, and find $\lim_{t\to\infty} S(t)$.

17. A tank is empty at $t=0$. Water is added to the tank at the rate of 10 gal/min, but it leaks out at a rate (in gallons per minute) equal to the number of gallons in the tank. What is the smallest capacity the tank can have if this process is to continue forever?

18. A person deposits $25,000 in a bank that pays 5% per year interest, compounded continuously. The person continuously withdraws from the account at the rate of $750 per year. Find $V(t)$, the value of the account at time t after the initial deposit.

19. A person has a fortune that grows at rate proportional to the square root of its worth. Find the worth W of the fortune as a function of t if it was $1 million 6 months ago and is $4 million today.

20. Let $p = p(t)$ be the quantity of a product present at time t. The product is manufactured continuously at a rate proportional to p, with proportionality constant 1/2, and it is consumed continuously at a rate proportional to p^2, with proportionality constant 1/8. Find $p(t)$ if $p(0) = 100$.

21.

a. In the situation of Example 4.1.6 find the exact value P(t) of the person's account after t years, where t is an integer. Assume that each year has exactly 52 weeks, and include the year-end deposit in the computation.

HINT: At time t the initial \$1000 has been on deposit for t years. There have been $52t$ deposits of \$50 each. The first \$50 has been on deposit for $t-1/52$ years, the second for $t-2/52$ years ... in general, the j th \$50 has been on deposit for $t-j/52$ years ($1 \leq j \leq 52t$). Find the present value of each \$50 deposit assuming 6% interest compounded continuously, and use the formula

$$1+x+x^2+\ldots+x^n = \frac{1-x^{n+1}}{1-x}(x \neq 1) \tag{3.1E.1}$$

to find their total value.

b. Let

$$p(t) = \frac{Q(t)-P(t)}{P(t)} \tag{3.1E.2}$$

be the relative error after t years. Find

$$p(\infty) = \lim_{t\to\infty} p(t). \tag{3.1E.3}$$

22. A homebuyer borrows P_0 dollars at an annual interest rate r, agreeing to repay the loan with equal monthly payments of M dollars per month over N years.

a. Derive a differential equation for the loan principal (amount that the homebuyer owes) $P(t)$ at time $t > 0$, making the simplifying assumption that the homebuyer repays the loan continuously rather than in discrete steps. (See Example 4.1.6.)

b. Solve the equation derived in (a).

c. Use the result of (b) to determine an approximate value for M assuming that each year has exactly 12 months of equal length.

d. It can be shown that the exact value of M is given by

$$M = \frac{rP_0}{12}\left(1-(1+r/12)^{-12N}\right)^{-1}. \tag{3.1E.4}$$

Compare the value of M obtained from the answer in (c) to the exact value if (i) $P_0 = \$50,000$, \(r=7{1\over2}\)%, $N = 20$ (ii) $P_0 = \$150,000$, $r = 9.0\%$, $N = 30$.

23. Assume that the homebuyer of *Exercise 4.1.22* elects to repay the loan continuously at the rate of αM dollars per month, where α is a constant greater than 1. (This is called *accelerated payment*.)

a. Determine the time $T(\alpha)$ when the loan will be paid off and the amount $S(\alpha)$ that the homebuyer will save.
b. Suppose $P_0 = \$50,000$, $r = 8\%$, and $N = 15$. Compute the savings realized by accelerated payments with $\alpha = 1.05, 1.10$, and 1.15.

24. A benefactor wishes to establish a trust fund to pay a researcher's salary for T years. The salary is to start at S_0 dollars per year and increase at a fractional rate of a per year. Find the amount of money P_0 that the benefactor must deposit in a trust fund paying interest at a rate r per year. Assume that the researcher's salary is paid continuously, the interest is compounded continuously, and the salary increases are granted continuously.

25. A radioactive substance with decay constant k is produced at the rate of

$$\frac{at}{1+btQ(t)} \tag{3.1E.5}$$

units of mass per unit time, where a and b are positive constants and $Q(t)$ is the mass of the substance present at time t; thus, the rate of production is small at the start and tends to slow when Q is large.

a. Set up a differential equation for Q.

b. Choose your own positive values for a, b, k, and $Q_0 = Q(0)$. Use a numerical method to discover what happens to $Q(t)$ as $t \to \infty$. (Be precise, expressing your conclusions in terms of a, b, k. However, no proof is required.)

26. Follow the instructions of *Exercise 4.1.25*, assuming that the substance is produced at the rate of $at/(1+bt(Q(t))^2)$ units of mass per unit of time.

27. Follow the instructions of *Exercise 4.1.25*, assuming that the substance is produced at the rate of $at/(1+bt)$ units of mass per unit of time.

3.2: Cooling and Mixing

Newton’s Law of Cooling

Newton’s law of cooling states that if an object with temperature $T(t)$ at time t is in a medium with temperature $T_m(t)$, the rate of change of T at time t is proportional to $T(t)-T_m(t)$; thus, T satisfies a differential equation of the form

$$T' = -k(T-T_m). \tag{3.2.1}$$

Here $k>0$, since the temperature of the object must decrease if $T>T_m$, or increase if $T<T_m$. We’ll call k the *temperature decay constant of the medium*.

For simplicity, in this section we’ll assume that the medium is maintained at a constant temperature T_m. This is another example of building a simple mathematical model for a physical phenomenon. Like most mathematical models it has its limitations. For example, it is reasonable to assume that the temperature of a room remains approximately constant if the cooling object is a cup of coffee, but perhaps not if it is a huge cauldron of molten metal. (For more on this see *Exercise 4.2.17*.)

To solve Equation 3.2.1, we rewrite it as

$$T' + kT = kT_m.$$

Since e^{-kt} is a solution of the complementary equation, the solutions of this equation are of the form $T=ue^{-kt}$, where $u'e^{-kt}=kT_m$, so $u'=kT_me^{kt}$. Hence,

$$u = T_me^{kt}+c,$$

so

$$T = ue^{-kt} = T_m + ce^{-kt}.$$

If $T(0)=T_0$, setting $t=0$ here yields $c=T_0-T_m$, so

$$T = T_m + (T_0-T_m)e^{-kt}. \tag{3.2.2}$$

Note that $T-T_m$ decays exponentially, with decay constant k.

Below is a video on Newton's Law of Cooling.

Example 3.2.1

A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C. After 4 minutes the temperature of the insulator is 200°C. What is its temperature after 8 minutes?

Solution

Here $T_0=400$ and $T_m=25$, so Equation 3.2.2 becomes

$$T = 25 + 375e^{-kt}. \tag{3.2.3}$$

We determine k from the stated condition that $T(4) = 200$; that is,

$$200 = 25 + 375e^{-4k};$$

hence,

$$e^{-4k} = \frac{175}{375} = \frac{7}{15}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{4}\ln\frac{7}{15} = \frac{1}{4}\ln\frac{15}{7}.$$

Substituting this into Equation 3.2.3 yields

$$T = 25 + 375e^{-\frac{t}{4}\ln\frac{15}{7}}$$

(Figure 3.2.1). Therefore the temperature of the insulator after 8 minutes is

$$\begin{aligned} T(8) &= 25 + 375e^{-2\ln\frac{15}{7}} \\ &= 25 + 375\left(\tfrac{7}{15}\right)^2 \approx 107^\circ\mathrm{C}. \end{aligned}$$

Example 3.2.2

An object with temperature 72°F is placed outside, where the temperature is -20°F. At 11:05 the temperature of the object is 60°F and at 11:07 its temperature is 50°F. At what time was the object placed outside?

Solution

Let $T(t)$ be the temperature of the object at time t. For convenience, we choose the origin $t_0 = 0$ of the time scale to be 11:05 so that $T_0 = 60$. We must determine the time τ when $T(\tau) = 72$. Substituting $T_0 = 60$ and $T_m = -20$ into Equation 3.2.2 yields

$$T = -20 + \big(60 - (-20)\big)e^{-kt}$$

or

$$T = -20 + 80e^{-kt}. \tag{3.2.4}$$

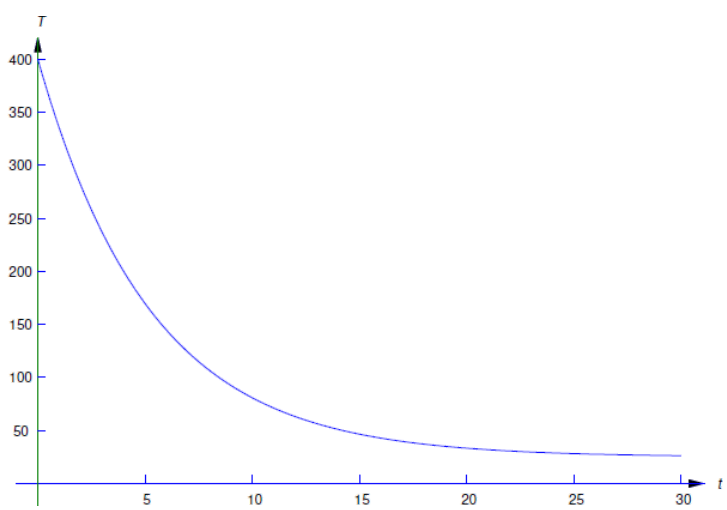


Figure 3.2.1 : $T = 25 + 375e^{-(t/4)\ln 15/7}$

We obtain k from the stated condition that the temperature of the object is 50°F at 11:07. Since 11:07 is $t=2$ on our time scale, we can determine k by substituting $T=50$ and $t=2$ into Equation 3.2.4 to obtain

$$50=-20+80e^{-2k}$$

This is shown in Figure 3.2.2 .

Hence,

$$e^{-2k}=\frac{70}{80}=\frac{7}{8}.$$

Taking logarithms and solving for k yields

$$k=-\frac{1}{2}\ln\frac{7}{8}=\frac{1}{2}\ln\frac{8}{7}.$$

Substituting this into Equation 3.2.4 yields

$$T=-20+80e^{-\frac{t}{2}\ln\frac{8}{7}},$$

and the condition $T(\tau)=72$ implies that

$$72=-20+80e^{-\frac{\tau}{2}\ln\frac{8}{7}};$$

hence,

$$e^{-\frac{\tau}{2}\ln\frac{8}{7}}=\frac{92}{80}=\frac{23}{20}.$$

Taking logarithms and solving for τ yields

$$\tau=-\frac{2\ln\frac{23}{20}}{\ln\frac{8}{7}}\approx-2.09\text{ min}.$$

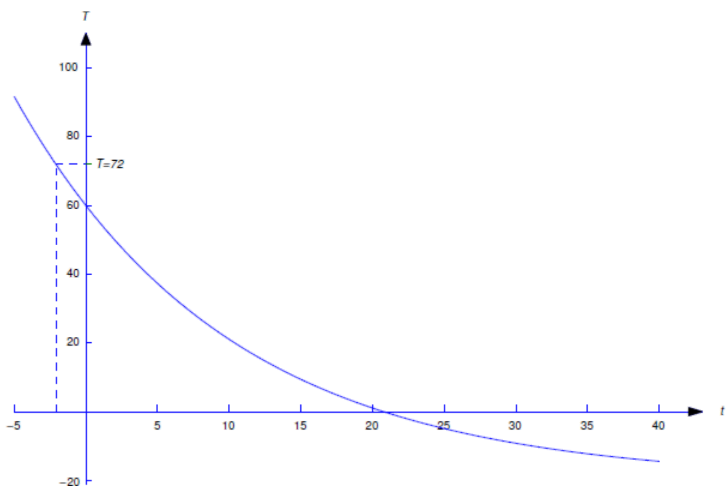


Figure 3.2.2 : $-20+80e^{-\frac{t}{2}\ln\frac{8}{7}}$

Therefore the object was placed outside about 2 minutes and 5 seconds before 11:05; that is, at 11:02:55.

Mixing Problems

In the next two examples a saltwater solution with a given concentration (weight of salt per unit volume of solution) is added at a specified rate to a tank that initially contains saltwater with a different concentration. The problem is to determine the quantity of salt in the tank as a function of time. This is an example of a *mixing problem*. To construct a tractable mathematical model for mixing problems we assume in our examples (and most exercises) that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. *Exercises 4.2.22* and *4.2.23* deal with situations where this isn't so, but the distribution of salt becomes approximately uniform as $t\to\infty$.

Example 3.2.3

A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains 1/2 pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate (Figure 3.2.3).

a. Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.
b. Find $\lim_{t\to\infty} Q(t)$.

Solution a

To find a differential equation for Q, we must use the given information to derive an expression for Q'. But Q' is the rate of change of the quantity of salt in the tank changes with respect to time; thus, if *rate in* denotes the rate at which salt enters the tank and *rate out* denotes the rate by which it leaves, then

$$Q' = \text{rate in} - \text{rate out}. \tag{3.2.5}$$

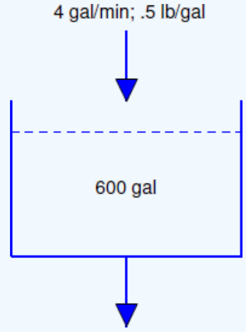


Figure 3.2.3 : A mixing problem

The rate in is

$$\left(\frac{1}{2}\ \text{lb/gal}\right) \times (4\ \text{gal/min}) = 2\ \text{lb/min}.$$

Determining the rate out requires a little more thought. We're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank; that is, we are removing $1/150$ of the mixture per minute. Since the salt is evenly distributed in the mixture, we are also removing $1/150$ of the salt per minute. Therefore, if there are $Q(t)$ pounds of salt in the tank at time t, the rate out at any time t is $Q(t)/150$. Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned} \text{rate out } &= (\text{concentration}) \times (\text{rate of flow out}) \\ &= (\text{lb/gal}) \times (\text{gal/min}) \\ &= \frac{Q(t)}{600} \times 4 \\ &= \frac{Q(t)}{150}. \end{aligned}$$

We can now write Equation 3.2.5 as

$$Q' = 2 - \frac{Q}{150}.$$

This first order equation can be rewritten as

$$Q' + \frac{Q}{150} = 2.$$

Since $e^{-t/150}$ is a solution of the complementary equation, the solutions of this equation are of the form $Q = ue^{-t/150}$, where $u'e^{-t/150} = 2$, so $u' = 2e^{t/150}$. Hence,

$$u = 300e^{t/150} + c,$$

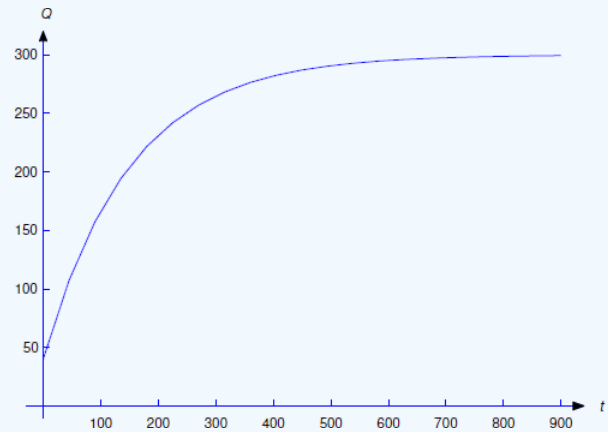


Figure 3.2.4 : $Q = 300 - 260e^{-t/150}$

so

$$Q = ue^{-t/150} = 300 + ce^{-t/150} \tag{3.2.6}$$

(Figure 3.2.4). Since $Q(0) = 40$, $c = -260$; therefore,

$$Q = 300 - 260e^{-t/150}.$$

Solution b

From Equation 3.2.6, we see that that $\lim_{t\to\infty} Q(t) = 300$ for any value of $Q(0)$. This is intuitively reasonable, since the incoming solution contains 1/2 pound of salt per gallon and there are always 600 gallons of water in the tank.

Below is a video on mixing.

Example 3.2.4

A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Starting at $t_0 = 0$, water that contains 1/4 g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min (Figure [figure:4.2.5}). Find a differential equation for the quantity $Q(t)$ of salt in the tank at time t prior to the time when the tank overflows and find the concentration $K(t)$ (g/liter) of salt in the tank at any such time.

Solution

We first determine the amount $W(t)$ of solution in the tank at any time t prior to overflow. Since $W(0) = 200$ and we are adding 4 liters/min while removing only 2 liters/min, there's a net gain of 2 liters/min in the tank; therefore,

$$W(t) = 2t + 200.$$

Since $W(150) = 500$ liters (capacity of the tank), this formula is valid for $0 \leq t \leq 150$.

Now let $Q(t)$ be the number of grams of salt in the tank at time t, where $0 \leq t \leq 150$. As in Example 3.2.3

$$Q' = \text{rate in} - \text{rate out}. \tag{3.2.7}$$

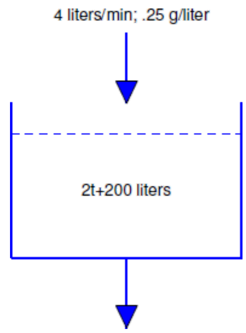


Figure 3.2.5 : Another mixing problem

The rate in is

$$\left(\frac{1}{4}\ \text{g}/\text{liter}\right) \times (4\ \text{liters}/\text{min}) = 1\ \text{g}/\text{min}. \tag{3.2.8}$$

To determine the rate out, we observe that since the mixture is being removed from the tank at the constant rate of 2 liters/min and there are $2t + 200$ liters in the tank at time t, the fraction of the mixture being removed per minute at time t is

$$\frac{2}{2t+200} = \frac{1}{t+100}.$$

We're removing this same fraction of the salt per minute. Therefore, since there are $Q(t)$ grams of salt in the tank at time t,

$$\text{rate out} = \frac{Q(t)}{t+100}. \tag{3.2.9}$$

Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned} \text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) = (\text{g}/\text{liter}) \times (\text{liters}/\text{min}) \\ &= \tfrac{Q(t)}{2t+200} \times 2 = \tfrac{Q(t)}{t+100}. \end{aligned}$$

Substituting Equation 3.2.8 and Equation 3.2.9 into Equation 3.2.7 yields

$$Q' = 1 - \frac{Q}{t+100}, \quad \text{so} \quad Q' + \frac{1}{t+100}Q = 1. \tag{3.2.10}$$

By separation of variables, $1/(t+100)$ is a solution of the complementary equation, so the solutions of Equation 3.2.10 are of the form

$$Q = \frac{u}{t+100}, \quad \text{where} \quad \frac{u'}{t+100=1}, \quad \text{so} \quad u' = t + 100.$$

Hence,

$$u = \frac{(t+100)^2}{2} + c. \tag{3.2.11}$$

Since $Q(0) = 10$ and $u = (t+100)Q$, Equation 3.2.11 implies that

$$(100)(10) = \frac{(100)^2}{2} + c,$$

so

$$c = 100(10) - \frac{(100)^2}{2} = -4000$$

and therefore

$$u = \frac{(t+100)^2}{2} - 4000.$$

Hence,

$$Q = \frac{u}{t+200} = \frac{t+100}{2} - \frac{4000}{t+100}.$$

Now let $K(t)$ be the concentration of salt at time t. Then

$$K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$$

This is shown in Figure 3.2.6 .

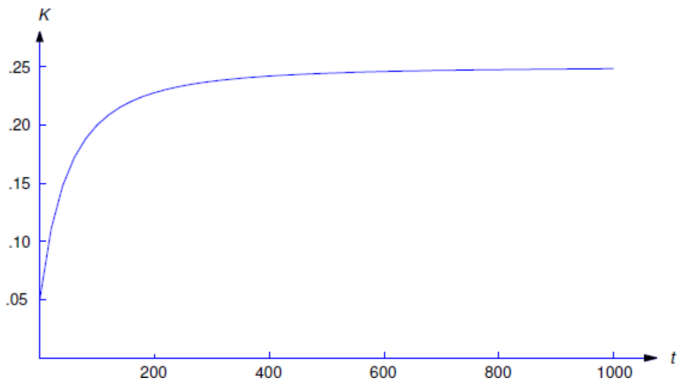


Figure 3.2.6 : $K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$

Below is a video on salt in a tank and differential equations.

3.2E: Cooling and Mixing (Exercises)

Q4.2.1

1. A thermometer is moved from a room where the temperature is 70°F to a freezer where the temperature is $12^{\circ}F$. After 30 seconds the thermometer reads 40°F. What does it read after 2 minutes?

2. A fluid initially at 100°C is placed outside on a day when the temperature is -10°C, and the temperature of the fluid drops 20°C in one minute. Find the temperature $T(t)$ of the fluid for $t>0$.

3. At 12:00 pm a thermometer reading 10°F is placed in a room where the temperature is 70°F. It reads 56° when it is placed outside, where the temperature is 5°F, at 12:03. What does it read at 12:05 pm?

4. A thermometer initially reading 212°F is placed in a room where the temperature is 70°F. After 2 minutes the thermometer reads 125°F.

a. What does the thermometer read after 4 minutes?
b. When will the thermometer read 72°F?
c. When will the thermometer read 69°F?

5. An object with initial temperature 150°C is placed outside, where the temperature is 35°C. Its temperatures at 12:15 and 12:20 are 120°C and 90°C, respectively.

a. At what time was the object placed outside?
b. When will its temperature be 40°C?

6. An object is placed in a room where the temperature is 20°C. The temperature of the object drops by 5°C in 4 minutes and by 7°C in 8 minutes. What was the temperature of the object when it was initially placed in the room?

7. A cup of boiling water is placed outside at 1:00 pm. One minute later the temperature of the water is 152°F. After another minute its temperature is 112°F. What is the outside temperature?

8. A tank initially contains 40 gallons of pure water. A solution with 1 gram of salt per gallon of water is added to the tank at 3 gal/min, and the resulting solution drains out at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t>0$.

9. A tank initially contains a solution of 10 pounds of salt in 60 gallons of water. Water with $1/2$ pound of salt per gallon is added to the tank at 6 gal/min, and the resulting solution leaves at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t>0$.

10. A tank initially contains 100 liters of a salt solution with a concentration of $.1$ g/liter. A solution with a salt concentration of $.3$ g/liter is added to the tank at 5 liters/min, and the resulting mixture is drained out at the same rate. Find the concentration $K(t)$ of salt in the tank as a function of t.

11. A 200 gallon tank initially contains 100 gallons of water with 20 pounds of salt. A salt solution with $1/4$ pound of salt per gallon is added to the tank at 4 gal/min, and the resulting mixture is drained out at 2 gal/min. Find the quantity of salt in the tank as it is about to overflow.

12. Suppose water is added to a tank at 10 gal/min, but leaks out at the rate of $1/5$ gal/min for each gallon in the tank. What is the smallest capacity the tank can have if the process is to continue indefinitely?

13. A chemical reaction in a laboratory with volume V (in ft^3) produces q_1 ft^3/min of a noxious gas as a byproduct. The gas is dangerous at concentrations greater than $\overline{c}$, but harmless at concentrations $\le \overline{c}$. Intake fans at one end of the laboratory pull in fresh air at the rate of q_2 ft^3/min and exhaust fans at the other end exhaust the mixture of gas and air from the laboratory at the same rate. Assuming that the gas is always uniformly distributed in the room and its initial concentration c_0 is at a safe level, find the smallest value of q_2 required to maintain safe conditions in the laboratory for all time.

14. A 1200-gallon tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0=0$, water that contains $1/2$ pound of salt per gallon is added to the tank at the rate of 6 gal/min and the resulting mixture is drained from the tank at 4 gal/min. Find the quantity $Q(t)$ of salt in the tank at any time $t>0$ prior to overflow.

15. Tank T_1 initially contain 50 gallons of pure water. Starting at $t_0=0$, water that contains 1 pound of salt per gallon is poured into T_1 at the rate of 2 gal/min. The mixture is drained from T_1 at the same rate into a second tank T_2, which initially contains 50

gallons of pure water. Also starting at $t_0=0$, a mixture from another source that contains 2 pounds of salt per gallon is poured into T_2 at the rate of 2 gal/min. The mixture is drained from T_2 at the rate of 4 gal/min.

a. Find a differential equation for the quantity $Q(t)$ of salt in tank T_2 at time $t>0$.
b. Solve the equation derived in (a) to determine $Q(t)$.
c. Find $\lim_{t\to\infty} Q(t)$.

16. Suppose an object with initial temperature T_0 is placed in a sealed container, which is in turn placed in a medium with temperature T_m. Let the initial temperature of the container be S_0. Assume that the temperature of the object does not affect the temperature of the container, which in turn does not affect the temperature of the medium. (These assumptions are reasonable, for example, if the object is a cup of coffee, the container is a house, and the medium is the atmosphere.)

a. Assuming that the container and the medium have distinct temperature decay constants k and k_m respectively, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t.
b. Assuming that the container and the medium have the same temperature decay constant k, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t.
c. Find $\lim._{t\to\infty} S(t)$ and $\lim_{t\to\infty} T(t)$.

17. In our previous examples and exercises concerning Newton's law of cooling we assumed that the temperature of the medium remains constant. This model is adequate if the heat lost or gained by the object is insignificant compared to the heat required to cause an appreciable change in the temperature of the medium. If this isn't so, we must use a model that accounts for the heat exchanged between the object and the medium. Let $T=T(t)$ and $T_m=T_m(t)$ be the temperatures of the object and the medium, respectively, and let T_0 and T_{m0} be their initial values. Again, we assume that T and T_m are related by Newton's law of cooling,

$$T'=-k(T-T_m). \tag{A}$$

We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T-T_0)$ and that the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m-T_{m0})$, where a and a_m are positive constants depending upon the masses and thermal properties of the object and medium, respectively. If we assume that the total heat of the system consisting of the object and the medium remains constant (that is, energy is conserved), then

$$a(T-T_0)+a_m(T_m-T_{m0})=0. \tag{B}$$

a. Equation (A) involves two unknown functions T and T_m. Use (A) and (B) to derive a differential equation involving only T.
b. Find $T(t)$ and $T_m(t)$ for $t>0$.
c. Find $\lim_{t\to\infty} T(t)$ and $\lim_{t\to\infty} T_m(t)$.

18. Control mechanisms allow fluid to flow into a tank at a rate proportional to the volume V of fluid in the tank, and to flow out at a rate proportional to V^2. Suppose $V(0)=V_0$ and the constants of proportionality are a and b, respectively. Find $V(t)$ for $t>0$ and find $\lim_{t\to\infty} V(t)$.

19. Identical tanks T_1 and T_2 initially contain W gallons each of pure water. Starting at $t_0=0$, a salt solution with constant concentration c is pumped into T_1 at r gal/min and drained from T_1 into T_2 at the same rate. The resulting mixture in T_2 is also drained at the same rate. Find the concentrations $c_1(t)$ and $c_2(t)$ in tanks T_1 and T_2 for $t>0$.

20. An infinite sequence of identical tanks T_1, T_2, …, T_n, …, initially contain W gallons each of pure water. They are hooked together so that fluid drains from T_n into T_{n+1} $(n=1,2,\cdots)$. A salt solution is circulated through the tanks so that it enters and leaves each tank at the constant rate of r gal/min. The solution has a concentration of c pounds of salt per gallon when it enters T_1.

a. Find the concentration $c_n(t)$ in tank T_n for $t>0$.
b. Find $\lim_{t\to\infty} c_n(t)$ for each n.

21. Tanks T_1 and T_2 have capacities W_1 and W_2 liters, respectively. Initially they are both full of dye solutions with concentrations c_1 and c_2 grams per liter. Starting at $t_0=0$, the solution from T_1 is pumped into T_2 at a rate of r liters per minute, and the solution from T_2 is pumped into T_1 at the same rate.

a. Find the concentrations $c_1(t)$ and $c_2(t)$ of the dye in T_1 and T_2 for $t>0$.
b. Find $\lim_{t\to\infty} c_1(t)$ and $\lim_{t\to\infty} c_2(t)$.

22. Consider the mixing problem of Example 4.2.3, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t\to\infty$. In this

case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{150}Q = 2 \tag{3.2E.1}$$

where $\lim_{t\to\infty} a(t) = 1$.

a. Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t\to\infty} Q(t)$?.
b. Use numerical methods to confirm your guess in the these cases:

$$\text{(i) } a(t) = t/(1+t) \quad \text{(ii) } a(t) = 1 - e^{-t^2} \quad \text{(iii) } a(t) = 1 - \sin(e^{-t}). \tag{3.2E.2}$$

23. Consider the mixing problem of Example 4.2.4 in a tank with infinite capacity, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \to \infty$. In this case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{t+100}Q = 1 \tag{3.2E.3}$$

where $\lim_{t\to\infty} a(t) = 1$.

a. Let $K(t)$ be the concentration of salt at time t. Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t\to\infty} K(t)$?
b. Use numerical methods to confirm your guess in the these cases:

$$\text{(i) } a(t) = t/(1+t) \quad \text{(ii) } a(t) = 1 - e^{-t^2} \quad \text{(iii) } a(t) = 1 + \sin(e^{-t}). \tag{3.2E.4}$$

3.3: Elementary Mechanics

Newton's Second Law of Motion

In this section we consider an object with constant mass m moving along a line under a force F. Let $y = y(t)$ be the displacement of the object from a reference point on the line at time t, and let $v = v(t)$ and $a = a(t)$ be the velocity and acceleration of the object at time t. Thus, $v = y'$ and $a = v' = y''$, where the prime denotes differentiation with respect to t. Newton's second law of motion asserts that the force F and the acceleration a are related by the equation

$$F = ma. \tag{3.3.1}$$

Note: Units

In applications there are three main sets of units in use for length, mass, force, and time: the cgs, mks, and British systems. All three use the second as the unit of time. Table 3.3.1 shows the other units. Consistent with Equation 3.3.1, the unit of force in each system is defined to be the force required to impart an acceleration of (one unit of length)$/s^2$ to one unit of mass.

Table 3.3.1

Set	Length	Force	Mass
cgs	centimeter (cm)	dyne (d)	gram (g)
mks	meter (m)	newton (N)	kilogram (kg)
British	foot (ft)	pound (lb)	slug (sl)

If we assume that Earth is a perfect sphere with constant mass density, Newton's law of gravitation (discussed later in this section) asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth. However, if the object remains sufficiently close to Earth's surface, we may assume that the gravitational force is constant and equal to its value at the surface. The magnitude of this force is mg, where g is called the *acceleration due to gravity*. (To be completely accurate, g should be called the *magnitude of the acceleration due to gravity at Earth's surface*.) This quantity has been determined experimentally. Approximate values of g are

$$\begin{aligned} g &= 980 \text{ cm/s}^2 && \text{(cgs)} \\ g &= 9.8 \text{ m/s}^2 && \text{(mks)} \\ g &= 32 \text{ ft/s}^2 && \text{(British).} \end{aligned}$$

In general, the force F in Equation 3.3.1 may depend upon t, y, and y'. Since $a = y''$, Equation 3.3.1 can be written in the form

$$my'' = F(t, y, y'), \tag{3.3.2}$$

which is a second order equation. We'll consider this equation with restrictions on F later; however, since Chapter 2 dealt only with first order equations, we consider here only problems in which Equation 3.3.2 can be recast as a first order equation. This is possible if F does not depend on y, so Equation 3.3.2 is of the form

$$my'' = F(t, y').$$

Letting $v = y'$ and $v' = y''$ yields a first order equation for v:

$$mv' = F(t, v). \tag{3.3.3}$$

Solving this equation yields v as a function of t. If we know $y(t_0)$ for some time t_0, we can integrate v to obtain y as a function of t.

Equations of the form Equation 3.3.3 occur in problems involving motion through a resisting medium.

Motion Through a Resisting Medium Under Constant Gravitational Force

Now we consider an object moving vertically in some medium. We assume that the only forces acting on the object are gravity and resistance from the medium. We also assume that the motion takes place close to Earth's surface and take the upward direction to be positive, so the gravitational force can be assumed to have the constant value $-mg$. We'll see that, under reasonable assumptions on the resisting force, the velocity approaches a limit as $t \to \infty$. We call this limit the *terminal velocity*.

Example 3.3.1

An object with mass m moves under constant gravitational force through a medium that exerts a resistance with magnitude proportional to the speed of the object. (Recall that the speed of an object is $|v|$, the absolute value of its velocity v.) Find the velocity of the object as a function of t, and find the terminal velocity. Assume that the initial velocity is v_0.

Solution

The total force acting on the object is

$$F = -mg + F_1, \tag{3.3.4}$$

where $-mg$ is the force due to gravity and F_1 is the resisting force of the medium, which has magnitude $k|v|$, where k is a positive constant. If the object is moving downward ($v \le 0$), the resisting force is upward (Figure 3.3.1a), so

$$F_1 = k|v| = k(-v) = -kv.$$

On the other hand, if the object is moving upward ($v \ge 0$), the resisting force is downward (Figure 3.3.1b , so

$$F_1 = -k|v| = -kv.$$

Thus, Equation 3.3.4 can be written as

$$F = -mg - kv, \tag{3.3.5}$$

regardless of the sign of the velocity.

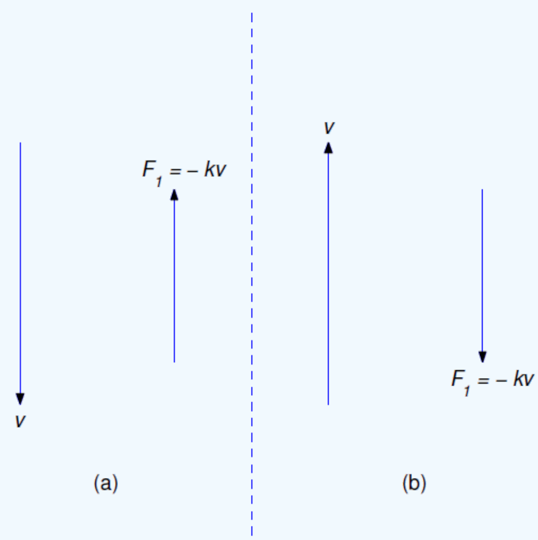


Figure 3.3.1 : Resistive forces

From Newton's second law of motion,

$$F = ma = mv',$$

so Equation 3.3.5 yields

$$mv' = -mg - kv,$$

or

$$v' + \frac{k}{m}v = -g. \tag{3.3.6}$$

Since $e^{-kt/m}$ is a solution of the complementary equation, the solutions of Equation 3.3.6 are of the form $v = ue^{-kt/m}$, where $u'e^{-kt/m} = -g$, so $u' = -ge^{kt/m}$. Hence,

$$u = -\frac{mg}{k}e^{kt/m} + c,$$

so

$$v = ue^{-kt/m} = -\frac{mg}{k} + ce^{-kt/m}. \tag{3.3.7}$$

Since $v(0) = v_0$,

$$v_0 = -\frac{mg}{k} + c,$$

so

$$c = v_0 + \frac{mg}{k}$$

and Equation 3.3.7 becomes

$$v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

Letting $t \to \infty$ here shows that the terminal velocity is

$$\lim_{t\to\infty} v(t) = -\frac{mg}{k},$$

which is independent of the initial velocity v_0 (Figure 3.3.2).

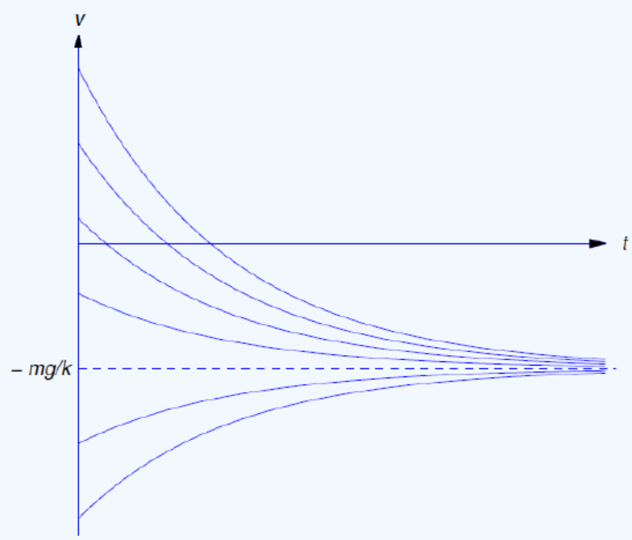


Figure 3.3.2 : Solutions of $mv' = -mg - kv$

✔ Example 3.3.2 : Terminal Velocity

A 960-lb object is given an initial upward velocity of 60 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 3 lb for each ft/s of speed. Assuming that the only other force acting on the object is constant gravity, find its velocity v as a function of t, and find its terminal velocity.

Solution

Since $mg = 960$ and $g = 32$, $m = 960/32 = 30$. The atmospheric resistance is $-3v$ lb if v is expressed in feet per second. Therefore

$$30v' = -960 - 3v,$$

which we rewrite as

$$v' + \frac{1}{10}v = -32.$$

Since $e^{-t/10}$ is a solution of the complementary equation, the solutions of this equation are of the form $v = ue^{-t/10}$, where $u'e^{-t/10} = -32$, so $u' = -32e^{t/10}$. Hence,

$$u = -320e^{t/10} + c,$$

so

$$v = ue^{-t/10} = -320 + ce^{-t/10}. \tag{3.3.8}$$

The initial velocity is 60 ft/s in the upward (positive) direction; hence, $v_0 = 60$. Substituting $t = 0$ and $v = 60$ in Equation 3.3.8 yields

$$60 = -320 + c,$$

so $c = 380$, and Equation 3.3.8 becomes

$$v = -320 + 380e^{-t/10} \text{ ft/s}$$

The terminal velocity is

$$\lim_{t\to\infty} v(t) = -320 \text{ ft/s}.$$

Example 3.3.3

A 10 kg mass is given an initial velocity $v_0 \le 0$ near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 N if the speed is 2 m/s, find the velocity of the object as a function of t, and find the terminal velocity.

Solution

Since the object is falling, the resistance is in the upward (positive) direction. Hence,

$$mv' = -mg + kv^2, \tag{3.3.9}$$

where k is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s,

$$k(2^2) = 8,$$

so $k = 2$ N-s^2/m^2. Since $m = 10$ and $g = 9.8$, Equation 3.3.9 becomes

$$10v' = -98 + 2v^2 = 2(v^2 - 49). \tag{3.3.10}$$

If $v_0 = -7$, then $v \equiv -7$ for all $t \ge 0$. If $v_0 \ne -7$, we separate variables to obtain

$$\frac{1}{v^2 - 49}v' = \frac{1}{5}, \tag{3.3.11}$$

which is convenient for the required partial fraction expansion

$$\frac{1}{v^2 - 49} = \frac{1}{(v-7)(v+7)} = \frac{1}{14}\left[\frac{1}{v-7} - \frac{1}{v+7}\right]. \tag{3.3.12}$$

Substituting Equation 3.3.12 into Equation 3.3.11 yields

$$\frac{1}{14}\left[\frac{1}{v-7}-\frac{1}{v+7}\right]v'=\frac{1}{5},$$

so

$$\left[\frac{1}{v-7}-\frac{1}{v+7}\right]v'=\frac{14}{5}.$$

Integrating this yields

$$\ln|v-7|-\ln|v+7|=14t/5+k.$$

Therefore

$$\left|\frac{v-7}{v+7}\right|=e^ke^{14t/5}.$$

Since Theorem 2.3.1 implies that $(v-7)/(v+7)$ cannot change sign (why?), we can rewrite the last equation as

$$\frac{v-7}{v+7}=ce^{14t/5}, \tag{3.3.13}$$

which is an implicit solution of Equation 3.3.10. Solving this for v yields

$$v=-7\frac{c+e^{-14t/5}}{c-e^{-14t/5}}. \tag{3.3.14}$$

Since $v(0)=v_0$, it Equation 3.3.13 implies that

$$c=\frac{v_0-7}{v_0+7}.$$

Substituting this into Equation 3.3.14 and simplifying yields

$$v=-7\frac{v_0(1+e^{-14t/5}-7(1-e^{-14t/5}}{v_0(1-e^{-14t/5}-7(1+e^{-14t/5}}.$$

Since $v_0\le 0$, v is defined and negative for all $t>0$. The terminal velocity is

$$\lim_{t\to\infty}v(t)=-7\text{ m/s},$$

independent of v_0. More generally, it can be shown (*Exercise 4.3.11*) that if v is any solution of Equation 3.3.9 such that $v_0\le 0$ then

$$\lim_{t\to\infty}v(t)=-\sqrt{\frac{mg}{k}}.$$

This is demonstrated in Figure 3.3.3 .

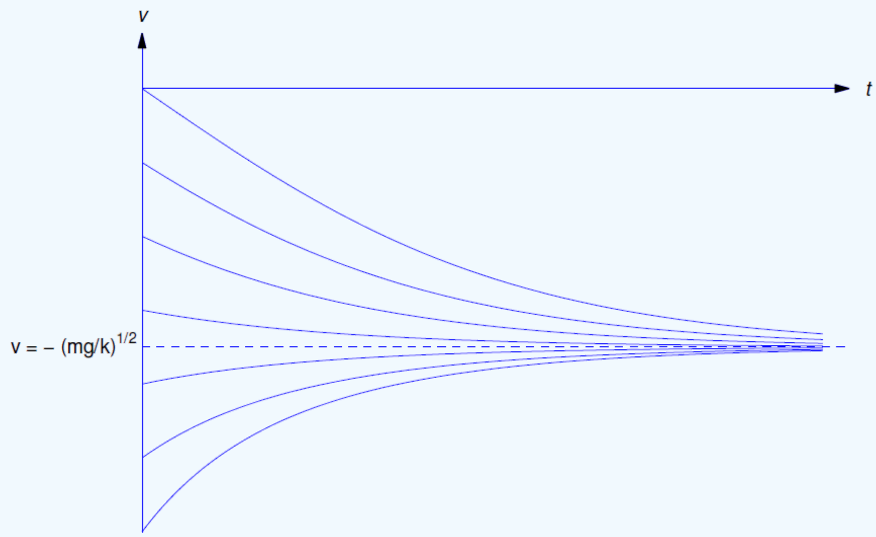


Figure 3.3.3 : Solutions of $mv' = -mg + kv^2,\ v(0) = v_0 \le 0$

Example 3.3.4

A 10-kg mass is launched vertically upward from Earth's surface with an initial velocity of v_0 m/s. The only forces acting on the mass are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the atmospheric resistance is 8 N if the speed is 2 m/s, find the time T required for the mass to reach maximum altitude.

Solution

The mass will climb while $v > 0$ and reach its maximum altitude when $v = 0$. Therefore $v > 0$ for $0 \le t < T$ and $v(T) = 0$; therefore, we replace Equation 3.3.10 by

$$10v' = -98 - 2v^2. \tag{3.3.15}$$

Separating variables yields

$$\frac{5}{v^2+49}v' = -1,$$

and integrating this yields

$$\frac{5}{7}\tan^{-1}\frac{v}{7} = -t + c.$$

(Recall that $\tan^{-1} u$ is the number θ such that $-\pi/2 < \theta < \pi/2$ and $\tan\theta = u$.) Since $v(0) = v_0$,

$$c = \frac{5}{7}\tan^{-1}\frac{v_0}{7},$$

so v is defined implicitly by

$$\frac{5}{7}\tan^{-1}\frac{v}{7} = -t + \frac{5}{7}\tan^{-1}\frac{v_0}{7}, \quad 0 \le t \le T. \tag{3.3.16}$$

Solving this for v yields

$$v = 7\tan\left(-\frac{7t}{5} + \tan^{-1}\frac{v_0}{7}\right). \tag{3.3.17}$$

Using the identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

with $A = \tan^{-1}(v_0/7)$ and $B = 7t/5$, and noting that $\tan(\tan^{-1}\theta) = \theta$, we can simplify Equation 3.3.17 to

$$v = 7\frac{v_0 - 7\tan(7t/5)}{7 + v_0\tan(7t/5)}.$$

Since $v(T) = 0$ and $\tan^{-1}(0) = 0$, Equation 3.3.16 implies that

$$-T + \frac{5}{7}\tan^{-1}\frac{v_0}{7} = 0.$$

Therefore

$$T = \frac{5}{7}\tan^{-1}\frac{v_0}{7}.$$

Since $\tan^{-1}(v_0/7) < \pi/2$ for all v_0, the time required for the mass to reach its maximum altitude is less than

$$\frac{5\pi}{14} \approx 1.122 \text{ s}$$

regardless of the initial velocity. Figure 3.3.4 shows graphs of v over $[0, T]$ for various values of v_0.

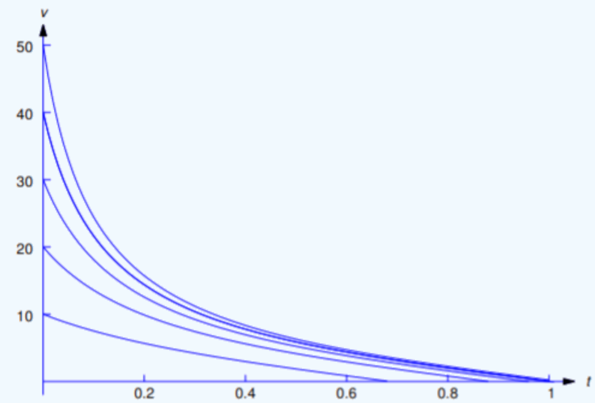


Figure 3.3.4 : Solutions of 3.3.15 for various $v_0 > 0$

Escape Velocity

Suppose a space vehicle is launched vertically and its fuel is exhausted when the vehicle reaches an altitude h above Earth, where h is sufficiently large so that resistance due to Earth's atmosphere can be neglected. Let $t = 0$ be the time when burnout occurs. Assuming that the gravitational forces of all other celestial bodies can be neglected, the motion of the vehicle for $t > 0$ is that of an object with constant mass m under the influence of Earth's gravitational force, which we now assume to vary inversely with the square of the distance from Earth's center; thus, if we take the upward direction to be positive then gravitational force on the vehicle at an altitude y above Earth is

$$F = -\frac{K}{(y+R)^2}, \tag{3.3.18}$$

where R is Earth's radius (Figure 3.3.5).

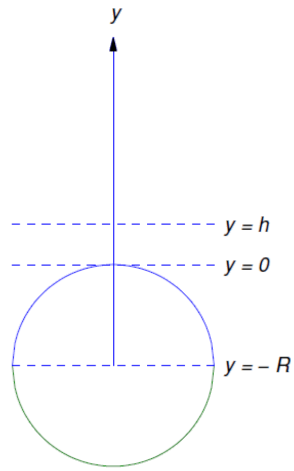


Figure 3.3.5 : Escape velocity

Since $F=-mg$ when $y=0$, setting $y=0$ in Equation 3.3.18 yields

$$-mg=-\frac{K}{R^2};$$

therefore $K=mgR^2$ and Equation 3.3.18 can be written more specifically as

$$F=-\frac{mgR^2}{(y+R)^2}. \tag{3.3.19}$$

From Newton's second law of motion,

$$F=m\frac{d^2y}{dt^2},$$

so Equation 3.3.19 implies that

$$\frac{d^2y}{dt^2}=-\frac{gR^2}{(y+R)^2}. \tag{3.3.20}$$

We'll show that there's a number v_e, called the *escape velocity*, with these properties:

1. If $v_0\ge v_e$ then $v(t)>0$ for all $t>0$, and the vehicle continues to climb for all $t>0$; that is, it "escapes" Earth. (Is it really so obvious that $\lim_{t\to\infty} y(t)=\infty$ in this case? For a proof, see *Exercise 4.3.20*.)
2. If $v_0<v_e$ then $v(t)$ decreases to zero and becomes negative. Therefore the vehicle attains a maximum altitude y_m and falls back to Earth.

Since Equation 3.3.20 is second order, we cannot solve it by methods discussed so far. However, we are concerned with v rather than y, and v is easier to find. Since $v=y'$ the chain rule implies that

$$\frac{d^2y}{dt^2}=\frac{dv}{dt}=\frac{dv}{dy}\frac{dy}{dt}=v\frac{dv}{dy}.$$

Substituting this into Equation 3.3.20 yields the first order separable equation

$$v\frac{dv}{dy}=-\frac{gR^2}{(y+R)^2}. \tag{3.3.21}$$

When $t=0$, the velocity is v_0 and the altitude is h. Therefore we can obtain v as a function of y by solving the initial value problem

$$v\frac{dv}{dy}=-\frac{gR^2}{(y+R)^2}, \quad v(h)=v_0.$$

Integrating Equation 3.3.21 with respect to y yields

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + c. \tag{3.3.22}$$

Since $v(h) = v_0$,

$$c = \frac{v_0^2}{2} - \frac{gR^2}{h+R},$$

so Equation 3.3.22 becomes

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right). \tag{3.3.23}$$

If

$$v_0 \geq \left(\frac{2gR^2}{h+R}\right)^{1/2},$$

the parenthetical expression in Equation 3.3.23 is nonnegative, so $v(y) > 0$ for $y > h$. This proves that there's an escape velocity v_e. We'll now prove that

$$v_e = \left(\frac{2gR^2}{h+R}\right)^{1/2}$$

by showing that the vehicle falls back to Earth if

$$v_0 < \left(\frac{2gR^2}{h+R}\right)^{1/2}. \tag{3.3.24}$$

If Equation 3.3.24 holds then the parenthetical expression in Equation 3.3.23 is negative and the vehicle will attain a maximum altitude $y_m > h$ that satisfies the equation

$$0 = \frac{gR^2}{y_m+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right).$$

The velocity will be zero at the maximum altitude, and the object will then fall to Earth under the influence of gravity.

Below is a video on solving a differential equation that models a falling object.

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3.3E: Elementary Mechanics (Exercises)

Except where directed otherwise, assume that the magnitude of the gravitational force on an object with mass m is constant and equal to mg. In exercises involving vertical motion take the upward direction to be positive.

Q4.3.1

1. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to his speed, with $k=2.5$ lb-s/ft. Assuming that he starts from rest, find his velocity as a function of time and find his terminal velocity.

2. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to her speed, with constant of proportionality k. Find k, given that her terminal velocity is -16 ft/s, and then find her velocity v as a function of t. Assume that she starts from rest.

3. A boat weighs $64,000$ lb. Its propellor produces a constant thrust of $50,000$ lb and the water exerts a resistive force with magnitude proportional to the speed, with $k=2000$ lb-s/ft. Assuming that the boat starts from rest, find its velocity as a function of time, and find its terminal velocity.

4. A constant horizontal force of 10 N pushes a 20 kg-mass through a medium that resists its motion with $.5$ N for every m/s of speed. The initial velocity of the mass is 7 m/s in the direction opposite to the direction of the applied force. Find the velocity of the mass for $t>0$.

5. A stone weighing $1/2$ lb is thrown upward from an initial height of 5 ft with an initial speed of 32 ft/s. Air resistance is proportional to speed, with $k=1/128$ lb-s/ft. Find the maximum height attained by the stone.

6. A 3200-lb car is moving at 64 ft/s down a 30-degree grade when it runs out of fuel. Find its velocity after that if friction exerts a resistive force with magnitude proportional to the square of the speed, with $k=1$ lb-s$^2/ft^2$. Also find its terminal velocity.

7. A 96 lb weight is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the speed. Find its velocity as a function of time if its terminal velocity is -128 ft/s.

8. An object with mass m moves vertically through a medium that exerts a resistive force with magnitude proportional to the speed. Let $y=y(t)$ be the altitude of the object at time t, with $y(0)=y_0$. Use the results of Example 4.3.1 to show that

$$y(t)=y_0+\frac{m}{k}(v_0-v-gt).$$

9. An object with mass m is launched vertically upward with initial velocity v_0 from Earth's surface ($y_0=0$) in a medium that exerts a resistive force with magnitude proportional to the speed. Find the time T when the object attains its maximum altitude y_m. Then use the result of *Exercise 4.3.8* to find y_m.

10. An object weighing 256 lb is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the square of the speed. The magnitude of the resisting force is 1 lb when $|v|=4$ ft$/$s. Find v for $t>0$, and find its terminal velocity.

11. An object with mass m is given an initial velocity $v_0\le 0$ in a medium that exerts a resistive force with magnitude proportional to the square of the speed. Find the velocity of the object for $t>0$, and find its terminal velocity.

12. An object with mass m is launched vertically upward with initial velocity v_0 in a medium that exerts a resistive force with magnitude proportional to the square of the speed.

a. Find the time T when the object reaches its maximum altitude.
b. Use the result of *Exercise 4.3.11* to find the velocity of the object for $t>T$.

13. An object with mass m is given an initial velocity $v_0\le 0$ in a medium that exerts a resistive force of the form $a|v|/(1+|v|)$, where a is positive constant.

a. Set up a differential equation for the speed of the object.
b. Use your favorite numerical method to solve the equation you found in (a), to convince yourself that there's a unique number a_0 such that $\lim_{t\to\infty} s(t)=\infty$ if $a\le a_0$ and $\lim_{t\to\infty} s(t)$ exists (finite) if $a>a_0$. (We say that a_0 is the *bifurcation value* of a.) Try to find a_0 and $\lim_{t\to\infty} s(t)$ in the case where $a>a_0$.

14. An object of mass m falls in a medium that exerts a resistive force $f = f(s)$, where $s = |v|$ is the speed of the object. Assume that $f(0) = 0$ and f is strictly increasing and differentiable on $(0, \infty)$.

a. Write a differential equation for the speed $s = s(t)$ of the object. Take it as given that all solutions of this equation with $s(0) \geq 0$ are defined for all $t > 0$ (which makes good sense on physical grounds).
b. Show that if $\lim_{s\to\infty} f(s) \leq mg$ then $\lim_{t\to\infty} s(t) = \infty$.
c. Show that if $\lim_{s\to\infty} f(s) > mg$ then $\lim_{t\to\infty} s(t) = s_T$ (terminal speed), where $f(s_T) = mg$..

15. A 100-g mass with initial velocity $v_0 \leq 0$ falls in a medium that exerts a resistive force proportional to the fourth power of the speed. The resistance is .1 N if the speed is 3 m/s.

a. Set up the initial value problem for the velocity v of the mass for $t > 0$.
b. Use *Exercise 4.3.14* (c) to determine the terminal velocity of the object.
c. To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 1]$ (seconds) of the initial value problem of (a) , with initial values $v_0 = 0$, -2, -4, …, -12. Present your results in graphical form similar to Figure 4.3.3.

16. A 64-lb object with initial velocity $v_0 \leq 0$ falls through a dense fluid that exerts a resistive force proportional to the square root of the speed. The resistance is 64 lb if the speed is 16 ft/s.

a. Set up the initial value problem for the velocity v of the mass for $t > 0$.
b. Use *Exercise 4.3.14* (c) to determine the terminal velocity of the object.
c. To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 4]$ (seconds) of the initial value problem of (a), with initial values $v_0 = 0$, -5, -10, …, -30. Present your results in graphical form similar to Figure 4.3.3.

Q4.3.2

In *Exercises 4.3.17-4.3.20*, assume that the force due to gravity is given by Newton's law of gravitation. Take the upward direction to be positive.

17. A space probe is to be launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.

18. A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s^2. Find the escape velocity in miles/s.

19.

a. Show that (Equation 4.3.27) can be rewritten as

$$v^2 = \frac{h-y}{y+R} v_e^2 + v_0^2.$$

b. Show that if $v_0 = \rho v_e$ with $0 \leq \rho < 1$, then the maximum altitude y_m attained by the space vehicle is

$$y_m = \frac{h + R\rho^2}{1-\rho^2}.$$

c. By requiring that $v(y_m) = 0$, use (Equation 4.3.26) to deduce that if $v_0 < v_e$ then

$$|v| = v_e \left[\frac{(1-\rho^2)(y_m - y)}{y+R}\right]^{1/2},$$

where y_m and ρ are as defined in (b) and $y \geq h$.
d. Deduce from (c) that if $v < v_e$, the vehicle takes equal times to climb from $y = h$ to $y = y_m$ and to fall back from $y = y_m$ to $y = h$.

20. In the situation considered in the discussion of escape velocity, show that $\lim_{t\to\infty} y(t) = \infty$ if $v(t) > 0$ for all $t > 0$.

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CHAPTER OVERVIEW

4: Higher order linear ODEs

We have already studied the basics of differential equations, including separable first-order equations. In this chapter, we go a little further and look at second-order equations, which are equations containing second derivatives of the dependent variable. The solution methods we examine are different from those discussed earlier, and the solutions tend to involve trigonometric functions as well as exponential functions. Here we concentrate primarily on second-order equations with constant coefficients.

4.1: Second order linear ODEs

Let us consider the general second order linear differential equation

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We usually divide through by $A(x)$ to get

$$y'' + p(x)y' + q(x)y = f(x),$$

where $p(x) = \frac{B(x)}{A(x)}$, $q(x) = \frac{C(x)}{A(x)}$, and $f(x) = \frac{F(x)}{A(x)}$. The word *linear* means that the equation contains no powers nor functions of y, y', and y''.

In the special case when $f(x) = 0$ we have a so-called *homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0, \tag{4.1.1}$$

We have already seen some second order linear homogeneous equations:

$$\begin{array}{lll} y'' + k^2 y = 0 & \text{Two solutions are:} & y_1 = \cos(kx), \quad y_2 = \sin(kx). \\ y'' - k^2 y = 0 & \text{Two solutions are:} & y_1 = e^{kx}, \quad y_2 = e^{-kx}. \end{array}$$

If we know two solutions of a linear homogeneous equation, we know a lot more of them.

Theorem: Superposition

Suppose y_1 and y_2 are two solutions of the homogeneous equation (4.1.1). Then

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

also solves (4.1.1) for arbitrary constants C_1 and C_2.

That is, we can add solutions together and multiply them by constants to obtain new and different solutions. We call the expression $C_1y_1 + C_2y_2$ a *linear combination* of y_1 and y_2. Let us prove this theorem; the proof is very enlightening and illustrates how linear equations work.

Proof

Let $y = C_1y_1 + C_2y_2$. Then

$$\begin{aligned} y'' + py' + qy &= (C_1y_1 + C_2y_2)'' + p(C_1y_1 + C_2y_2)' + q(C_1y_1 + C_2y_2) \\ &= C_1y_1'' + C_2y_2'' + C_1py_1' + C_2py_2' + C_1qy_1 + C_2qy_2 \\ &= C_1(y_1'' + py_1' + qy_1) + C_2(y_2'' + py_2' + qy_2) \\ &= C_1.0 + C_2.0 = 0 \end{aligned}$$

The proof becomes even simpler to state if we use the operator notation. An *operator* is an object that eats functions and spits out functions (kind of like what a function, which eats numbers and spits out numbers). Define the operator L by

$$Ly = y'' + py' + qy.$$

The differential equation now becomes $Ly = 0$. The operator (and the equation) L being linear means that $L(C_1y_1 + C_2y_2) = C_1Ly_1 + C_2Ly_2$. The proof above becomes

$$Ly = L(C_1y_1 + C_2y_2) = C_1Ly_1 + C_2Ly_2 = C_1.0 + C_2.0 = 0$$

Two different solutions to the second equation $y'' - k^2y = 0$ are $y_1 = \cosh(kx)$ and $y_2 = \sinh(kx)$. Let us remind ourselves of the definition, $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$. Therefore, these are solutions by superposition as they are linear combinations of the two exponential solutions.

The functions $\sinh$ and $\cosh$ are sometimes more convenient to use than the exponential. Let us review some of their properties.

$$\begin{aligned} &\cosh 0 = 1 && \sinh 0 = 0, \\ &\tfrac{d}{dx}[\cosh x] = \sinh x, && \tfrac{d}{dx}[\sinh x] = \cosh x, \\ &\cosh^2 x - \sinh^2 x = 1. && \end{aligned}$$

Below is a video on finding the constants of the exponential solution to a differential equation

Exercise

Derive these properties using the definitions of sinh and cosh in terms of exponentials.

Linear equations have nice and simple answers to the existence and uniqueness question.

Theorem: Existence and Uniqueness

Suppose $p(x)$, $q(x)$, and $f(x)$ are continuous functions on some interval I containing a with a, b_0 and b_1 constants. The equation

$$y'' + p(x)y' + q(x)y = f(x).$$

has exactly *one* solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1.$$

For example, the equation $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx)$$

The equation $y'' - k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cosh(kx) + \frac{b_1}{k} \sinh(kx)$$

Using cosh and sinh in this solution allows us to solve for the initial conditions in a cleaner way than if we have used the exponentials.

The initial conditions for a second order ODE consist of two equations. Common sense tells us that if we have two arbitrary constants and two equations, then we should be able to solve for the constants and find a solution to the differential equation satisfying the initial conditions.

Exercise

Suppose we find two different solutions y_1 and y_2 to the homogeneous equation (4.1.1). Can every solution be written (using superposition) in the form $y = C_1 y_1 + C_2 y_2$?

Answer

Answer is affirmative! Provided that y_1 and y_2 are different enough in the following sense. We will say y_1 and y_2 are linearly independent if one is not a constant multiple of the other.

Theorem

Let $p(x)$ and $q(x)$ be continuous functions and let y_1 and y_2 be two linearly independent solutions to the homogeneous equation (4.1.1). Then every other solution is of the form

$$y = C_1 y_1 + C_2 y_2.$$

That is, $y = C_1 y_1 + C_2 y_2$ is the general solution.

For example, we found the solutions $y_1 = \sin x$ and $y_2 = \cos x$ for the equation $y'' + y = 0$. It is not hard to see that sine and cosine are not constant multiples of each other. If $\sin x = A \cos x$ for some constant A, we let $x = 0$ and this would imply $A = 0$. But then $\sin x = 0$ for all x, which is preposterous. So y_1 and y_2 are linearly independent. Hence,

$$y = C_1 \cos x + C_2 \sin x$$

is the general solution to $y'' + y = 0$.

For two functions, checking linear independence is rather simple. Let us see another example. Consider $y'' - 2x^{-2} y = 0$. Then $y_1 = x^2$ and $y_2 = \frac{1}{x}$ are solutions. To see that they are linearly independent, suppose one is a multiple of the other: $y_1 = A y_2$, we just have to find out that A cannot be a constant. In this case we have $A = \frac{y_1}{y_2} = x^3$, this most decidedly not a constant. So $y = C_1 x^2 + C_2 \frac{1}{x}$ is the general solution.

If you have one solution to a second order linear homogeneous equation, then you can find another one. This is the *reduction of order method*. The idea is that if we somehow found y_1 as a solution of $y'' + p(x)y' + q(x)y = 0$ we try a second solution of the form $y_2(x) = y_1(x)v(x)$. We just need to find v. We plug y_2 into the equation:

$$\begin{aligned} 0 = y_2'' + p(x)y_2' + q(x)y_2 &= y_1''v + 2y_1'v' + y_1 v'' + p(x)(y_1'v + y_1 v') + q(z)y_1 v \\ &= y_1 v'' + (2y_1' + p(x)y_1)v' + \cancelto{0}{\left(y_1'' + p(x)y_1' + q(x)y_1\right)} v. \end{aligned} \tag{4.1.2}$$

In other words, $y_1 v'' + (2y_1' + p(x)y_1)v' = 0$. Using $w = v'$ we have the first order linear equation $y_1 w' + (2y_1' + p(x)y_1)w = 0$. After solving this equation for w (integrating factor), we find v by antidifferentiating w. We then form y_2 by computing $y_1 v$. For example, suppose we somehow know $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$. The equation for w is then $xw' + 3w = 0$. We find a solution, $w = Cx^{-3}$, and we find an antiderivative $v = \frac{-C}{2x^2}$. Hence $y_2 = y_1 v = \frac{-C}{2x}$. Any C works and so $C = -2$ makes $y_2 = \frac{1}{x}$. Thus, the general solution is $y = C_1 x + C_2 \frac{1}{x}$.

Since we have a formula for the solution to the first order linear equation, we can write a formula for y_2:

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{\left(y_1(x)\right)^2}\,dx$$

However, it is much easier to remember that we just need to try $y_2(x) = y_1(x)v(x)$ and find $v(x)$ as we did above. Also, the technique works for higher order equations too: you get to reduce the order for each solution you find. So it is better to remember how to do it rather than a specific formula.

We will study the solution of nonhomogeneous equations in Section 2.5. We will first focus on finding general solutions to homogeneous equations.

4.2: The Method of Undetermined Coefficients I

In this section we consider the constant coefficient equation

$$ay''+by'+cy=e^{\alpha x}G(x), \tag{4.2.1}$$

where α is a constant and G is a polynomial.

From Theorem 5.3.2, the general solution of Equation 4.2.1 is $y=y_p+c_1y_1+c_2y_2$, where y_p is a particular solution of Equation 4.2.1 and $\{y_1,y_2\}$ is a fundamental set of solutions of the complementary equation

$$ay''+by'+cy=0.$$

In Section 5.2 we showed how to find $\{y_1,y_2\}$. In this section we'll show how to find y_p. The procedure that we'll use is called *the method of undetermined coefficients*. Our first example is similar to *Exercises 5.3.16-5.3.21*.

Example 4.2.1 :

Find a particular solution of

$$y''-7y'+12y=4e^{2x}. \tag{4.2.2}$$

Then find the general solution.

Solution

Substituting $y_p=Ae^{2x}$ for y in Equation 4.2.2 will produce a constant multiple of Ae^{2x} on the left side of Equation 4.2.2, so it may be possible to choose A so that y_p is a solution of Equation 4.2.2. Let's try it; if $y_p=Ae^{2x}$ then

$$y_p''-7y_p'+12y_p=4Ae^{2x}-14Ae^{2x}+12Ae^{2x}=2Ae^{2x}=4e^{2x}$$

if $A=2$. Therefore $y_p=2e^{2x}$ is a particular solution of Equation 4.2.2. To find the general solution, we note that the characteristic polynomial of the complementary equation

$$y''-7y'+12y=0 \tag{4.2.3}$$

is $p(r)=r^2-7r+12=(r-3)(r-4)$, so $\{e^{3x},e^{4x}\}$ is a fundamental set of solutions of Equation 4.2.3. Therefore the general solution of Equation 4.2.2 is

$$y=2e^{2x}+c_1e^{3x}+c_2e^{4x}.$$

Below is a video on the method of undetermined coefficients

Example 4.2.2

Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{4.2.4}$$

Then find the general solution.

Solution

Fresh from our success in finding a particular solution of Equation 4.2.2 — where we chose $y_p = Ae^{2x}$ because the right side of Equation 4.2.2 is a constant multiple of e^{2x} — it may seem reasonable to try $y_p = Ae^{4x}$ as a particular solution of Equation 4.2.4. However, this will not work, since we saw in Example 4.2.1 that e^{4x} is a solution of the complementary equation Equation 4.2.3, so substituting $y_p = Ae^{4x}$ into the left side of Equation 4.2.4) produces zero on the left, no matter how we choose A. To discover a suitable form for y_p, we use the same approach that we used in Section 5.2 to find a second solution of

$$ay'' + by' + cy = 0$$

in the case where the characteristic equation has a repeated real root: we look for solutions of Equation 4.2.4 in the form $y = ue^{4x}$, where u is a function to be determined. Substituting

$$y = ue^{4x}, \quad y' = u'e^{4x} + 4ue^{4x}, \quad \text{and} \quad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x} \tag{4.2.5}$$

into Equation 4.2.4 and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 7(u' + 4u) + 12u = 5,$$

or

$$u'' + u' = 5.$$

By inspection we see that $u_p = 5x$ is a particular solution of this equation, so $y_p = 5xe^{4x}$ is a particular solution of Equation 4.2.4. Therefore

$$y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$$

is the general solution.

Below is a video on using the method of undetermined coefficients to solve a differential equation.

Example 4.2.3

Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{4.2.6}$$

Solution

Since the characteristic polynomial of the complementary equation

$$y'' - 8y' + 16y = 0 \tag{4.2.7}$$

is $p(r)=r^2-8r+16=(r-4)^2$, both $y_1=e^{4x}$ and $y_2=xe^{4x}$ are solutions of Equation 4.2.7. Therefore Equation 4.2.6) does not have a solution of the form $y_p=Ae^{4x}$ or $y_p=Axe^{4x}$. As in Example 4.2.2 , we look for solutions of Equation 4.2.6 in the form $y=ue^{4x}$, where u is a function to be determined. Substituting from Equation 4.2.5 into Equation 4.2.6 and canceling the common factor e^{4x} yields

$$(u''+8u'+16u)-8(u'+4u)+16u=2,$$

or

$$u''=2.$$

Integrating twice and taking the constants of integration to be zero shows that $u_p=x^2$ is a particular solution of this equation, so $y_p=x^2e^{4x}$ is a particular solution of Equation 4.2.4 Therefore

$$y=e^{4x}(x^2+c_1+c_2x)$$

is the general solution.

The preceding examples illustrate the following facts concerning the form of a particular solution y_p of a constant coefficent equation

$$ay''+by'+cy=ke^{\alpha x},$$

where k is a nonzero constant:

a. If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay''+by'+cy=0, \tag{4.2.8}$$

then $y_p=Ae^{\alpha x}$, where A is a constant. (See Example 4.2.1).

b. If $e^{\alpha x}$ is a solution of Equation 4.2.8 but $xe^{\alpha x}$ is not, then $y_p=Axe^{\alpha x}$, where A is a constant. (See Example 4.2.2 .)

c. If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of Equation 4.2.8, then $y_p=Ax^2e^{\alpha x}$, where A is a constant. (See Example 4.2.3 .)

See *Exercise 5.4.30* for the proofs of these facts.

In all three cases you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p''+by_p'+cy_p=ke^{\alpha x},$$

and solve for the constant A, as we did in Example 4.2.1 . (See *Exercises 5.4.31-5.4.33*.) However, if the equation is

$$ay''+by'+cy=ke^{\alpha x}G(x),$$

where G is a polynomial of degree greater than zero, we recommend that you use the substitution $y=ue^{\alpha x}$ as we did in Examples 4.2.2 and 4.2.3 . The equation for u will turn out to be

$$au''+p'(\alpha)u'+p(\alpha)u=G(x), \tag{4.2.9}$$

where $p(r)=ar^2+br+c$ is the characteristic polynomial of the complementary equation and $p'(r)=2ar+b$ (*Exercise 5.4.30*); however, you shouldn't memorize this since it is easy to derive the equation for u in any particular case. Note, however, that if $e^{\alpha x}$ is a solution of the complementary equation then $p(\alpha)=0$, so Equation 4.2.9 reduces to

$$au''+p'(\alpha)u'=G(x),$$

while if both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation then $p(r)=a(r-\alpha)^2$ and $p'(r)=2a(r-\alpha)$, so $p(\alpha)=p'(\alpha)=0$ and Equation 4.2.9) reduces to

$$au''=G(x).$$

✓ Example 4.2.4

Find a particular solution of

$$y''-3y'+2y=e^{3x}(-1+2x+x^2). \tag{4.2.10}$$

Solution

Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and} y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into Equation 4.2.10) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 3(u' + 3u) + 2u = -1 + 2x + x^2,$$

or

$$u'' + 3u' + 2u = -1 + 2x + x^2. \tag{4.2.11}$$

As in Example 5.3.2, in order to guess a form for a particular solution of Equation 4.2.11), we note that substituting a second degree polynomial $u_p = A + Bx + Cx^2$ for u in the left side of Equation 4.2.11) produces another second degree polynomial with coefficients that depend upon A, B, and C; thus,

$$\text{if} \quad u_p = A + Bx + Cx^2 \quad \text{then} \quad u_p' = B + 2Cx \quad \text{and} \quad u_p'' = 2C.$$

If u_p is to satisfy Equation 4.2.11), we must have

$$\begin{aligned} u_p'' + 3u_p' + 2u_p \quad &= 2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) \\ &= (2C + 3B + 2A) + (6C + 2B)x + 2Cx^2 = -1 + 2x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} 2C \quad &= 1 \\ 2B + 6C \quad &= 2 \\ 2A + 3B + 2C \quad &= -1. \end{aligned}$$

Solving these equations for C, B, and A (in that order) yields $C = 1/2, B = -1/2, A = -1/4$. Therefore

$$u_p = -\frac{1}{4}(1 + 2x - 2x^2)$$

is a particular solution of Equation 4.2.11, and

$$y_p = u_p e^{3x} = -\frac{e^{3x}}{4}(1 + 2x - 2x^2)$$

is a particular solution of Equation 4.2.10.

Example 4.2.5

Find a particular solution of

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2). \tag{4.2.12}$$

Solution

Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and } y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into Equation 4.2.12) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 4(u' + 3u) + 3u = 6 + 8x + 12x^2,$$

or

$$u'' + 2u' = 6 + 8x + 12x^2. \tag{4.2.13}$$

There's no u term in this equation, since e^{3x} is a solution of the complementary equation for Equation 4.2.12). (See *Exercise 5.4.30*.) Therefore Equation 4.2.13) does not have a particular solution of the form $u_p = A + Bx + Cx^2$ that we used successfully in Example 4.2.4 , since with this choice of u_p,

$$u_p'' + 2u_p' = 2C + (B + 2Cx)$$

can't contain the last term ($12x^2$) on the right side of Equation 4.2.13). Instead, let's try $u_p = Ax + Bx^2 + Cx^3$ on the grounds that

$$u_p' = A + 2Bx + 3Cx^2 \quad \text{and} \quad u_p'' = 2B + 6Cx$$

together contain all the powers of x that appear on the right side of Equation 4.2.13).

Substituting these expressions in place of u' and u'' in Equation 4.2.13) yields

$$(2B + 6Cx) + 2(A + 2Bx + 3Cx^2) = (2B + 2A) + (6C + 4B)x + 6Cx^2 = 6 + 8x + 12x^2.$$

Comparing coefficients of like powers of x on the two sides of the last equality shows that u_p satisfies Equation 4.2.13) if

$$\begin{aligned} 6C &= 12 \\ 4B + 6C &= 8 \\ 2A + 2B &= 6. \end{aligned}$$

Solving these equations successively yields $C = 2$, $B = -1$, and $A = 4$. Therefore

$$u_p = x(4 - x + 2x^2)$$

is a particular solution of Equation 4.2.13), and

$$y_p = u_p e^{3x} = xe^{3x}(4 - x + 2x^2)$$

is a particular solution of Equation 4.2.12).

Example 4.2.6

Find a particular solution of

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2). \tag{4.2.14}$$

Solution

Substituting

$$y = ue^{-x/2}, \quad y' = u'e^{-x/2} - \frac{1}{2}ue^{-x/2}, \quad \text{and} \quad y'' = u''e^{-x/2} - u'e^{-x/2} + \frac{1}{4}ue^{-x/2}$$

into Equation 4.2.14) and canceling $e^{-x/2}$ yields

$$4\left(u'' - u' + \frac{u}{4}\right) + 4\left(u' - \frac{u}{2}\right) + u = 4u'' = -8 + 48x + 144x^2,$$

or

$$u'' = -2 + 12x + 36x^2, \tag{4.2.15}$$

which does not contain u or u' because $e^{-x/2}$ and $xe^{-x/2}$ are both solutions of the complementary equation. (See *Exercise 5.4.30*.) To obtain a particular solution of Equation 4.2.15) we integrate twice, taking the constants of integration to be zero; thus,

$$u_p' = -2x + 6x^2 + 12x^3 \quad \text{and} \quad u_p = -x^2 + 2x^3 + 3x^4 = x^2(-1 + 2x + 3x^2).$$

Therefore

$$y_p = u_p e^{-x/2} = x^2 e^{-x/2}(-1 + 2x + 3x^2)$$

is a particular solution of Equation 4.2.14).

Below is a video on using the method of undetermined coefficients to solve a nonhomogeneous differential equation.

Summary

The preceding examples illustrate the following facts concerning particular solutions of a constant coefficent equation of the form

$$ay''+by'+cy=e^{\alpha x}G(x),$$

where G is a polynomial (see *Exercise 5.4.30*):

a. If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay''+by'+cy=0, \tag{4.2.16}$$

then $y_p=e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 4.2.4).
b. If $e^{\alpha x}$ is a solution of Equation 4.2.16 but $xe^{\alpha x}$ is not, then $y_p=xe^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 4.2.5 .)
c. If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of Equation 4.2.16, then $y_p=x^2e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 4.2.6 .)

In all three cases, you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p''+by_p'+cy_p=e^{\alpha x}G(x),$$

and solve for the coefficients of the polynomial Q. However, if you try this you will see that the computations are more tedious than those that you encounter by making the substitution $y=ue^{\alpha x}$ and finding a particular solution of the resulting equation for u. (See *Exercises 5.4.34-5.4.36*.) In Case (a) the equation for u will be of the form

$$au''+p'(\alpha)u'+p(\alpha)u=G(x),$$

with a particular solution of the form $u_p=Q(x)$, a polynomial of the same degree as G, whose coefficients can be found by the method used in Example 4.2.4 . In Case (b) the equation for u will be of the form

$$au''+p'(\alpha)u'=G(x)$$

(no u term on the left), with a particular solution of the form $u_p=xQ(x)$, where Q is a polynomial of the same degree as G whose coefficents can be found by the method used in Example 4.2.5 . In Case (c), the equation for u will be of the form

$$au''=G(x)$$

with a particular solution of the form $u_p=x^2Q(x)$ that can be obtained by integrating $G(x)/a$ twice and taking the constants of integration to be zero, as in Example 4.2.6 .

Using the Principle of Superposition

The next example shows how to combine the method of undetermined coefficients and Theorem 5.3.3, the principle of superposition.

Example 4.2.7

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}. \tag{4.2.17}$$

Solution

In Example 4.2.1 we found that $y_{p_1} = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x},$$

and in Example 4.2.2 we found that $y_{p_2} = 5xe^{4x}$ is a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

Therefore the principle of superposition implies that $y_p = 2e^{2x} + 5xe^{4x}$ is a particular solution of Equation 4.2.17).

4.2E: The Method of Undetermined Coefficients I (Exercises)

Q5.4.1

In *Exercises 5.4.1-5.4.14* find a particular solution.

1. $y''-3y'+2y=e^{3x}(1+x)$
2. $y''-6y'+5y=e^{-3x}(35-8x)$
3. $y''-2y'-3y=e^{x}(-8+3x)$
4. $y''+2y'+y=e^{2x}(-7-15x+9x^2)$
5. $y''+4y=e^{-x}(7-4x+5x^2)$
6. $y''-y'-2y=e^{x}(9+2x-4x^2)$
7. $y''-4y'-5y=-6xe^{-x}$
8. $y''-3y'+2y=e^{x}(3-4x)$
9. $y''+y'-12y=e^{3x}(-6+7x)$
10. $2y''-3y'-2y=e^{2x}(-6+10x)$
11. $y''+2y'+y=e^{-x}(2+3x)$
12. $y''-2y'+y=e^{x}(1-6x)$
13. $y''-4y'+4y=e^{2x}(1-3x+6x^2)$
14. $9y''+6y'+y=e^{-x/3}(2-4x+4x^2)$

Q5.4.2

In *Exercises 5.4.15-5.4.19* find the general solution.

15. $y''-3y'+2y=e^{3x}(1+x)$
16. $y''-6y'+8y=e^{x}(11-6x)$
17. $y''+6y'+9y=e^{2x}(3-5x)$
18. $y''+2y'-3y=-16xe^{x}$
19. $y''-2y'+y=e^{x}(2-12x)$

Q5.4.3

In *Exercises 5.4.20-5.4.23* solve the initial value problem and plot the solution.

20. $y''-4y'-5y=9e^{2x}(1+x), \quad y(0)=0, \quad y'(0)=-10$
21. $y''+3y'-4y=e^{2x}(7+6x), \quad y(0)=2, \quad y'(0)=8$
22. $y''+4y'+3y=-e^{-x}(2+8x), \quad y(0)=1, \quad y'(0)=2$
23. $y''-3y'-10y=7e^{-2x}, \quad y(0)=1, \quad y'(0)=-17$

Q5.4.4

In *Exercises 5.4.24-5.4.29* use the principle of superposition to find a particular solution.

24. $y''+y'+y=xe^{x}+e^{-x}(1+2x)$
25. $y''-7y'+12y=-e^{x}(17-42x)-e^{3x}$
26. $y''-8y'+16y=6xe^{4x}+2+16x+16x^2$
27. $y''-3y'+2y=-e^{2x}(3+4x)-e^{x}$

28. $y''-2y'+2y=e^{x}(1+x)+e^{-x}(2-8x+5x^2)$
29. $y''+y=e^{-x}(2-4x+2x^2)+e^{3x}(8-12x-10x^2)$

Q5.4.5

30.

a. Prove that y is a solution of the constant coefficient equation

$$ay''+by'+cy=e^{\alpha x}G(x) \tag{A}$$

if and only if $y=ue^{\alpha x}$, where u satisfies

$$au''+p'(\alpha)u'+p(\alpha)u=G(x) \tag{B}$$

and $p(r)=ar^2+br+c$ is the characteristic polynomial of the complementary equation

$$ay''+by'+cy=0.$$

For the rest of this exercise, let G be a polynomial. Give the requested proofs for the case where

$$G(x)=g_0+g_1x+g_2x^2+g_3x^3.$$

b. Prove that if $e^{\alpha x}$ isn't a solution of the complementary equation then (B) has a particular solution of the form $u_p=A(x)$, where A is a polynomial of the same degree as G, as in Example 5.4.4. Conclude that (A) has a particular solution of the form $y_p=e^{\alpha x}A(x)$.
c. Show that if $e^{\alpha x}$ is a solution of the complementary equation and $xe^{\alpha x}$ isn't , then (B) has a particular solution of the form $u_p=xA(x)$, where A is a polynomial of the same degree as G, as in Example 5.4.5. Conclude that (A) has a particular solution of the form $y_p=xe^{\alpha x}A(x)$.
d. Show that if $e^{\alpha x}$ and $xe^{\alpha x}$ are both solutions of the complementary equation then (B) has a particular solution of the form $u_p=x^2A(x)$, where A is a polynomial of the same degree as G, and $x^2A(x)$ can be obtained by integrating G/a twice, taking the constants of integration to be zero, as in Example 5.4.6. Conclude that (A) has a particular solution of the form $y_p=x^2e^{\alpha x}A(x)$.

Q5.4.6

Exercises 5.4.31–5.4.36 treat the equations considered in Examples 5.4.1–5.4.6. Substitute the suggested form of y_p into the equation and equate the resulting coefficients of like functions on the two sides of the resulting equation to derive a set of simultaneous equations for the coefficients in y_p. Then solve for the coefficients to obtain y_p. Compare the work you've done with the work required to obtain the same results in Examples 5.4.1–5.4.6.

31. Compare with Example 5.4.1:

$$y''-7y'+12y=4e^{2x};\quad y_p=Ae^{2x}$$

32. Compare with Example 5.4.2:

$$y''-7y'+12y=5e^{4x};\quad y_p=Axe^{4x}$$

33. Compare with Example 5.4.3:

$$y''-8y'+16y=2e^{4x};\quad y_p=Ax^2e^{4x}$$

34. Compare with Example 5.4.4:

$$y''-3y'+2y=e^{3x}(-1+2x+x^2),\quad y_p=e^{3x}(A+Bx+Cx^2)$$

35. Compare with Example 5.4.5:

$$y''-4y'+3y=e^{3x}(6+8x+12x^2),\quad y_p=e^{3x}(Ax+Bx^2+Cx^3)$$

36. Compare with Example 5.4.6:

$$4y''+4y'+y=e^{-x/2}(-8+48x+144x^2),\quad y_p=e^{-x/2}(Ax^2+Bx^3+Cx^4)$$

Q5.4.7

37. Write $y = ue^{\alpha x}$ to find the general solution.

a. $y'' + 2y' + y = \frac{e^{-x}}{\sqrt{x}}$
b. $y'' + 6y' + 9y = e^{-3x} \ln x$
c. $y'' - 4y' + 4y = \frac{e^{2x}}{1+x}$
d. $4y'' + 4y' + y = 4e^{-x/2} \left(\frac{1}{x} + x\right)$

38. Suppose $\alpha \neq 0$ and k is a positive integer. In most calculus books integrals like $\int x^k e^{\alpha x}\, dx$ are evaluated by integrating by parts k times. This exercise presents another method. Let

$$y = \int e^{\alpha x} P(x)\, dx$$

with

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k$$

(where $p_k \neq 0$).

a. Show that $y = e^{\alpha x} u$, where

$$u' + \alpha u = P(x). \tag{A}$$

b. Show that (A) has a particular solution of the form

$$u_p = A_0 + A_1 x + \cdots + A_k x^k,$$

where A_k, A_{k-1}, …, A_0 can be computed successively by equating coefficients of $x^k, x^{k-1}, \ldots, 1$ on both sides of the equation

$$u_p' + \alpha u_p = P(x).$$

c. Conclude that

$$\int e^{\alpha x} P(x)\, dx = \left(A_0 + A_1 x + \cdots + A_k x^k\right) e^{\alpha x} + c,$$

where c is a constant of integration.

39. Use the method of *Exercise 5.4.38* to evaluate the integral.

a. $\int e^x (4 + x) dx$
b. $\int e^{-x} (-1 + x^2) dx$
c. $\int x^3 e^{-2x} dx$
d. $\int e^x (1 + x)^2 dx$
e. $\int e^{3x} (-14 + 30x + 27x^2) dx$
f. $\int e^{-x} (1 + 6x^2 - 14x^3 + 3x^4) dx$

40. Use the method suggested in *Exercise 5.4.38* to evaluate $\int x^k e^{\alpha x}\, dx$, where k is an arbitrary positive integer and $\alpha \neq 0$.

4.3: The Method of Undetermined Coefficients II

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{4.3.1}$$

where λ and ω are real numbers, $\omega \neq 0$, and P and Q are polynomials. We want to find a particular solution of Equation 4.3.1. As in Section 5.4, the procedure that we will use is called *the method of undetermined coefficients*.

Forcing Functions Without Exponential Factors

We begin with the case where $\lambda = 0$ in Equation 4.3.1; thus, we we want to find a particular solution of

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x, \tag{4.3.2}$$

where P and Q are polynomials.

Differentiating $x^r\cos\omega x$ and $x^r\sin\omega x$ yields

$$\frac{d}{dx}x^r\cos\omega x = -\omega x^r\sin\omega x + rx^{r-1}\cos\omega x$$

and

$$\frac{d}{dx}x^r\sin\omega x = \quad \omega x^r\cos\omega x + rx^{r-1}\sin\omega x.$$

This implies that if

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where A and B are polynomials, then

$$ay_p'' + by_p' + cy_p = F(x)\cos\omega x + G(x)\sin\omega x,$$

where F and G are polynomials with coefficients that can be expressed in terms of the coefficients of A and B. This suggests that we try to choose A and B so that $F = P$ and $G = Q$, respectively. Then y_p will be a particular solution of Equation 4.3.2. The next theorem tells us how to choose the proper form for y_p. For the proof see *Exercise 5.5.37*.

Below is a video writing the form of the particular solution to a differential equation.

Theorem 4.3.1

Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{4.3.3}$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \text{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation. The solutions of

$$a(y''+\omega^2 y)=P(x)\cos\omega x+Q(x)\sin\omega x$$

for which $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation are of the form of Equation 4.3.3, where

$$A(x)=A_0x+A_1x^2+\cdots+A_kx^{k+1} \quad \text{and} \quad B(x)=B_0x+B_1x^2+\cdots+B_kx^{k+1}.$$

For an analog of this theorem that's applicable to Equation 4.3.1, see *Exercise 5.5.38*.

Below is a video on writing the form of the particular solution to a differential equation.

✓ Example 4.3.1

Find a particular solution of

$$y''-2y'+y=5\cos 2x+10\sin 2x. \tag{4.3.4}$$

Solution

In Equation 4.3.4 the coefficients of $\cos 2x$ and $\sin 2x$ are both zero degree polynomials (constants). Therefore Theorem 4.3.1 implies that Equation 4.3.4 has a particular solution

$$y_p=A\cos 2x+B\sin 2x.$$

Since

$$y_p'=-2A\sin 2x+2B\cos 2x \quad \text{and} \quad y_p''=-4(A\cos 2x+B\sin 2x),$$

replacing y by y_p in Equation 4.3.4 yields

$$\begin{aligned} y_p''-2y_p'+y_p &= -4(A\cos 2x+B\sin 2x)-4(-A\sin 2x+B\cos 2x) \\ &\qquad +(A\cos 2x+B\sin 2x) \\ &= (-3A-4B)\cos 2x+(4A-3B)\sin 2x. \end{aligned}$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ here with the corresponding coefficients on the right side of Equation 4.3.4 shows that y_p is a solution of Equation 4.3.4 if

$$\begin{aligned} -3A-4B &= \ \ 5 \\ 4A-3B &= 10. \end{aligned}$$

Solving these equations yields $A=1$, $B=-2$. Therefore

$$y_p=\cos 2x-2\sin 2x$$

is a particular solution of Equation 4.3.4.

Below is a video on solving an initial value problem using the particular solution and the method of undetermined coefficients.

Below is a video on finding the finding the general solution to a nonhomogeneous differential equation using undetermined coefficients.

Example 4.3.2

Find a particular solution of

$$y'' + 4y = 8\cos 2x + 12\sin 2x. \tag{4.3.5}$$

Solution

The procedure used in Example 4.3.1 doesn't work here; substituting $y_p = A\cos 2x + B\sin 2x$ for y in Equation 4.3.5 yields

$$y_p'' + 4y_p = -4(A\cos 2x + B\sin 2x) + 4(A\cos 2x + B\sin 2x) = 0$$

for any choice of A and B, since $\cos 2x$ and $\sin 2x$ are both solutions of the complementary equation for Equation 4.3.5. We're dealing with the second case mentioned in Theorem 4.3.1 , and should therefore try a particular solution of the form

$$y_p = x(A\cos 2x + B\sin 2x). \tag{4.3.6}$$

Then

$$\begin{aligned} y_p' &= A\cos 2x + B\sin 2x + 2x(-A\sin 2x + B\cos 2x) \\ \text{and} y_p'' &= -4A\sin 2x + 4B\cos 2x - 4x(A\cos 2x + B\sin 2x) \\ &= -4A\sin 2x + 4B\cos 2x - 4y_p \text{ (see (4.3.6))}, \end{aligned}$$

so

$$y_p'' + 4y_p = -4A\sin 2x + 4B\cos 2x.$$

Therefore y_p is a solution of Equation 4.3.5if

$$-4A\sin 2x + 4B\cos 2x = 8\cos 2x + 12\sin 2x,$$

which holds if $A = -3$ and $B = 2$. Therefore

$$y_p = -x(3\cos 2x - 2\sin 2x)$$

is a particular solution of Equation 4.3.5.

Below is a video on using the method of undetermined coefficients to solve a nonhomogeneous differential equation.

✓ Example 4.3.3

Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{4.3.7}$$

Solution

The coefficients of $\cos x$ and $\sin x$ in Equation 4.3.7 are polynomials of degree one and zero, respectively. Therefore Theorem 4.3.1 tells us to look for a particular solution of Equation 4.3.7 of the form

$$y_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x. \tag{4.3.8}$$

Then

$$y_p' = (A_1 + B_0 + B_1x)\cos x + (B_1 - A_0 - A_1x)\sin x \tag{4.3.9}$$

and

$$y_p'' = (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x, \tag{4.3.10}$$

so

$$y_p'' + 3y_p' + 2y_p \quad = [A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x]\cos x + [B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x]\sin x. \tag{4.3.11}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in Equation 4.3.7shows that y_p is a solution of Equation 4.3.7if

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + \ \ B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + \ \ B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

Solving the first two equations yields $A_1 = 2$, $B_1 = 6$. Substituting these into the last two equations yields

$$\begin{aligned} A_0 + 3B_0 &= 16 - 3A_1 - 2B_1 = -2 \\ -3A_0 + \ \ B_0 &= 10 + 2A_1 - 3B_1 = -4. \end{aligned}$$

Solving these equations yields $A_0 = 1$, $B_0 = -1$. Substituting $A_0 = 1$, $A_1 = 2$, $B_0 = -1$, $B_1 = 6$ into Equation 4.3.8shows that

$$y_p = (1 + 2x)\cos x - (1 - 6x)\sin x$$

is a particular solution of Equation 4.3.7.

A Useful Observation

In Equations 4.3.9, 4.3.10, and 4.3.11 the polynomials multiplying $\sin x$ can be obtained by replacing A_0, A_1, B_0, and B_1 by B_0, B_1, $-A_0$, and $-A_1$, respectively, in the polynomials mutiplying $\cos x$. An analogous result applies in general, as follows (*Exercise 5.5.36*).

Theorem 4.3.2

If

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x,$$

where $A(x)$ and $B(x)$ are polynomials with coefficients A_0 …, A_k and B_0, …, B_k, then the polynomials multiplying $\sin\omega x$ in

$$y_p', \quad y_p'', \quad ay_p'' + by_p' + cy_p \quad \text{and} \quad y_p'' + \omega^2 y_p$$

can be obtained by replacing A_0, …, A_k by B_0, …, B_k and B_0, …, B_k by $-A_0$, …, $-A_k$ in the corresponding polynomials multiplying $\cos\omega x$.

We will not use this theorem in our examples, but we recommend that you use it to check your manipulations when you work the exercises.

Example 4.3.4

Find a particular solution of

$$y'' + y = (8-4x)\cos x - (8+8x)\sin x. \tag{4.3.12}$$

Solution

According to Theorem 4.3.1 , we should look for a particular solution of the form

$$y_p = (A_0x + A_1x^2)\cos x + (B_0x + B_1x^2)\sin x, \tag{4.3.13}$$

since $\cos x$ and $\sin x$ are solutions of the complementary equation. However, let's try

$$y_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x \tag{4.3.14}$$

first, so you can see why it doesn't work. From Equation 4.3.10,

$$y_p'' = (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x,$$

which together with Equation 4.3.14implies that

$$y_p'' + y_p = 2B_1\cos x - 2A_1\sin x.$$

Since the right side of this equation does not contain $x\cos x$ or $x\sin x$, Equation 4.3.14can't satisfy Equation 4.3.12no matter how we choose A_0, A_1, B_0, and B_1.

Now let y_p be as in Equation 4.3.13. Then

$$\begin{aligned} y_p' &= \left[A_0 + (2A_1 + B_0)x + B_1x^2\right]\cos x \\ &+ \left[B_0 + (2B_1 - A_0)x - A_1x^2\right]\sin x \end{aligned}$$

and

$$\begin{aligned} y_p'' &= \left[2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1x^2\right]\cos x \\ &+ \left[2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1x^2\right]\sin x, \end{aligned}$$

so

$$y_p'' + y_p = (2A_1 + 2B_0 + 4B_1x)\cos x + (2B_1 - 2A_0 - 4A_1x)\sin x.$$

Comparing the coefficients of $\cos x$ and $\sin x$ here with the corresponding coefficients in Equation 4.3.12shows that y_p is a solution of Equation 4.3.12if

$$\begin{aligned} 4B_1 &= -4 \\ -4A_1 &= -8 \\ 2B_0 + 2A_1 &= 8 \\ -2A_0 + 2B_1 &= -8. \end{aligned}$$

The solution of this system is $A_1 = 2$, $B_1 = -1$, $A_0 = 3$, $B_0 = 2$. Therefore

$$y_p = x\left[(3+2x)\cos x + (2-x)\sin x\right]$$

is a particular solution of Equation 4.3.12.

Forcing Functions with Exponential Factors

To find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{4.3.15}$$

when $\lambda \neq 0$, we recall from Section 5.4 that substituting $y = ue^{\lambda x}$ into Equation 4.3.15 will produce a constant coefficient equation for u with the forcing function $P(x)\cos\omega x + Q(x)\sin\omega x$. We can find a particular solution u_p of this equation by the procedure that we used in Examples 4.3.1 -4.3.4 . Then $y_p = u_p e^{\lambda x}$ is a particular solution of Equation 4.3.15.

Example 4.3.5

Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right]. \tag{4.3.16}$$

Let $y = ue^{-2x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x}\left[(u'' - 4u' + 4u) - 3(u' - 2u) + 2u\right] \\ &= e^{-2x}(u'' - 7u' + 12u) \\ &= e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right] \end{aligned}$$

if

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x. \tag{4.3.17}$$

Since $\cos 3x$ and $\sin 3x$ aren't solutions of the complementary equation

$$u'' - 7u' + 12u = 0,$$

Theorem 4.3.1 tells us to look for a particular solution of Equation 4.3.17 of the form

$$u_p = (A_0 + A_1 x)\cos 3x + (B_0 + B_1 x)\sin 3x. \tag{4.3.18}$$

Then

$$\begin{aligned} u_p' &= (A_1 + 3B_0 + 3B_1 x)\cos 3x + (B_1 - 3A_0 - 3A_1 x)\sin 3x \\ \text{and} \qquad u_p'' &= (-9A_0 + 6B_1 - 9A_1 x)\cos 3x - (9B_0 + 6A_1 + 9B_1 x)\sin 3x, \end{aligned}$$

so

$$\begin{aligned} u_p'' - 7u_p' + 12u_p &= \left[3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x\right]\cos 3x \\ &+ \left[21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x\right]\sin 3x. \end{aligned}$$

Comparing the coefficients of $x\cos 3x$, $x\sin 3x$, $\cos 3x$, and $\sin 3x$ here with the corresponding coefficients on the right side of Equation 4.3.17 shows that u_p is a solution of Equation 4.3.17 if

$$\begin{aligned} 3A_1 - 21B_1 \qquad &= 0 \\ 21A_1 + \; 3B_1 \qquad &= 150 \\ 3A_0 - 21B_0 - 7A_1 + \; 6B_1 &= \quad 2 \\ 21A_0 + \; 3B_0 - 6A_1 - \; 7B_1 &= -34. \end{aligned} \tag{4.3.19}$$

Solving the first two equations yields $A_1 = 7$, $B_1 = 1$. Substituting these values into the last two equations of Equation 4.3.19 yields

$$\begin{aligned} 3A_0 - 21B_0 &= \quad 2 + 7A_1 - 6B_1 = 45 \\ 21A_0 + \; 3B_0 &= -34 + 6A_1 + 7B_1 = 15. \end{aligned}$$

Solving this system yields $A_0 = 1$, $B_0 = -2$. Substituting $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$ into Equation 4.3.18 shows that

$$u_p = (1 + 7x)\cos 3x - (2 - x)\sin 3x$$

is a particular solution of Equation 4.3.17. Therefore

$$y_p = e^{-2x}\left[(1 + 7x)\cos 3x - (2 - x)\sin 3x\right]$$

is a particular solution of Equation 4.3.16.

Example 4.3.6

Find a particular solution of

$$y'' + 2y' + 5y = e^{-x}\left[(6 - 16x)\cos 2x - (8 + 8x)\sin 2x\right]. \tag{4.3.20}$$

Solution

Let $y = ue^{-x}$. Then

$$\begin{aligned} y''+2y'+5y &= e^{-x}\left[(u''-2u'+u)+2(u'-u)+5u\right] \\ &= e^{-x}(u''+4u) \\ &= e^{-x}\left[(6-16x)\cos 2x-(8+8x)\sin 2x\right] \end{aligned}$$

if

$$u''+4u=(6-16x)\cos 2x-(8+8x)\sin 2x. \tag{4.3.21}$$

Since $\cos 2x$ and $\sin 2x$ are solutions of the complementary equation

$$u''+4u=0,$$

Theorem 4.3.1 tells us to look for a particular solution of Equation 4.3.21of the form

$$u_p=(A_0x+A_1x^2)\cos 2x+(B_0x+B_1x^2)\sin 2x.$$

Then

$$\begin{aligned} u_p' &= \left[A_0+(2A_1+2B_0)x+2B_1x^2\right]\cos 2x \\ &+\left[B_0+(2B_1-2A_0)x-2A_1x^2\right]\sin 2x \end{aligned}$$

and

$$\begin{aligned} u_p'' &= \left[2A_1+4B_0-(4A_0-8B_1)x-4A_1x^2\right]\cos 2x \\ &+\left[2B_1-4A_0-(4B_0+8A_1)x-4B_1x^2\right]\sin 2x, \end{aligned}$$

so

$$u_p''+4u_p=(2A_1+4B_0+8B_1x)\cos 2x+(2B_1-4A_0-8A_1x)\sin 2x.$$

Equating the coefficients of $x\cos 2x$, $x\sin 2x$, $\cos 2x$, and $\sin 2x$ here with the corresponding coefficients on the right side of Equation 4.3.21 shows that u_p is a solution of Equation 4.3.21if

$$\begin{aligned} 8B_1 &= -16 \\ -8A_1 &= -\ \ 8 \\ 4B_0+2A_1 &= 6 \\ -4A_0+2B_1 &= -8. \end{aligned} \tag{4.3.22}$$

The solution of this system is $A_1=1$, $B_1=-2$, $B_0=1$, $A_0=1$. Therefore

$$u_p=x[(1+x)\cos 2x+(1-2x)\sin 2x]$$

is a particular solution of Equation 4.3.21, and

$$y_p=xe^{-x}\left[(1+x)\cos 2x+(1-2x)\sin 2x\right]$$

is a particular solution of Equation 4.3.20.

You can also find a particular solution of Equation 4.3.20by substituting

$$y_p=xe^{-x}\left[(A_0+A_1x)\cos 2x+(B_0+B_1x)\sin 2x\right]$$

for y in Equation 4.3.20and equating the coefficients of $xe^{-x}\cos 2x$, $xe^{-x}\sin 2x$, $e^{-x}\cos 2x$, and $e^{-x}\sin 2x$ in the resulting expression for

$$y_p''+2y_p'+5y_p$$

with the corresponding coefficients on the right side of Equation 4.3.20. (See *Exercise 5.5.38*). This leads to the same system Equation 4.3.22 of equations for A_0, A_1, B_0, and B_1 that we obtained in Example 4.3.6 . However, if you try this approach you'll see that deriving Equation 4.3.22this way is much more tedious than the way we did it in Example 4.3.6 .

Below is a video on using the method of undetermined coefficients to find the general solution to a differential equation.

4.3E: The Method of Undetermined Coefficients II (Exercises)

Q5.5.1

In *Exercises 5.5.1-5.5.17* find a particular solution.

1. $y'' + 3y' + 2y = 7\cos x - \sin x$

2. $y'' + 3y' + y = (2 - 6x)\cos x - 9\sin x$

3. $y'' + 2y' + y = e^x(6\cos x + 17\sin x)$

4. $y'' + 3y' - 2y = -e^{2x}(5\cos 2x + 9\sin 2x)$

5. $y'' - y' + y = e^x(2 + x)\sin x$

6. $y'' + 3y' - 2y = e^{-2x}\left[(4 + 20x)\cos 3x + (26 - 32x)\sin 3x\right]$

7. $y'' + 4y = -12\cos 2x - 4\sin 2x$

8. $y'' + y = (-4 + 8x)\cos x + (8 - 4x)\sin x$

9. $4y'' + y = -4\cos x/2 - 8x\sin x/2$

10. $y'' + 2y' + 2y = e^{-x}(8\cos x - 6\sin x)$

11. $y'' - 2y' + 5y = e^x\left[(6 + 8x)\cos 2x + (6 - 8x)\sin 2x\right]$

12. $y'' + 2y' + y = 8x^2\cos x - 4x\sin x$

13. $y'' + 3y' + 2y = (12 + 20x + 10x^2)\cos x + 8x\sin x$

14. $y'' + 3y' + 2y = (1 - x - 4x^2)\cos 2x - (1 + 7x + 2x^2)\sin 2x$

15. $y'' - 5y' + 6y = -e^x\left[(4 + 6x - x^2)\cos x - (2 - 4x + 3x^2)\sin x\right]$

16. $y'' - 2y' + y = -e^x\left[(3 + 4x - x^2)\cos x + (3 - 4x - x^2)\sin x\right]$

17. $y'' - 2y' + 2y = e^x\left[(2 - 2x - 6x^2)\cos x + (2 - 10x + 6x^2)\sin x\right]$

Q5.5.2

In *Exercises 5.5.18-5.5.21* find a particular solution and graph it.

18. $y'' + 2y' + y = e^{-x}\left[(5 - 2x)\cos x - (3 + 3x)\sin x\right]$

19. $y'' + 9y = -6\cos 3x - 12\sin 3x$

20. $y'' + 3y' + 2y = (1 - x - 4x^2)\cos 2x - (1 + 7x + 2x^2)\sin 2x$

21. $y'' + 4y' + 3y = e^{-x}\left[(2 + x + x^2)\cos x + (5 + 4x + 2x^2)\sin x\right]$

Q5.5.3

In *Exercises 5.5.22-5.5.26* solve the initial value problem.

22. $y'' - 7y' + 6y = -e^x(17\cos x - 7\sin x), \quad y(0) = 4,\ y'(0) = 2$

23. $y'' - 2y' + 2y = -e^x(6\cos x + 4\sin x), \quad y(0) = 1,\ y'(0) = 4$

24. $y'' + 6y' + 10y = -40e^x\sin x, \quad y(0) = 2, \quad y'(0) = -3$

25. $y'' - 6y' + 10y = -e^{3x}(6\cos x + 4\sin x), \quad y(0) = 2, \quad y'(0) = 7$

26. $y'' - 3y' + 2y = e^{3x}\left[21\cos x - (11 + 10x)\sin x\right], \ y(0) = 0, \quad y'(0) = 6$

Q5.5.4

In *Exercises 5.5.27-5.5.32* use the principle of superposition to find a particular solution. Where indicated, solve the initial value problem.

27. $y'' - 2y' - 3y = 4e^{3x} + e^x(\cos x - 2\sin x)$

28. $y''+y=4\cos x-2\sin x+xe^{x}+e^{-x}$
29. $y''-3y'+2y=xe^{x}+2e^{2x}+\sin x$
30. $y''-2y'+2y=4xe^{x}\cos x+xe^{-x}+1+x^{2}$
31. $y''-4y'+4y=e^{2x}(1+x)+e^{2x}(\cos x-\sin x)+3e^{3x}+1+x$
32. $y''-4y'+4y=6e^{2x}+25\sin x, \quad y(0)=5,\ y'(0)=3$

Q5.5.5

In *Exercises 5.5.33-5.5.35* solve the initial value problem and graph the solution.

33. $y''+4y=-e^{-2x}\left[(4-7x)\cos x+(2-4x)\sin x\right],\ y(0)=3, \quad y'(0)=1$
34. $y''+4y'+4y=2\cos 2x+3\sin 2x+e^{-x}, \quad y(0)=-1,\ y'(0)=2$
35. $y''+4y=e^{x}(11+15x)+8\cos 2x-12\sin 2x, \quad y(0)=3,\ y'(0)=5$

Q5.5.6

36.

a. Verify that if

$$y_p=A(x)\cos\omega x+B(x)\sin\omega x \tag{4.3E.1}$$

where A and B are twice differentiable, then

$$\begin{aligned} y_p' &=(A'+\omega B)\cos\omega x+(B'-\omega A)\sin\omega x \quad \text{and} \\ y_p'' &=(A''+2\omega B'-\omega^2 A)\cos\omega x+(B''-2\omega A'-\omega^2 B)\sin\omega x. \end{aligned}$$

b. Use the results of (a) to verify that

$$\begin{aligned} ay_p''+by_p'+cy_p= &\left[(c-a\omega^2)A+b\omega B+2a\omega B'+bA'+aA''\right]\cos\omega x+ \\ &\left[-b\omega A+(c-a\omega^2)B-2a\omega A'+bB'+aB''\right]\sin\omega x. \end{aligned}$$

c. Use the results of (a) to verify that

$$y_p''+\omega^2 y_p=(A''+2\omega B')\cos\omega x+(B''-2\omega A')\sin\omega x. \tag{4.3E.2}$$

d. Prove Theorem 5.5.2.

37. Let a, b, c, and ω be constants, with $a\neq 0$ and $\omega>0$, and let

$$P(x)=p_0+p_1x+\cdots+p_kx^k \quad \text{and} \quad Q(x)=q_0+q_1x+\cdots+q_kx^k, \tag{4.3E.3}$$

where at least one of the coefficients p_k, q_k is nonzero, so k is the larger of the degrees of P and Q.

a. Show that if $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation

$$ay''+by'+cy=0, \tag{4.3E.4}$$

then there are polynomials

$$A(x)=A_0+A_1x+\cdots+A_kx^k \quad \text{and} \quad B(x)=B_0+B_1x+\cdots+B_kx^k \tag{A}$$

such that

$$\begin{aligned} (c-a\omega^2)A+b\omega B+2a\omega B'+bA'+aA'' &=P \\ -b\omega A+(c-a\omega^2)B-2a\omega A'+bB'+aB'' &=Q, \end{aligned} \tag{4.3E.5}$$

where (A_k,B_k), (A_{k-1},B_{k-1}), …,(A_0,B_0) can be computed successively by solving the systems

$$\begin{aligned} (c-a\omega^2)A_k+b\omega B_k &=p_k \\ -b\omega A_k+(c-a\omega^2)B_k &=q_k, \end{aligned} \tag{4.3E.6}$$

and, if $1 \leq r \leq k$,

$$\begin{aligned} (c-a\omega^2)A_{k-r} + b\omega B_{k-r} &= p_{k-r} + \cdots \\ -b\omega A_{k-r} + (c-a\omega^2)B_{k-r} &= q_{k-r} + \cdots, \end{aligned} \tag{4.3E.7}$$

where the terms indicated by "$\cdots$" depend upon the previously computed coefficients with subscripts greater than $k-r$. Conclude from this and *Exercise 5.5.36b* that

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x \tag{B}$$

is a particular solution of

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x. \tag{4.3E.8}$$

b. Conclude from *Exercise 5.5.36c* that the equation

$$a(y'' + \omega^2 y) = P(x)\cos\omega x + Q(x)\sin\omega x \tag{C}$$

does not have a solution of the form (B) with A and B as in (A). Then show that there are polynomials

$$A(x) = A_0 x + A_1 x^2 + \cdots + A_k x^{k+1} \quad \text{and} \quad B(x) = B_0 x + B_1 x^2 + \cdots + B_k x^{k+1} \tag{4.3E.9}$$

such that

$$\begin{aligned} a(A'' + 2\omega B') &= P \\ a(B'' - 2\omega A') &= Q, \end{aligned} \tag{4.3E.10}$$

where the pairs (A_k, B_k), (A_{k-1}, B_{k-1}), …, (A_0, B_0) can be computed successively as follows:

$$\begin{aligned} A_k &= -\frac{q_k}{2a\omega(k+1)} \\ B_k &= \frac{p_k}{2a\omega(k+1)}, \end{aligned}$$

and, if $k \geq 1$,

$$\begin{aligned} A_{k-j} &= -\frac{1}{2\omega}\left[\frac{q_{k-j}}{a(k-j+1)} - (k-j+2)B_{k-j+1}\right] \\ B_{k-j} &= \frac{1}{2\omega}\left[\frac{p_{k-j}}{a(k-j+1)} - (k-j+2)A_{k-j+1}\right] \end{aligned}$$

for $1 \leq j \leq k$. Conclude that (B) with this choice of the polynomials A and B is a particular solution of (C).

38. Show that Theorem 5.5.1 implies the next theorem:

Theorem 5.5E.1

Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{4.3E.11}$$

has a particular solution

$$y_p = e^{\lambda x}\left(A(x)\cos\omega x + B(x)\sin\omega x\right), \tag{A}$$

where

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \text{and} \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k, \tag{4.3E.12}$$

provided that $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are not solutions of the complementary equation. The equation

$$a\left[y'' - 2\lambda y' + (\lambda^2 + \omega^2)y\right] = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{4.3E.13}$$

(for which $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are solutions of the complementary equation) has a particular solution of the form (A), where

$$A(x)=A_0x+A_1x^2+\cdots+A_kx^{k+1} \quad \text{and} \quad B(x)=B_0x+B_1x^2+\cdots+B_kx^{k+1}. \tag{4.3E.14}$$

39. This exercise presents a method for evaluating the integral

$$y=\int e^{\lambda x}\left(P(x)\cos\omega x+Q(x)\sin\omega x\right)\,dx \tag{4.3E.15}$$

where $\omega\neq 0$ and

$$P(x)=p_0+p_1x+\cdots+p_kx^k,\quad Q(x)=q_0+q_1x+\cdots+q_kx^k. \tag{4.3E.16}$$

a. Show that $y=e^{\lambda x}u$, where

$$u'+\lambda u=P(x)\cos\omega x+Q(x)\sin\omega x. \tag{A}$$

b. Show that (A) has a particular solution of the form

$$u_p=A(x)\cos\omega x+B(x)\sin\omega x, \tag{4.3E.17}$$

where

$$A(x)=A_0+A_1x+\cdots+A_kx^k,\quad B(x)=B_0+B_1x+\cdots+B_kx^k, \tag{4.3E.18}$$

and the pairs of coefficients (A_k,B_k), (A_{k-1},B_{k-1}), …,(A_0,B_0) can be computed successively as the solutions of pairs of equations obtained by equating the coefficients of $x^r\cos\omega x$ and $x^r\sin\omega x$ for $r=k$, $k-1$, …, 0.

c. Conclude that

$$\int e^{\lambda x}\left(P(x)\cos\omega x+Q(x)\sin\omega x\right)\,dx=e^{\lambda x}\left(A(x)\cos\omega x+B(x)\sin\omega x\right)+c, \tag{4.3E.19}$$

where c is a constant of integration.

40. Use the method of *Exercise 5.5.39* to evaluate the integral.

a. $\int x^2\cos x\,dx$
b. $\int x^2e^x\cos x\,dx$
c. $\int xe^{-x}\sin 2x\,dx$
d. $\int x^2e^{-x}\sin x\,dx$
e. $\int x^3e^x\sin x\,dx$
f. $\int e^x[x\cos x-(1+3x)\sin x]dx$
g. $\int e^{-x}[(1+x^2)\cos x+(1_x^2)\sin x]dx$

4.4: Constant coefficient second order linear ODEs

4.4.1: Solving Constant Coefficient Equations

Suppose we have the problem

$$y'' - 6y' + 8y = 0, y(0) = -2, y'(0) = 6$$

This is a second order linear homogeneous equation with constant coefficients. Constant coefficients means that the functions in front of y'', y', and y are constants and do not depend on x.

To guess a solution, think of a function that you know stays essentially the same when we differentiate it, so that we can take the function and its derivatives, add some multiples of these together, and end up with zero.

Let us try[1] a solution of the form $y = e^{rx}$. Then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. Plug in to get

$$\begin{aligned} y'' - 6y' + 8y &= 0, \\ \underbrace{r^2 e^{rx}}_{y''} - 6\underbrace{re^{rx}}_{y'} + 8\underbrace{e^{rx}}_{y} &= 0, \\ r^2 - 6r + 8 &= 0 \qquad \text{(divide through by } e^{rx}\text{)}, \\ (r-2)(r-4) &= 0. \end{aligned} \tag{4.4.1}$$

Hence, if $r = 2$ or $r = 4$, then e^{rx} is a solution. So let $y_1 = e^{2x}$ and $y_2 = e^{4x}$.

Exercise 4.4.1

Check that y_1 and y_2 are solutions.

Solution

The functions e^{2x} and e^{4x} are linearly independent. If they were not linearly independent we could write $e^{4x} = Ce^{2x}$ for some constant C, implying that $e^{2x} = C$ for all x, which is clearly not possible. Hence, we can write the general solution as

$$y = C_1 e^{2x} + C_2 e^{4x}$$

We need to solve for C_1 and C_2. To apply the initial conditions we first find $y' = 2C_1 e^{2x} + 4C_2 e^{4x}$. We plug in $x = 0$ and solve.

$$\begin{aligned} -2 &= y(0) = C_1 + C_2 \\ 6 &= y'(0) = 2C_1 + 4C_2 \end{aligned} \tag{4.4.2}$$

Either apply some matrix algebra, or just solve these by high school math. For example, divide the second equation by 2 to obtain $3 = C_1 + 2C_2$, and subtract the two equations to get $5 = C_2$. Then $C_1 = -7$ as $-2 = C_1 + 5$. Hence, the solution we are looking for is

$$y = -7e^{2x} + 5e^{4x}$$

Let us generalize this example into a method. Suppose that we have an equation

$$ay'' + by' + cy = 0, \tag{4.4.3}$$

where a, b, c are constants. Try the solution $y = e^{rx}$ to obtain

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

Divide by e^{rx} to obtain the so-called *characteristic equation* of the ODE:

$$ar^2 + br + c = 0$$

Solve for the r by using the quadratic formula.

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore, we have $e^{r_1 x}$ and $e^{r_2 x}$ as solutions. There is still a difficulty if $r_1 = r_2$, but it is not hard to overcome.

Theorem 4.4.1

Suppose that r_1 and r_2 are the roots of the characteristic equation.

If r_1 and r_2 are distinct and real (when $b^2 - 4ac > 0$), then (4.4.3) has the general solution

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $r_1 = r_2$ (happens when $b^2 - 4ac = 0$), then (4.4.3) has the general solution

$$y = (C_1 + C_2 x) e^{r_1 x}$$

For another example of the first case, take the equation $y'' - k^2 y = 0$. Here the characteristic equation is $r^2 - k^2 = 0$ or $(r-k)(r+k) = 0$. Consequently, e^{-kx} and e^{kx} are the two linearly independent solutions.

Below is a video on the characteristic equation of a differential equation.

Example 4.4.1

Solve

$$y'' - k^2 y = 0.$$

Solution

The characteristic equation is $r^2 - k^2 = 0$ or $(r-k)(r+k) = 0$. Consequently, e^{-kx} and e^{kx} are the two linearly independent solutions, and the general solution is

$$y = C_1 e^{kx} + C_2 e_{-kx}.$$

Since $\cosh s = \frac{e^s + e^{-s}}{2}$ and $\sinh s = \frac{e^s - e^{-s}}{2}$, we can also write the general solution as

$$y = D_1 \cosh(kx) + D_2 \sinh(kx).$$

Below is a video on finding the finding the general solution to a differential equation.

> **Example 4.4.2:**
>
> Find the general solution of
>
> $$y'' - 8y' + 16y = 0$$
>
> **Solution**
>
> The characteristic equation is $r^2 - 8r + 16 = (r-4)^2 = 0$. The equation has a double root $r_1 = r_2 = 4$. The general solution is, therefore,
>
> $$y = (C_1 + C_2 x)e^{4x} = C_1 e^{4x} + C_2 x e^{4x}$$

Below is a video on finding the general solution to a differential equation involving two real irrational roots.

> **Exercise 4.4.2: Linear Independence**
>
> Check that e^{4x} and xe^{4x} *are linearly independent.*
>
> **Answer**
>
> That e^{4x} solves the equation is clear. If xe^{4x} solves the equation, then we know we are done. Let us compute $y' = e^{4x} + 4xe^{4x}$ and $y'' = 8e^{4x} + 16xe^{4x}$. Plug in
>
> $$y'' - 8y' + 16y = 8e^{4x} + 16xe^{4x} - 8(e^{4x} + 4xe^{4x}) + 16xe^{4x} = 0$$
>
> We should note that in practice, doubled root rarely happens. If coefficients are picked truly randomly we are very unlikely to get a doubled root.

Below is a video on finding the the general solution to a differential equation.

Let us give a short proof for why the solution xe^{rx} works when the root is doubled. This case is really a limiting case of when the two roots are distinct and very close. Note that $\frac{e^{r}2^{x}-e^{x}1^{x}}{r_2-r_1}$ is a solution when the roots are distinct. When we take the limit as r_1 goes to r_2, we are really taking the derivative of e^{rx} using r as the variable. Therefore, the limit is xe^{rx}, and hence this is a solution in the doubled root case.

Below is a video on finding the solution to a differential equation involving repeated roots.

4.4.2: Complex numbers and Euler’s formula

It may happen that a polynomial has some complex roots. For example, the equation $r^2+1=0$ has no real roots, but it does have two complex roots. Here we review some properties of complex numbers.

Complex numbers may seem a strange concept, especially because of the terminology. There is nothing imaginary or really complicated about complex numbers. A complex number is simply a pair of real numbers, (a,b). We can think of a complex number as a point in the plane. We add complex numbers in the straightforward way, $(a,b)+(c,d)=(a+c,b+d)$. We define multiplication by

$$(a,b)\times(c,d)\overset{\text{def}}{=}(ac-bd,ad+bc).$$

It turns out that with this multiplication rule, all the standard properties of arithmetic hold. Further, and most importantly $(0,1)\times(0,1)=(-1,0)$.

Generally we just write (a,b) as $(a+ib)$, and we treat i as if it were an unknown. We do arithmetic with complex numbers just as we would with polynomials. The property we just mentioned becomes $i^2=-1$. So whenever we see i^2, we replace it by -1. The numbers i and $-i$ are the two roots of $r^2+1=0$.

Note that engineers often use the letter j instead of i for the square root of -1. We will use the mathematicians' convention and use i.

? Exercise 4.4.3

Make sure you understand (that you can justify) the following identities:

a. $i^2 = -1, i^3 = -1, i^4 = 1$,
b. $\frac{1}{i} = -i$,
c. $(3-7i)(-2-9i) = \cdots = -69-13i$,
d. $(3-2i)(3+2i) = 3^2 - (2i)^2 = 3^2 + 2^2 = 13$,
e. $\frac{1}{3-2i} = \frac{1}{3-2i}\frac{3+2i}{3+2i} = \frac{3+2i}{13} = \frac{3}{13} + \frac{2}{13}i$.

We can also define the exponential e^{a+ib} of a complex number. We do this by writing down the Taylor series and plugging in the complex number. Because most properties of the exponential can be proved by looking at the Taylor series, these properties still hold for the complex exponential. For example the very important property: $e^{x+y} = e^x e^y$. This means that $e^{a+ib} = e^a e^{ib}$. Hence if we can compute e^{ib}, we can compute e^{a+ib}. For e^{ib} we use the so-called Euler's formula.

Theorem 4.4.2

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \textit{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

In other words, $e^{a+ib} = e^a(\cos(b) + i\sin(b)) = e^a\cos(b) + ie^a\sin(b)$.

? Exercise 4.4.4:

Using Euler's formula, check the identities:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$$

? Exercise 4.4.5

Double angle identities: Start with $e^{i(2\theta)} = \left(e^{i\theta}\right)^2$. Use Euler on each side and deduce:

Answer

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad \text{and} \quad \sin(2\theta) = 2\sin\theta\cos\theta$$

For a complex number $a+ib$ we call a the real part and b the imaginary part of the number. Often the following notation is used,

$$\mathrm{Re}(a+ib) = a \quad \text{and} \quad \mathrm{Im}(a+ib) = b$$

4.4.3: Complex roots

Suppose that the equation $ay'' + by' + cy = 0$ has the characteristic equation $ar^2 + br + c = 0$ that has complex roots. By the quadratic formula, the roots are $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$. These roots are complex if $b^2 - 4ac < 0$. In this case the roots are

$$r_1, r_2 = \frac{-b}{2a} \pm i\frac{\sqrt{4ac-b^2}}{2a}$$

As you can see, we always get a pair of roots of the form $\alpha \pm i\beta$. In this case we can still write the solution as

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

However, the exponential is now complex valued. We would need to allow C_1 and C_2 to be complex numbers to obtain a real-valued solution (which is what we are after). While there is nothing particularly wrong with this approach, it can make calculations harder and it is generally preferred to find two real-valued solutions.

Here we can use Euler's formula. Let

$$y_1 = e^{(\alpha+i\beta)x} \quad \text{and} \quad y_2 = e^{(\alpha-i\beta)x}$$

Then note that

$$\begin{aligned} y_1 &= e^{ax}\cos(\beta x) + ie^{ax}\sin(\beta x) \\ y_2 &= e^{ax}\cos(\beta x) - ie^{ax}\sin(\beta x) \end{aligned} \tag{4.4.4}$$

Linear combinations of solutions are also solutions. Hence,

$$\begin{aligned} y_3 &= \frac{y_1+y_2}{2} = e^{ax}\cos(\beta x) \\ y_4 &= \frac{y_1-y_2}{2i} = e^{ax}\sin(\beta x) \end{aligned} \tag{4.4.5}$$

are also solutions. Furthermore, they are real-valued. It is not hard to see that they are linearly independent (not multiples of each other). Therefore, we have the following theorem.

Theorem 4.4.3

For the homegneous second order ODE

$$ay'' + by' + cy = 0$$

If the characteristic equation has the roots $\alpha \pm i\beta$ (when $b^2 - 4ac < 0$), then the general solution is

$$y = C_1e^{ax}\cos(\beta x) + C_2e^{ax}\sin(\beta x)$$

Below is a video on finding the solution to a differential equation using the principal of superposition.

Example 4.4.3

Find the general solution of $y'' + k^2y = 0$, for a constant $k > 0$.

Solution

The characteristic equation is $r^2 + k^2 = 0$. Therefore, the roots are $r = \pm ik$ and by the theorem we have the general solution

$$y = C_1\cos(kx) + C_2\sin(kx)$$

Below is a video on finding the solution to a differential equation involving complex roots.

✓ Example 4.4.4

Find the solution of $y''-6y'+13y=0, y(0)=0, y'(0)=10$.

Solution

The characteristic equation is $r^2-6r+13=0$. By completing the square we get $(r-3)^2+2^2=0$ and hence the roots are $r=3\pm 2i$. By the theorem we have the general solution

$$y=C_1e^{3x}\cos(2x)+C_2e^{3x}\sin(2x)$$

To find the solution satisfying the initial conditions, we first plug in zero to get

$$0=y(0)=C_1e^0\cos 0+C_2e^0\sin 0=C_1$$

Hence $C_1=0$ and $y=C_2e^{3x}\sin(2x)$. We differentiate

$$y'=3C_2e^{3x}\sin(2x)+2C_2e^{3x}\cos(2x)$$

We again plug in the initial condition and obtain $10=y'(0)=2C_2$, or $C_2=5$. Hence the solution we are seeking is

$$y=5e^{3x}\sin(2x)$$

Below is a video on finding the solution to an initial value problem.

Below is a video on finding the solution to a differential equation given initial values.

Below is another video on finding the solution to a differential equation given initial values.

4.4.4: Footnotes

[1] Making an educated guess with some parameters to solve for is such a central technique in differential equations, that people sometimes use a fancy name for such a guess: *ansatz*, German for “initial placement of a tool at a work piece.” Yes, the Germans have a word for that.

4.5: Higher order linear ODEs

Equations that appear in applications tend to be second order, although higher order equations do appear from time to time. Hence, it is a generally assumed that the world is “second order” from a modern physics perspective. The basic results about linear ODEs of higher order are essentially the same as for second order equations, with 2 replaced by n. The important concept of linear independence is somewhat more complicated when more than two functions are involved.

For higher order constant coefficient ODEs, the methods are also somewhat harder to apply, but we will not dwell on these complications. We can always use the methods for systems of linear equations to solve higher order constant coefficient equations. So let us start with a general homogeneous linear equation:

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots+p_1(x)y'+p_o(x)y=f(x) \tag{4.5.1}$$

Theorem 4.5.1

Superposition

Suppose $y_1, y_2, \ldots, y_n$ are solutions of the homogeneous equation (Equation 4.5.1). Then

$$y(x)=C_1y_1(x)+C_2y_2(x)+\ldots+C_ny_n(x)$$

also solves Equation 4.5.1 for arbitrary constants $C_1, \ldots. C_n$.

In other words, a linear combination of solutions to Equation 4.5.1 is also a solution to Equation 4.5.1. We also have the existence and uniqueness theorem for nonhomogeneous linear equations.

Theorem 4.5.2

Existence and Uniqueness

Suppose p_o through p_{n-1}, and f are continuous functions on some interval I, a is a number in I, and $b_0, b_1, \ldots, b_{n-1}$ are constants. The equation

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots+p_1(x)y'+p_o(x)y=f(x)$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a)=b_0, \quad y'(a)=b_1, \quad \ldots, \quad y^{(n-1)}(a)=b_{n-1}$$

4.5.1: Linear Independence

When we had two functions y_1 and y_2 we said they were linearly independent if one was not the multiple of the other. Same idea holds for n functions. In this case it is easier to state as follows. The functions $y_1, y_2, \ldots, y_n$ are linearly independent if

$$c_1y_1+c_2y_2+\cdots+c_ny_n=0$$

has only the trivial solution $c_1=c_2=\cdots=c_n=0$, where the equation must hold for all x. If we can solve equation with some constants where for example $c_1 \neq 0$, then we can solve for y_1 as a linear combination of the others. If the functions are not linearly independent, they are linearly dependent.

Example 4.5.1

Show that e^x, e^{2x}, and e^{3x} are linearly independent functions.

Solution

Let us give several ways to show this fact. Many textbooks introduce Wronskians, but that is really not necessary to solve this example. Let us write down

$$c_1e^x+c_2e^{2x}+c_3e^{3x}=0$$

We use rules of exponentials and write $z=e^x$. Then we have

$$c_1 z + c_2 z^2 + c_3 z^3 = 0$$

The left hand side is a third degree polynomial in z. It can either be identically zero, or it can have at most 3 zeros. Therefore, it is identically zero, $c_1 = c_2 = c_3 = 0$, and the functions are linearly independent.

Let us try another way. As before we write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0$$

This equation has to hold for all x. What we could do is divide through by e^{3x} to get

$$c_1 e^{-2x} + c_2 e^{-x} + c_3 = 0$$

As the equation is true for all x, let $x \to \infty$. After taking the limit we see that $c_3 = 0$. Hence our equation becomes

$$c_1 e^x + c_2 e^{2x} = 0$$

Rinse, repeat!

How about yet another way. We again write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0$$

We can evaluate the equation and its derivatives at different values of x to obtain equations for c_1, c_2, and c_3. Let us first divide by e^x for simplicity.

$$c_1 + c_2 e^x + c_3 e^{2x} = 0$$

We set $x = 0$ to get the equation $c_1 + c_2 + c_3 = 0$. Now differentiate both sides

$$c_2 e^x + 2 c_3 e^{2x} = 0$$

We set $x = 0$ to get $c_2 + 2c_3 = 0$. We divide by e^x again and differentiate to get $2c_3 e^x = 0$. It is clear that c_3 is zero. Then c_2 must be zero as $c_2 = -2c_3$, and c_1 must be zero because $c_1 + c_2 + c_3 = 0$.

There is no one best way to do it. All of these methods are perfectly valid. The important thing is to understand why the functions are linearly independent.

Example $4.5.2$

On the other hand, the functions e^x, e^{-x} and $\cosh x$ are linearly dependent. Simply apply definition of the hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{or} \quad 2\cosh x - e^x - e^{-x} = 0$$

4.5.2: Constant Coefficient Higher Order ODEs

When we have a higher order constant coefficient homogeneous linear equation, the song and dance is exactly the same as it was for second order. We just need to find more solutions. If the equation is n^{th} order we need to find n linearly independent solutions. It is best seen by example.

Example $4.5.3$: Third order ODE with Constant Coefficients

Find the general solution to

$$y''' - 3'' - y' + 3y = 0 \tag{4.5.2}$$

Solution

Try: $y = e^{rx}$. We plug in and get

$$\underbrace{r^3 e^{rx}}_{y'''} - 3\underbrace{r^2 e^{rx}}_{y''} - \underbrace{re^{rx}}_{y'} + 3\underbrace{e^{rx}}_{y} = 0.$$

We divide through by e^{rx}. Then

$$r^3 - 3r^2 - r + 3 = 0$$

The trick now is to find the roots. There is a formula for the roots of degree 3 and 4 polynomials, but it is very complicated. There is no formula for higher degree polynomials. That does not mean that the roots do not exist. There are always n roots for an n^{th} degree polynomial. They may be repeated and they may be complex. Computers are pretty good at finding roots approximately for reasonable size polynomials.

A good place to start is to plot the polynomial and check where it is zero. We can also simply try plugging in. We just start plugging in numbers $r = -2, -1, 0, 1, 2, \ldots$ and see if we get a hit (we can also try complex numbers). Even if we do not get a hit, we may get an indication of where the root is. For example, we plug $r = -2$ into our polynomial and get -15; we plug in $r = 0$ and get 3. That means there is a root between $r = -2$ and $r = 0$, because the sign changed. If we find one root, say r_1, then we know $(r - r_1)$ is a factor of our polynomial. Polynomial long division can then be used.

A good strategy is to begin with $r = -1$, 1, or 0. These are easy to compute. Our polynomial happens to have two such roots, $r_1 = -1$ and $r_2 = 1$ and. There should be three roots and the last root is reasonably easy to find. The constant term in a monic [1] polynomial such as this is the multiple of the negations of all the roots because $r^3 - 3r^2 - r + 3 = (r - r_1)(r - r_2)(r - r_3)$. So

$$3 = (-r_1)(-r_2)(-r_3) = (1)(-1)(-r_3) = r_3$$

You should check that $r_3 = 3$ really is a root. Hence we know that e^{-x}, e^{x}, and e^{3x} are solutions to (4.5.2). They are linearly independent as can easily be checked, and there are three of them, which happens to be exactly the number we need. Hence the general solution is

$$y = C_1 e^{-x} + C_2 e^{x} + C_3 e^{3x}$$

Suppose we were given some initial conditions $y(0) = 1$, $y'(0) = 2$, and $y''(0) = 3$. Then

$$\begin{aligned} 1 &= y(0) = C_1 + C_2 + C_3 \\ 2 &= y'(0) = -C_1 + C_2 + 3C_3 \\ 3 &= y''(0) = C_1 + C_2 + 9C_3 \end{aligned} \tag{4.5.3}$$

It is possible to find the solution by high school algebra, but it would be a pain. The sensible way to solve a system of equations such as this is to use matrix algebra, see Section 3.2 or Appendix A. For now we note that the solution is $C_1 = -\frac{1}{4}$, $C_2 = 1$, and $C_3 = \frac{1}{4}$. The specific solution to the ODE is

$$y = -\frac{1}{4} e^{-x} + e^{x} + \frac{1}{4} e^{3x}$$

Next, suppose that we have real roots, but they are repeated. Let us say we have a root r repeated k times. In the spirit of the second order solution, and for the same reasons, we have the solutions

$$e^{rx}, xe^{rx}, x^2 e^{rx}, \ldots, x^{k-1} e^{rx}$$

We take a linear combination of these solutions to find the general solution.

Example 4.5.4

Solve

$$y^{(4)} - 3y''' + 3y'' - y' = 0$$

Solution

We note that the characteristic equation is

$$r^4 - 3r^3 + 3r^2 - r = 0$$

By inspection we note that $r^4 - 3r^3 + 3r^2 - r = r(r-1)^3$. Hence the roots given with multiplicity are $r = 0, 1, 1, 1$. Thus the general solution is

$$y=\underbrace{(C_1+C_2+C_3x^2)e^x}_{\text{terms coming from r = 1}}+\underbrace{C_4}_{\text{from r = 0}}$$

The case of complex roots is similar to second order equations. Complex roots always come in pairs $r=\alpha\pm i\beta$. Suppose we have two such complex roots, each repeated k times. The corresponding solution is

$$(C_0+C_1x+\cdots+C_{k-1}x^{k-1})e^{ax}\cos(\beta x)+(D_0+D_1x+\cdots+D_{k-1}x^{k-1})e^{ax}\sin(\beta x)$$

where $C_0,\ldots,C_{k-1},D_0,\ldots,D_{k-1}$ are arbitrary constants.

Below is a video on finding the solution to a differential equation given initial values.

Example 4.5.5

Solve

$$y^{(4)}-4y'''+8y''-8y'+4y=0$$

Solution

The characteristic equation is

$$\begin{aligned} r^4-4r^3+8r^2-8r+4 &=0 \\ (r^2-2r+2)^2 &=0 \\ ((r-1)^2+1)^2 &=0 \end{aligned} \tag{4.5.4}$$

Hence the roots are $1\pm i$, both with multiplicity 2. Hence the general solution to the ODE is

$$y=(C_1+C_2x)e^x\cos x+(C_3+C_4x)e^x\sin x$$

The way we solved the characteristic equation above is really by guessing or by inspection. It is not so easy in general. We could also have asked a computer or an advanced calculator for the roots.

4.5.3: Footnotes

[1] The word monic means that the coefficient of the top degree r^d, in our case r^3, is 1.

4.5.4: Outside Links

- After reading this lecture, it may be good to try Project III from the IODE website: www.math.uiuc.edu/iode/.

4.6: Reduction of Order

In this section we give a method for finding the general solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{4.6.1}$$

if we know a nontrivial solution y_1 of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{4.6.2}$$

The method is called *reduction of order* because it reduces the task of solving Equation 4.6.1 to solving a first order equation. Unlike the method of undetermined coefficients, it does not require P_0, P_1, and P_2 to be constants, or F to be of any special form.

By now you shoudn't be surprised that we look for solutions of Equation 4.6.1 in the form

$$y = uy_1 \tag{4.6.3}$$

where u is to be determined so that y satisfies Equation 4.6.1. Substituting Equation 4.6.3 and

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

into Equation 4.6.1 yields

$$P_0(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_2(x)uy_1 = F(x).$$

Collecting the coefficients of u, u', and u'' yields

$$(P_0y_1)u'' + (2P_0y_1' + P_1y_1)u' + (P_0y_1'' + P_1y_1' + P_2y_1)u = F. \tag{4.6.4}$$

However, the coefficient of u is zero, since y_1 satisfies Equation 4.6.2. Therefore Equation 4.6.4 reduces to

$$Q_0(x)u'' + Q_1(x)u' = F, \tag{4.6.5}$$

with

$$Q_0 = P_0y_1 \quad \text{and} \quad Q_1 = 2P_0y_1' + P_1y_1.$$

(It isn't worthwhile to memorize the formulas for Q_0 and Q_1!) Since Equation 4.6.5 is a linear first order equation in u', we can solve it for u' by variation of parameters as in Section 1.2, integrate the solution to obtain u, and then obtain y from Equation 4.6.3.

Here is a video on using reduction of order to solve a differential equation.

Example 4.6.1

a. Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = x^2, \tag{4.6.6}$$

given that $y_1 = e^x$ is a solution of the complementary equation

$$xy'' - (2x+1)y' + (x+1)y = 0. \tag{4.6.7}$$

b. As a byproduct of (a), find a fundamental set of solutions of Equation 4.6.7.

Solution

a. If $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so

$$\begin{aligned} xy'' - (2x+1)y' + (x+1)y &= x(u''e^x + 2u'e^x + ue^x) - (2x+1)(u'e^x + ue^x) + (x+1)ue^x \\ &= (xu'' - u')e^x. \end{aligned}$$

Therefore $y = ue^x$ is a solution of Equation 4.6.6 if and only if

$$(xu'' - u')e^x = x^2,$$

which is a first order equation in u'. We rewrite it as

$$u'' - \frac{u'}{x} = xe^{-x}. \tag{4.6.8}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that Equation 4.6.8 becomes

$$z' - \frac{z}{x} = xe^{-x}. \tag{4.6.9}$$

We leave it to you to show (by separation of variables) that $z_1 = x$ is a solution of the complementary equation

$$z' - \frac{z}{x} = 0$$

for Equation 4.6.9. By applying variation of parameters as in Section 1.2, we can now see that every solution of Equation 4.6.9 is of the form

$$z = vx \quad \text{where} \quad v'x = xe^{-x}, \quad \text{so} \quad v' = e^{-x} \quad \text{and} \quad v = -e^{-x} + C_1.$$

Since $u' = z = vx$, u is a solution of Equation 4.6.8 if and only if

$$u' = vx = -xe^{-x} + C_1x.$$

Integrating this yields

$$u = (x+1)e^{-x} + \frac{C_1}{2}x^2 + C_2.$$

Therefore the general solution of Equation 4.6.6 is

$$y = ue^x = x + 1 + \frac{C_1}{2}x^2e^x + C_2e^x. \tag{4.6.10}$$

b. By letting $C_1 = C_2 = 0$ in Equation 4.6.10, we see that $y_{p_1} = x + 1$ is a solution of Equation 4.6.6. By letting $C_1 = 2$ and $C_2 = 0$, we see that $y_{p_2} = x + 1 + x^2e^x$ is also a solution of Equation 4.6.6. Since the difference of two solutions of Equation 4.6.6 is a solution of Equation 4.6.7, $y_2 = y_{p_1} - y_{p_2} = x^2e^x$ is a solution of Equation 4.6.7. Since y_2/y_1 is nonconstant and we already know that $y_1 = e^x$ is a solution of Equation 4.6.6, Theorem 5.1.6 implies that $\{e^x, x^2e^x\}$ is a fundamental set of solutions of Equation 4.6.7.

Although Equation 4.6.10 is a correct form for the general solution of Equation 4.6.6, it is silly to leave the arbitrary coefficient of x^2e^x as $C_1/2$ where C_1 is an arbitrary constant. Moreover, it is sensible to make the subscripts of the coefficients of $y_1 = e^x$ and $y_2 = x^2e^x$ consistent with the subscripts of the functions themselves. Therefore we rewrite Equation 4.6.10 as

$$y = x + 1 + c_1e^x + c_2x^2e^x$$

by simply renaming the arbitrary constants. We'll also do this in the next two examples, and in the answers to the exercises.

Here is a video on using reduction of order to solve a differential equation given one solution.

Example 4.6.2

a. Find the general solution of

$$x^2y''+xy'-y=x^2+1,$$

given that $y_1=x$ is a solution of the complementary equation

$$x^2y''+xy'-y=0. \tag{4.6.11}$$

As a byproduct of this result, find a fundamental set of solutions of Equation 4.6.11.

b. Solve the initial value problem

$$x^2y''+xy'-y=x^2+1, \quad y(1)=2,\ y'(1)=-3. \tag{4.6.12}$$

Solution

a. If $y=ux$, then $y'=u'x+u$ and $y''=u''x+2u'$, so

$$\begin{aligned} x^2y''+xy'-y &= x^2(u''x+2u')+x(u'x+u)-ux \\ &= x^3u''+3x^2u'. \end{aligned}$$

Therefore $y=ux$ is a solution of Equation 4.6.12 if and only if

$$x^3u''+3x^2u'=x^2+1,$$

which is a first order equation in u'. We rewrite it as

$$u''+\frac{3}{x}u'=\frac{1}{x}+\frac{1}{x^3}. \tag{4.6.13}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z=u'$, so that Equation 4.6.13 becomes

$$z'+\frac{3}{x}z=\frac{1}{x}+\frac{1}{x^3}. \tag{4.6.14}$$

We leave it to you to show by separation of variables that $z_1=1/x^3$ is a solution of the complementary equation

$$z'+\frac{3}{x}z=0$$

for Equation 4.6.14. By variation of parameters, every solution of Equation 4.6.14 is of the form

$$z = \frac{v}{x^3} \quad \text{where} \quad \frac{v'}{x^3} = \frac{1}{x} + \frac{1}{x^3}, \quad \text{so} \quad v' = x^2 + 1 \quad \text{and} \quad v = \frac{x^3}{3} + x + C_1.$$

Since $u' = z = v/x^3$, u is a solution of Equation 4.6.14 if and only if

$$u' = \frac{v}{x^3} = \frac{1}{3} + \frac{1}{x^2} + \frac{C_1}{x^3}.$$

Integrating this yields

$$u = \frac{x}{3} - \frac{1}{x} - \frac{C_1}{2x^2} + C_2.$$

Therefore the general solution of Equation 4.6.12 is

$$y = ux = \frac{x^2}{3} - 1 - \frac{C_1}{2x} + C_2 x. \tag{4.6.15}$$

Reasoning as in the solution of Example 4.6.1a, we conclude that $y_1 = x$ and $y_2 = 1/x$ form a fundamental set of solutions for Equation 4.6.11.

As we explained above, we rename the constants in Equation 4.6.15 and rewrite it as

$$y = \frac{x^2}{3} - 1 + c_1 x + \frac{c_2}{x}. \tag{4.6.16}$$

b. Differentiating Equation 4.6.16 yields

$$y' = \frac{2x}{3} + c_1 - \frac{c_2}{x^2}. \tag{4.6.17}$$

Setting $x = 1$ in Equation 4.6.16 and Equation 4.6.17 and imposing the initial conditions $y(1) = 2$ and $y'(1) = -3$ yields

$$\begin{aligned} c_1 + c_2 &= \frac{8}{3} \\ c_1 - c_2 &= -\frac{11}{3}. \end{aligned}$$

Solving these equations yields $c_1 = -1/2$, $c_2 = 19/6$. Therefore the solution of Equation 4.6.12 is

$$y = \frac{x^2}{3} - 1 - \frac{x}{2} + \frac{19}{6x}.$$

Here is a video on using reduction of order to solve a differential equation given one solution.

Using reduction of order to find the general solution of a homogeneous linear second order equation leads to a homogeneous linear first order equation in u' that can be solved by separation of variables. The next example illustrates this.

Example 4.6.3

Find the general solution and a fundamental set of solutions of

$$x^2y'' - 3xy' + 3y = 0, \tag{4.6.18}$$

given that $y_1 = x$ is a solution.

Solution

If $y = ux$ then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(u''x + 2u') - 3x(u'x + u) + 3ux \\ &= x^3u'' - x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of Equation 4.6.18if and only if

$$x^3u'' - x^2u' = 0.$$

Separating the variables u' and x yields

$$\frac{u''}{u'} = \frac{1}{x},$$

so

$$\ln|u'| = \ln|x| + k, \quad \text{or equivalently} \quad u' = C_1x.$$

Therefore

$$u = \frac{C_1}{2}x^2 + C_2,$$

so the general solution of Equation 4.6.18is

$$y = ux = \frac{C_1}{2}x^3 + C_2x,$$

which we rewrite as

$$y = c_1x + c_2x^3.$$

Therefore $\{x, x^3\}$ is a fundamental set of solutions of Equation 4.6.18.

Here is a video on using reduction of order to solve a differential equation given one solution.

4.6E: Reduction of Order (Exercises)

Q5.6.1

In *Exercises 5.6.1-5.6.17* find the general solution, given that y_1 satisfies the complementary equation. As a byproduct, find a fundamental set of solutions of the complementary equation.

1. $(2x+1)y''-2y'-(2x+3)y=(2x+1)^2; \quad y_1=e^{-x}$
2. $x^2y''+xy'-y=\frac{4}{x^2}; \quad y_1=x$
3. $x^2y''-xy'+y=x; \quad y_1=x$
4. $y''-3y'+2y=\frac{1}{1+e^{-x}}; \quad y_1=e^{2x}$
5. $y''-2y'+y=7x^{3/2}e^x; \quad y_1=e^x$
6. $4x^2y''+(4x-8x^2)y'+(4x^2-4x-1)y=4x^{1/2}e^x(1+4x); \quad y_1=x^{1/2}e^x$
7. $y''-2y'+2y=e^x\sec x; \quad y_1=e^x\cos x$
8. $y''+4xy'+(4x^2+2)y=8e^{-x(x+2)}; \quad y_1=e^{-x^2}$
9. $x^2y''+xy'-4y=-6x-4; \quad y_1=x^2$
10. $x^2y''+2x(x-1)y'+(x^2-2x+2)y=x^3e^{2x}; \quad y_1=xe^{-x}$
11. $x^2y''-x(2x-1)y'+(x^2-x-1)y=x^2e^x; \quad y_1=xe^x$
12. $(1-2x)y''+2y'+(2x-3)y=(1-4x+4x^2)e^x; \quad y_1=e^x$
13. $x^2y''-3xy'+4y=4x^4; \quad y_1=x^2$
14. $2xy''+(4x+1)y'+(2x+1)y=3x^{1/2}e^{-x}; \quad y_1=e^{-x}$
15. $xy''-(2x+1)y'+(x+1)y=-e^x; \quad y_1=e^x$
16. $4x^2y''-4x(x+1)y'+(2x+3)y=4x^{5/2}e^{2x}; \quad y_1=x^{1/2}$
17. $x^2y''-5xy'+8y=4x^2; \quad y_1=x^2$

Q5.6.2

In *Exercises 5.6.18-5.6.30* find a fundamental set of solutions, given that y_1 is a solution.

18. $xy''+(2-2x)y'+(x-2)y=0; \quad y_1=e^x$
19. $x^2y''-4xy'+6y=0; \quad y_1=x^2$
20. $x^2(\ln|x|)^2y''-(2x\ln|x|)y'+(2+\ln|x|)y=0; \quad y_1=\ln|x|$
21. $4xy''+2y'+y=0; \quad y_1=\sin\sqrt{x}$
22. $xy''-(2x+2)y'+(x+2)y=0; \quad y_1=e^x$
23. $x^2y''-(2a-1)xy'+a^2y=0; \quad y_1=x^a$
24. $x^2y''-2xy'+(x^2+2)y=0; \quad y_1=x\sin x$
25. $xy''-(4x+1)y'+(4x+2)y=0; \quad y_1=e^{2x}$
26. $4x^2(\sin x)y''-4x(x\cos x+\sin x)y'+(2x\cos x+3\sin x)y=0; \quad y_1=x^{1/2}$
27. $4x^2y''-4xy'+(3-16x^2)y=0; \quad y_1=x^{1/2}e^{2x}$
28. $(2x+1)xy''-2(2x^2-1)y'-4(x+1)y=0; \quad y_1=1/x$
29. $(x^2-2x)y''+(2-x^2)y'+(2x-2)y=0; \quad y_1=e^x$
30. $xy''-(4x+1)y'+(4x+2)y=0; \quad y_1=e^{2x}$

Q5.6.3

In *Exercises 5.6.31-5.6.33* solve the initial value problem, given that y_1 satisfies the complementary equation.

31. $x^2y''-3xy'+4y=4x^4, \quad y(-1)=7, \quad y'(-1)=-8; \quad y_1=x^2$

32. $(3x-1)y''-(3x+2)y'-(6x-8)y=0, \quad y(0)=2,\ y'(0)=3; \quad y_1=e^{2x}$

33. $(x+1)^2y''-2(x+1)y'-(x^2+2x-1)y=(x+1)^3e^x, \quad y(0)=1, \quad y'(0)=\ -1; \quad y_1=(x+1)e^x$

Q5.6.4

In *Exercises 5.6.34* and *5.6.35* solve the initial value problem and graph the solution, given that y_1 satisfies the complementary equation.

34. $x^2y''+2xy'-2y=x^2, \quad y(1)=\frac{5}{4},\ y'(1)=\frac{3}{2}; \quad y_1=x$

35. $(x^2-4)y''+4xy'+2y=x+2, \quad y(0)=-\frac{1}{3}, \quad y'(0)=-1; \quad y_1=\frac{1}{x-2}$

Q5.6.5

36. Suppose p_1 and p_2 are continuous on (a,b). Let y_1 be a solution of

$$y''+p_1(x)y'+p_2(x)y=0 \tag{A}$$

that has no zeros on (a,b), and let x_0 be in (a,b). Use reduction of order to show that y_1 and

$$y_2(x)=y_1(x)\int_{x_0}^{x}\frac{1}{y_1^2(t)}\exp\left(-\int_{x_0}^{t}p_1(s)\,ds\right)dt \tag{4.6E.1}$$

form a fundamental set of solutions of (A) on (a,b).

37. The nonlinear first order equation

$$y'+y^2+p(x)y+q(x)=0 \tag{A}$$

is a *Riccati equation*. (See *Exercise 2.4.55*.) Assume that p and q are continuous.

a. Show that y is a solution of (A) if and only if $y=z'/z$, where

$$z''+p(x)z'+q(x)z=0. \tag{B}$$

b. Show that the general solution of (A) is

$$y=\frac{c_1z_1'+c_2z_2'}{c_1z_1+c_2z_2}, \tag{C}$$

where $\{z_1,z_2\}$ is a fundamental set of solutions of (B) and c_1 and c_2 are arbitrary constants.

c. Does the formula (C) imply that the first order equation (A) has a two–parameter family of solutions? Explain your answer.

38. Use a method suggested by *Exercise 5.6.37* to find all solutions. of the equation.

a. $y'+y^2+k^2=0$
b. $y'+y^2-3y+2=0$
c. $y'+y^2+5y-6=0$
d. $y'+y^2+8y+7=0$
e. $y'+y^2+14y+50=0$
f. $6y'+6y^2-y-1=0$
g. $36y'+36y^2-12y+1=0$

39. Use a method suggested by *Exercise 5.6.37* and reduction of order to find all solutions of the equation, given that y_1 is a solution.

a. $x^2(y'+y^2)-x(x+2)y+x+2=0; \quad y_1=1/x$
b. $y'+y^2+4xy+4x^2+2=0; \quad y_1=-2x$
c. $(2x+1)(y'+y^2)-2y-(2x+3)=0; \quad y_1=-1$

d. $(3x-1)(y'+y^2)-(3x+2)y-6x+8=0; \quad y_1=2$
e. $x^2(y'+y^2)+xy+x^2-\frac{1}{4}=0; \quad y_1=-\tan x-\frac{1}{2x}$
f. $x^2(y'+y^2)-7xy+7=0; \quad y_1=1/x$

40. The nonlinear first order equation

$$y'+r(x)y^2+p(x)y+q(x)=0 \tag{A}$$

is the *generalized Riccati equation*. (See *Exercise 2.4.55*.) Assume that p and q are continuous and r is differentiable.

a. Show that y is a solution of (A) if and only if $y=z'/rz$, where

$$z''+\left[p(x)-\frac{r'(x)}{r(x)}\right]z'+r(x)q(x)z=0. \tag{B}$$

b. Show that the general solution of (A) is

$$y=\frac{c_1z_1'+c_2z_2'}{r(c_1z_1+c_2z_2)}, \tag{4.6E.2}$$

where $\{z_1,z_2\}$ is a fundamental set of solutions of (B) and c_1 and c_2 are arbitrary constants.

4.7: Variation of Parameters

In this section we give a method called *variation of parameters* for finding a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{4.7.1}$$

if we know a fundamental set $\{y_1, y_2\}$ of solutions of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{4.7.2}$$

Having found a particular solution y_p by this method, we can write the general solution of Equation 4.7.1 as

$$y = y_p + c_1y_1 + c_2y_2.$$

Since we need only one nontrivial solution of Equation 4.7.2 to find the general solution of Equation 4.7.1 by reduction of order, it is natural to ask why we are interested in variation of parameters, which requires two linearly independent solutions of Equation 4.7.2 to achieve the same goal. Here's the answer:

- If we already know two linearly independent solutions of Equation 4.7.2 then variation of parameters will probably be simpler than reduction of order.
- Variation of parameters generalizes naturally to a method for finding particular solutions of higher order linear equations (Section 9.4) and linear systems of equations (Section 10.7), while reduction of order doesn't.
- Variation of parameters is a powerful theoretical tool used by researchers in differential equations. Although a detailed discussion of this is beyond the scope of this book, you can get an idea of what it means from *Exercises* 5.7.37–5.7.39.

We'll now derive the method. As usual, we consider solutions of Equation 4.7.1 and Equation 4.7.2 on an interval (a, b) where P_0, P_1, P_2, and F are continuous and P_0 has no zeros. Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation Equation 4.7.2. We look for a particular solution of Equation 4.7.1 in the form

$$y_p = u_1y_1 + u_2y_2 \tag{4.7.3}$$

where u_1 and u_2 are functions to be determined so that y_p satisfies Equation 4.7.1. You may not think this is a good idea, since there are now two unknown functions to be determined, rather than one. However, since u_1 and u_2 have to satisfy only one condition (that y_p is a solution of Equation 4.7.1), we can impose a second condition that produces a convenient simplification, as follows.

Differentiating Equation 4.7.3 yields

$$y_p' = u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2. \tag{4.7.4}$$

As our second condition on u_1 and u_2 we require that

$$u_1'y_1 + u_2'y_2 = 0. \tag{4.7.5}$$

Then Equation 4.7.4 becomes

$$y_p' = u_1y_1' + u_2y_2'; \tag{4.7.6}$$

that is, Equation 4.7.5 permits us to differentiate y_p (once!) as if u_1 and u_2 are constants. Differentiating Equation 4.7.4 yields

$$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'. \tag{4.7.7}$$

(There are no terms involving u_1'' and u_2'' here, as there would be if we hadn't required Equation 4.7.5.) Substituting Equation 4.7.3, Equation 4.7.6, and Equation 4.7.7 into Equation 4.7.1 and collecting the coefficients of u_1 and u_2 yields

$$u_1(P_0y_1'' + P_1y_1' + P_2y_1) + u_2(P_0y_2'' + P_1y_2' + P_2y_2) + P_0(u_1'y_1' + u_2'y_2') = F.$$

As in the derivation of the method of reduction of order, the coefficients of u_1 and u_2 here are both zero because y_1 and y_2 satisfy the complementary equation. Hence, we can rewrite the last equation as

$$P_0(u_1'y_1' + u_2'y_2') = F. \tag{4.7.8}$$

Therefore y_p in Equation 4.7.3 satisfies Equation 4.7.1 if

$$\begin{aligned} u_1'y_1+u_2'y_2 &= 0 \\ u_1'y_1'+u_2'y_2' &= \frac{F}{P_0}, \end{aligned} \tag{4.7.9}$$

where the first equation is the same as Equation 4.7.5 and the second is from Equation 4.7.8.

We'll now show that you can always solve Equation 4.7.9 for u_1' and u_2'. (The method that we use here will always work, but simpler methods usually work when you're dealing with specific equations.) To obtain u_1', multiply the first equation in Equation 4.7.9 by y_2' and the second equation by y_2. This yields

$$\begin{aligned} u_1'y_1y_2'+u_2'y_2y_2' &= 0 \\ u_1'y_1'y_2+u_2'y_2'y_2 &= \frac{Fy_2}{P_0}. \end{aligned}$$

Subtracting the second equation from the first yields

$$u_1'(y_1y_2'-y_1'y_2) = -\frac{Fy_2}{P_0}. \tag{4.7.10}$$

Since $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 4.7.2 on (a,b), Theorem 5.1.6 implies that the Wronskian $y_1y_2'-y_1'y_2$ has no zeros on (a,b). Therefore we can solve Equation 4.7.10 for u_1', to obtain

$$u_1' = -\frac{Fy_2}{P_0(y_1y_2'-y_1'y_2)}. \tag{4.7.11}$$

We leave it to you to start from Equation 4.7.9 and show by a similar argument that

$$u_2' = \frac{Fy_1}{P_0(y_1y_2'-y_1'y_2)}. \tag{4.7.12}$$

We can now obtain u_1 and u_2 by integrating u_1' and u_2'. The constants of integration can be taken to be zero, since any choice of u_1 and u_2 in Equation 4.7.3 will suffice.

You should not memorize Equation 4.7.11 and Equation 4.7.12. On the other hand, you don't want to rederive the whole procedure for every specific problem. We recommend the a compromise:

a. Write

$$y_p = u_1y_1+u_2y_2 \tag{4.7.13}$$

to remind yourself of what you're doing.

b. Write the system

$$\begin{aligned} u_1'y_1+u_2'y_2 &= 0 \\ u_1'y_1'+u_2'y_2' &= \frac{F}{P_0} \end{aligned} \tag{4.7.14}$$

for the specific problem you're trying to solve.

c. Solve Equation 4.7.14 for u_1' and u_2' by any convenient method.
d. Obtain u_1 and u_2 by integrating u_1' and u_2', taking the constants of integration to be zero.
e. Substitute u_1 and u_2 into Equation 4.7.13 to obtain y_p.

Below is a video on using variation of parameters to solve a differential equation.

Below is another video on using variation of parameters to solve a differential equation.

Example 4.7.1

Find a particular solution y_p of

$$x^2y'' - 2xy' + 2y = x^{9/2}, \tag{4.7.15}$$

given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation

$$x^2y'' - 2xy' + 2y = 0.$$

Then find the general solution of Equation 4.7.15.

Solution

We set

$$y_p = u_1x + u_2x^2,$$

where

$$\begin{aligned} u_1'x + \ \ u_2'x^2 &= 0 \\ u_1' \ \ + 2u_2'x \ &= \frac{x^{9/2}}{x^2} = x^{5/2}. \end{aligned}$$

From the first equation, $u_1' = -u_2'x$. Substituting this into the second equation yields $u_2'x = x^{5/2}$, so $u_2' = x^{3/2}$ and therefore $u_1' = -u_2'x = -x^{5/2}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -\frac{2}{7}x^{7/2} \quad \text{and} \quad u_2 = \frac{2}{5}x^{5/2}.$$

Therefore

$$y_p = u_1 x + u_2 x^2 = -\frac{2}{7} x^{7/2} x + \frac{2}{5} x^{5/2} x^2 = \frac{4}{35} x^{9/2},$$

and the general solution of Equation 4.7.15 is

$$y = \frac{4}{35} x^{9/2} + c_1 x + c_2 x^2.$$

Below is a video on using variation of parameters to solve a differential equation.

Example 4.7.2

Find a particular solution y_p of

$$(x-1)y'' - xy' + y = (x-1)^2, \tag{4.7.16}$$

given that $y_1 = x$ and $y_2 = e^x$ are solutions of the complementary equation

$$(x-1)y'' - xy' + y = 0.$$

Then find the general solution of Equation 4.7.16.

Solution

We set

$$y_p = u_1 x + u_2 e^x,$$

where

$$\begin{aligned} u_1' x + u_2' e^x &= 0 \\ u_1' \quad + u_2' e^x &= \frac{(x-1)^2}{x-1} = x - 1. \end{aligned}$$

Subtracting the first equation from the second yields $-u_1'(x-1) = x-1$, so $u_1' = -1$. From this and the first equation, $u_2' = -xe^{-x} u_1' = xe^{-x}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -x \quad \text{and} \quad u_2 = -(x+1)e^{-x}.$$

Therefore

$$y_p = u_1 x + u_2 e^x = (-x)x + (-(x+1)e^{-x})e^x = -x^2 - x - 1,$$

so the general solution of Equation 4.7.16 is

$$y = y_p + c_1 x + c_2 e^x = -x^2 - x - 1 + c_1 x + c_2 e^x = -x^2 - 1 + (c_1 - 1)x + c_2 e^x. \tag{4.7.17}$$

However, since c_1 is an arbitrary constant, so is $c_1 - 1$; therefore, we improve the appearance of this result by renaming the constant and writing the general solution as

$$y = -x^2 - 1 + c_1 x + c_2 e^x. \tag{4.7.18}$$

There's nothing *wrong* with leaving the general solution of Equation 4.7.16 in the form Equation 4.7.17; however, we think you'll agree that Equation 4.7.18 is preferable. We can also view the transition from Equation 4.7.17 to Equation 4.7.18 differently. In this example the particular solution $y_p = -x^2 - x - 1$ contained the term $-x$, which satisfies the complementary equation. We can drop this term and redefine $y_p = -x^2 - 1$, since $-x^2 - x - 1$ is a solution of Equation 4.7.16 and x is a solution of the complementary equation; hence, $-x^2 - 1 = (-x^2 - x - 1) + x$ is also a solution of Equation 4.7.16. In general, it is always legitimate to drop linear combinations of $\{y_1, y_2\}$ from particular solutions obtained by variation of parameters. (See *Exercise 5.7.36* for a general discussion of this question.) We'll do this in the following examples and in the answers to exercises that ask for a particular solution. Therefore, don't be concerned if your answer to such an exercise differs from ours only by a solution of the complementary equation.

Below is a video on verifying a solution to a differential equation and then solving it using variation of parameters.

Example 4.7.3

Find a particular solution of

$$y'' + 3y' + 2y = \frac{1}{1+e^x}. \tag{4.7.19}$$

Then find the general solution.

Solution

The characteristic polynomial of the complementary equation

$$y'' + 3y' + 2y = 0 \tag{4.7.20}$$

is $p(r) = r^2 + 3r + 2 = (r+1)(r+2)$, so $y_1 = e^{-x}$ and $y_2 = e^{-2x}$ form a fundamental set of solutions of Equation 4.7.20. We look for a particular solution of Equation 4.7.19 in the form

$$y_p = u_1 e^{-x} + u_2 e^{-2x},$$

where

$$\begin{aligned} u_1' e^{-x} + \quad u_2' e^{-2x} &= 0 \\ -u_1' e^{-x} - 2u_2' e^{-2x} &= \frac{1}{1+e^x}. \end{aligned}$$

Adding these two equations yields

$$-u_2'e^{-2x} = \frac{1}{1+e^x}, \quad \text{so} \quad u_2' = -\frac{e^{2x}}{1+e^x}.$$

From the first equation,

$$u_1' = -u_2'e^{-x} = \frac{e^x}{1+e^x}.$$

Integrating by means of the substitution $v = e^x$ and taking the constants of integration to be zero yields

$$u_1 = \int \frac{e^x}{1+e^x}\,dx = \int \frac{dv}{1+v} = \ln(1+v) = \ln(1+e^x)$$

and

$$\begin{aligned} u_2 &= -\int \frac{e^{2x}}{1+e^x}\,dx = -\int \frac{v}{1+v}\,dv = \int \left[\frac{1}{1+v} - 1\right] dv \\ &= \ln(1+v) - v = \ln(1+e^x) - e^x. \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1e^{-x} + u_2e^{-2x} \\ &= [\ln(1+e^x)]e^{-x} + [\ln(1+e^x) - e^x]\,e^{-2x}, \end{aligned}$$

so

$$y_p = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x) - e^{-x}.$$

Since the last term on the right satisfies the complementary equation, we drop it and redefine

$$y_p = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x).$$

The general solution of Equation 4.7.19 is

$$y = y_p + c_1e^{-x} + c_2e^{-2x} = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x) + c_1e^{-x} + c_2e^{-2x}.$$

Example 4.7.4

Solve the initial value problem

$$(x^2-1)y'' + 4xy' + 2y = \frac{2}{x+1}, \quad y(0) = -1, \quad y'(0) = -5, \tag{4.7.21}$$

given that

$$y_1 = \frac{1}{x-1} \quad \text{and} \quad y_2 = \frac{1}{x+1}$$

are solutions of the complementary equation

$$(x^2-1)y'' + 4xy' + 2y = 0.$$

Solution

We first use variation of parameters to find a particular solution of

$$(x^2-1)y'' + 4xy' + 2y = \frac{2}{x+1}$$

on $(-1,1)$ in the form

$$y_p = \frac{u_1}{x-1} + \frac{u_2}{x+1},$$

where

$$\frac{u_1'}{x-1}+\frac{u_2'}{x+1}=0 \tag{4.7.22}$$

$$-\frac{u_1'}{(x-1)^2}-\frac{u_2'}{(x+1)^2}=\frac{2}{(x+1)(x^2-1)}$$

Multiplying the first equation by $1/(x-1)$ and adding the result to the second equation yields

$$\left[\frac{1}{x^2-1}-\frac{1}{(x+1)^2}\right]u_2'=\frac{2}{(x+1)(x^2-1)}. \tag{4.7.23}$$

Since

$$\left[\frac{1}{x^2-1}-\frac{1}{(x+1)^2}\right]=\frac{(x+1)-(x-1)}{(x+1)(x^2-1)}=\frac{2}{(x+1)(x^2-1)},$$

Equation 4.7.23 implies that $u_2'=1$. From Equation 4.7.22,

$$u_1'=-\frac{x-1}{x+1}u_2'=-\frac{x-1}{x+1}.$$

Integrating and taking the constants of integration to be zero yields

$$\begin{aligned} u_1 &= -\int \frac{x-1}{x+1}\,dx=-\int \frac{x+1-2}{x+1}\,dx \\ &= \int\left[\frac{2}{x+1}-1\right]dx=2\ln(x+1)-x \end{aligned}$$

and

$$u_2=\int dx=x.$$

Therefore

$$\begin{aligned} y_p &= \frac{u_1}{x-1}+\frac{u_2}{x+1}=[2\ln(x+1)-x]\frac{1}{x-1}+x\frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1}+x\left[\frac{1}{x+1}-\frac{1}{x-1}\right]=\frac{2\ln(x+1)}{x-1}-\frac{2x}{(x+1)(x-1)}. \end{aligned}$$

However, since

$$\frac{2x}{(x+1)(x-1)}=\left[\frac{1}{x+1}+\frac{1}{x-1}\right]$$

is a solution of the complementary equation, we redefine

$$y_p=\frac{2\ln(x+1)}{x-1}.$$

Therefore the general solution of Equation 4.7.24 is

$$y=\frac{2\ln(x+1)}{x-1}+\frac{c_1}{x-1}+\frac{c_2}{x+1}. \tag{4.7.24}$$

Differentiating this yields

$$y'=\frac{2}{x^2-1}-\frac{2\ln(x+1)}{(x-1)^2}-\frac{c_1}{(x-1)^2}-\frac{c_2}{(x+1)^2}.$$

Setting $x=0$ in the last two equations and imposing the initial conditions $y(0)=-1$ and $y'(0)=-5$ yields the system

$$\begin{aligned} -c_1+c_2 &= -1 \\ -2-c_1-c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = 2,\ c_2 = 1$. Substituting these into Equation 4.7.24 yields

$$\begin{aligned} y &= \frac{2\ln(x+1)}{x-1} + \frac{2}{x-1} + \frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1} \end{aligned}$$

as the solution of Equation 4.7.21. Figure 4.7.1 is a graph of the solution.

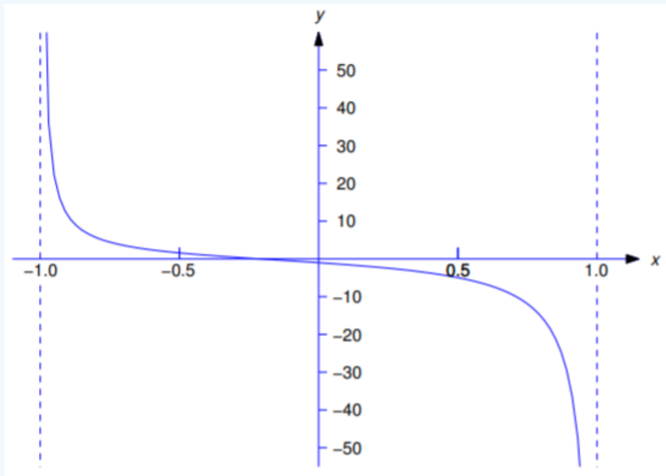

Figure 4.7.1 : $y = \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1}$

Below is a video on using variation of parameters to solve a differential equation.

We’ve now considered three methods for solving nonhomogeneous linear equations: undetermined coefficients, reduction of order, and variation of parameters. It’s natural to ask which method is best for a given problem. The method of undetermined coefficients should be used for constant coefficient equations with forcing functions that are linear combinations of polynomials multiplied by functions of the form $e^{\alpha x}$, $e^{\lambda x} \cos \omega x$, or $e^{\lambda x} \sin \omega x$. Although the other two methods can be used to solve such problems, they will be more difficult except in the most trivial cases, because of the integrations involved.

If the equation isn’t a constant coefficient equation or the forcing function isn’t of the form just specified, the method of undetermined coefficients does not apply and the choice is necessarily between the other two methods. The case could be made that reduction of order is better because it requires only one solution of the complementary equation while variation of parameters requires two. However, variation of parameters will probably be easier if you already know a fundamental set of solutions of the complementary equation.

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4.7E: Variation of Parameters (Exercises)

Q5.7.1

In *Exercises 5.7.1–5.7.6* use variation of parameters to find a particular solution.

1. $y'' + 9y = \tan 3x$
2. $y'' + 4y = \sin 2x \sec^2 2x$
3. $y'' - 3y' + 2y = \frac{4}{1+e^{-x}}$
4. $y'' - 2y' + 2y = 3e^x \sec x$
5. $y'' - 2y' + y = 14x^{3/2}e^x$
6. $y'' - y = \frac{4e^{-x}}{1-e^{-2x}}$

Q5.7.2

In *Exercises 5.7.7-5.7.29* use variation of parameters to find a particular solution, given the solutions y_1, y_2 of the complementary equation.

7. $x^2y'' + xy' - y = 2x^2 + 2; \quad y_1 = x, \quad y_2 = \frac{1}{x}$
8. $xy'' + (2-2x)y' + (x-2)y = e^{2x}; \quad y_1 = e^x, \quad y_2 = \frac{e^x}{x}$
9. $4x^2y'' + (4x - 8x^2)y' + (4x^2 - 4x - 1)y = 4x^{1/2}e^x, \quad x > 0 \;\; ; y_1 = x^{1/2}e^x,\; y_2 = x^{-1/2}e^x$
10. $y'' + 4xy' + (4x^2 + 2)y = 4e^{-x(x+2)}; \quad y_1 = e^{-x^2}, \quad y_2 = xe^{-x^2}$
11. $x^2y'' - 4xy' + 6y = x^{5/2},\; x > 0; \quad y_1 = x^2,\; y_2 = x^3$
12. $x^2y'' - 3xy' + 3y = 2x^4 \sin x; \quad y_1 = x,\; y_2 = x^3$
13. $(2x+1)y'' - 2y' - (2x+3)y = (2x+1)^2e^{-x}; \quad y_1 = e^{-x}, \quad y_2 = xe^x$
14. $4xy'' + 2y' + y = \sin\sqrt{x}; \quad y_1 = \cos\sqrt{x}, \quad y_2 = \sin\sqrt{x}$
15. $xy'' - (2x+2)y' + (x+2)y = 6x^3e^x; \quad y_1 = e^x, \quad y_2 = x^3e^x$
16. $x^2y'' - (2a-1)xy' + a^2y = x^{a+1}; \quad y_1 = x^a, \quad y_2 = x^a \ln x$
17. $x^2y'' - 2xy' + (x^2+2)y = x^3 \cos x; \quad y_1 = x\cos x, \quad y_2 = x\sin x$
18. $xy'' - y' - 4x^3y = 8x^5; \quad y_1 = e^{x^2},\; y_2 = e^{-x^2}$
19. $(\sin x)y'' + (2\sin x - \cos x)y' + (\sin x - \cos x)y = e^{-x}; \quad y_1 = e^{-x}, \quad y_2 = e^{-x}\cos x$
20. $4x^2y'' - 4xy' + (3 - 16x^2)y = 8x^{5/2}; \quad y_1 = \sqrt{x}e^{2x},\; y_2 = \sqrt{x}e^{-2x}$
21. $4x^2y'' - 4xy' + (4x^2 + 3)y = x^{7/2}; \quad y_1 = \sqrt{x}\sin x,\; y_2 = \sqrt{x}\cos x$
22. $x^2y'' - 2xy' - (x^2 - 2)y = 3x^4; \quad y_1 = xe^x,\; y_2 = xe^{-x}$
23. $x^2y'' - 2x(x+1)y' + (x^2 + 2x + 2)y = x^3e^x; \quad y_1 = xe^x, \quad y_2 = x^2e^x$
24. $x^2y'' - xy' - 3y = x^{3/2}; \quad y_1 = 1/x, \quad y_2 = x^3$
25. $x^2y'' - x(x+4)y' + 2(x+3)y = x^4e^x; \quad y_1 = x^2, \quad y_2 = x^2e^x$
26. $x^2y'' - 2x(x+2)y' + (x^2 + 4x + 6)y = 2xe^x; \quad y_1 = x^2e^x, \quad y_2 = x^3e^x$
27. $x^2y'' - 4xy' + (x^2 + 6)y = x^4; \quad y_1 = x^2\cos x, \quad y_2 = x^2\sin x$
28. $(x-1)y'' - xy' + y = 2(x-1)^2e^x; \quad y_1 = x, \quad y_2 = e^x$
29. $4x^2y'' - 4x(x+1)y' + (2x+3)y = x^{5/2}e^x; \quad y_1 = \sqrt{x}, \quad y_2 = \sqrt{x}e^x$

Q5.7.3

In *Exercises 5.7.30–5.7.32* use variation of parameters to solve the initial value problem, given y_1, y_2 are solutions of the complementary equation.

30. $(3x-1)y''-(3x+2)y'-(6x-8)y=(3x-1)^2e^{2x}, \quad y(0)=1,\ y'(0)=2$; $y_1=e^{2x},\ y_2=xe^{-x}$

31. $(x-1)^2y''-2(x-1)y'+2y=(x-1)^2, \quad y(0)=3, \quad y'(0)=-6$;

$y_1=x-1$, $y_2=x^2-1$

32. $(x-1)^2y''-(x^2-1)y'+(x+1)y=(x-1)^3e^x, \quad y(0)=4, \quad y'(0)=-6$;

$y_1=(x-1)e^x, \quad y_2=x-1$

Q5.7.4

In *Exercises 5.7.33-5.7.35* use variation of parameters to solve the initial value problem and graph the solution, given that y_1, y_2 are solutions of the complementary equation.

33. $(x^2-1)y''+4xy'+2y=2x, \quad y(0)=0,\ y'(0)=-2; \quad y_1=\frac{1}{x-1},\ y_2=\frac{1}{x+1}$

34. $x^2y''+2xy'-2y=-2x^2, \quad y(1)=1,\ y'(1)=-1; \quad y_1=x,\ y_2=\frac{1}{x^2}$

35. $(x+1)(2x+3)y''+2(x+2)y'-2y=(2x+3)^2, \quad y(0)=0, \quad y'(0)=0$; $y_1=x+2, \quad y_2=\frac{1}{x+1}$

Q5.7.5

36. Suppose

$$y_p=\bar{y}+a_1y_1+a_2y_2 \tag{4.7E.1}$$

is a particular solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F(x), \tag{A}$$

where y_1 and y_2 are solutions of the complementary equation

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0. \tag{4.7E.2}$$

Show that $\bar{y}$ is also a solution of (A).

37. Suppose p, q, and f are continuous on (a,b) and let x_0 be in (a,b). Let y_1 and y_2 be the solutions of

$$y''+p(x)y'+q(x)y=0 \tag{4.7E.3}$$

such that

$$y_1(x_0)=1, \quad y_1'(x_0)=0, \quad y_2(x_0)=0, \quad y_2'(x_0)=1. \tag{4.7E.4}$$

Use variation of parameters to show that the solution of the initial value problem

$$y''+p(x)y'+q(x)y=f(x), \quad y(x_0)=k_0,\ y'(x_0)=k_1, \tag{4.7E.5}$$

is

$$\begin{aligned} y(x) &= k_0y_1(x)+k_1y_2(x) \\ &+\int_{x_0}^x (y_1(t)y_2(x)-y_1(x)y_2(t))\,f(t)\exp\left(\int_{x_0}^t p(s)\,ds\right)dt. \end{aligned} \tag{4.7E.6}$$

HINT: *Use Abel's formula for the Wronskian of* $\{y_1,y_2\}$*, and integrate* u_1' *and* u_2' *from* x_0 *to* x.

Show also that

$$\begin{aligned} y'(x) &= k_0y_1'(x)+k_1y_2'(x) \\ &+\int_{x_0}^x (y_1(t)y_2'(x)-y_1'(x)y_2(t))\,f(t)\exp\left(\int_{x_0}^t p(s)\,ds\right)dt. \end{aligned} \tag{4.7E.7}$$

38. Suppose f is continuous on an open interval that contains $x_0 = 0$. Use variation of parameters to find a formula for the solution of the initial value problem

$$y'' - y = f(x), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{4.7E.8}$$

39. Suppose f is continuous on (a, ∞), where $a < 0$, so $x_0 = 0$ is in (a, ∞).

a. Use variation of parameters to find a formula for the solution of the initial value problem

$$y'' + y = f(x), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{4.7E.9}$$

HINT: *You will need the addition formulas for the sine and cosine.*

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B \end{aligned}$$

For the rest of this exercise assume that the improper integral $\int_0^\infty f(t)\,dt$ is absolutely convergent.

b. Show that if y is a solution of

$$y'' + y = f(x) \tag{A}$$

on (a, ∞), then

$$\lim_{x\to\infty} (y(x) - A_0 \cos x - A_1 \sin x) = 0 \tag{B}$$

and

$$\lim_{x\to\infty} (y'(x) + A_0 \sin x - A_1 \cos x) = 0, \tag{C}$$

where

$$A_0 = k_0 - \int_0^\infty f(t) \sin t\, dt \quad \text{and} \quad A_1 = k_1 + \int_0^\infty f(t) \cos t\, dt. \tag{4.7E.10}$$

HINT: Recall from calculus that if $\int_0^\infty f(t)dt$ *converges absolutely, then* $\lim_{x\to\infty} \int_x^\infty |f(t)|dt = 0$.

c. Show that if A_0 and A_1 are arbitrary constants, then there's a unique solution of $y'' + y = f(x)$ on (a, ∞) that satisfies (B) and (C).

4.8: Mechanical Vibrations

Let us look at some applications of linear second order constant coefficient equations.

4.8.1: Some examples

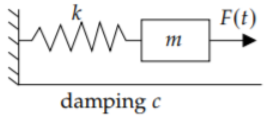

Figure 4.8.1

Our first example is a mass on a spring. Suppose we have a mass $m > 0$ (in kilograms) connected by a spring with spring constant $k > 0$ (in newtons per meter) to a fixed wall. There may be some external force $F(t)$ (in newtons) acting on the mass. Finally, there is some friction measured by $c \geq 0$ (in newton-seconds per meter) as the mass slides along the floor (or perhaps there is a damper connected).

Let x be the displacement of the mass ($x = 0$ is the rest position), with x growing to the right (away from the wall). The force exerted by the spring is proportional to the compression of the spring by Hooke's law. Therefore, it is kx in the negative direction. Similarly the amount of force exerted by friction is proportional to the velocity of the mass. By Newton's second law we know that force equals mass times acceleration and hence $mx'' = F(t) - cx' - kx$ or

$$mx'' + cx' + kx = F(t)$$

This is a linear second order constant coefficient ODE. We set up some terminology about this equation. We say the motion is

i. forced, if $F \not\equiv 0$ (if F is not identically zero),
ii. unforced or free, if $F \equiv 0$ (if F is identically zero),
iii. damped, if $c > 0$, and
iv. undamped, if $c = 0$.

This system appears in lots of applications even if it does not at first seem like it. Many real-world scenarios can be simplified to a mass on a spring. For example, a bungee jump setup is essentially a mass and spring system (you are the mass). It would be good if someone did the math before you jump off the bridge, right? Let us give two other examples.

Here is an example for electrical engineers. Consider the pictured RLC circuit. There is a resistor with a resistance of R ohms, an inductor with an inductance of L henries, and a capacitor with a capacitance of C farads. There is also an electric source (such as a battery) giving a voltage of $E(t)$ volts at time t (measured in seconds). Let $Q(t)$ be the charge in coulombs on the capacitor and $I(t)$ be the current in the circuit. The relation between the two is $Q' = I$. By elementary principles we find $LI' + RI + \frac{Q}{C} = E$. We differentiate to get

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t).$$

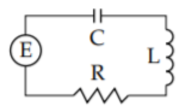

Figure 4.8.2

This is a nonhomogeneous second order constant coefficient linear equation. As L, R, and C are all positive, this system behaves just like the mass and spring system. Position of the mass is replaced by current. Mass is replaced by inductance, damping is replaced by resistance, and the spring constant is replaced by one over the capacitance. The change in voltage becomes the forcing function—for constant voltage this is an unforced motion.

Our next example behaves like a mass and spring system only approximately. Suppose a mass m hangs on a pendulum of length L. We seek an equation for the angle $\theta(t)$ (in radians). Let g be the force of gravity. Elementary physics mandates that the equation is

$$\theta'' + \frac{g}{L}\sin\theta = 0.$$

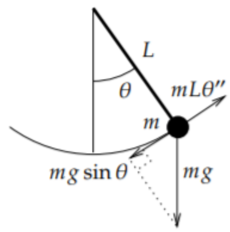

Figure 4.8.3

Let us derive this equation using Newton's second law: force equals mass times acceleration. The acceleration is $L\theta''$ and mass is m. So $mL\theta''$ has to be equal to the tangential component of the force given by the gravity, which is $mg\sin\theta$ in the opposite direction. So $mL\theta'' = -mg\sin\theta$. The m curiously cancels from the equation.

Now we make our approximation. For small θ we have that approximately $\sin\theta \approx \theta$. This can be seen by looking at the graph. In Figure $4.8.4$ we can see that for approximately $-0.5 < \theta < 0.5$ (in radians) the graphs of $\sin\theta$ and θ are almost the same.

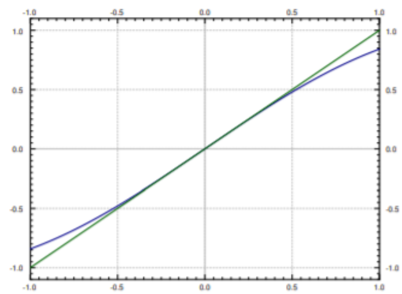

Figure 4.8.4: The graphs of $\sin\theta$ and θ (in radians).

Therefore, when the swings are small, θ is small and we can model the behavior by the simpler linear equation

$$\theta'' + \frac{g}{L}\theta = 0.$$

The errors from this approximation build up. So after a long time, the state of the real-world system might be substantially different from our solution. Also we will see that in a mass-spring system, the amplitude is independent of the period. This is not true for a pendulum. Nevertheless, for reasonably short periods of time and small swings (that is, only small angles θ), the approximation is reasonably good.

In real-world problems it is often necessary to make these types of simplifications. We must understand both the mathematics and the physics of the situation to see if the simplification is valid in the context of the questions we are trying to answer.

4.8.2: Free Undamped Motion

In this section we will only consider free or unforced motion, as we cannot yet solve nonhomogeneous equations. Let us start with undamped motion where $c = 0$. We have the equation

$$mx'' + kx = 0$$

If we divide by m and let $w_0 = \sqrt{\frac{k}{m}}$, then we can write the equation as

$$x'' + w_0^2 x = 0$$

The general solution to this equation is

$$x(t) = A\cos(w_0 t) + B\sin(w_0 t)$$

By a trigonometric identity, we have that for two different constants C and γ, we have

$$A\cos(w_0 t) + B\sin(w_0 t) = C\cos(w_0 t - \gamma)$$

It is not hard to compute that $C = \sqrt{A^2 + B^2}$ and $\tan\gamma = \frac{B}{A}$. Therefore, we let C and γ be our arbitrary constants and write $x(t) = C\cos(w_0 t - \gamma)$.

Below is a video on solving a differential equation that comes from a free undamped spring system.

? Exercise 4.8.1

Justify the above identity and verify the equations for C and γ. Hint: Start with $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ and multiply by C. Then think what should α and β be.

While it is generally easier to use the first form with A and B to solve for the initial conditions, the second form is much more natural. The constants C and γ have very nice interpretation. We look at the form of the solution

$$x(t) = C\cos(w_0 t - \gamma)$$

We can see that the amplitude is C, w_0 is the (angular) frequency, and γ is the so-called phase shift. The phase shift just shifts the graph left or right. We call w_0 the natural (angular) frequency. This entire setup is usually called simple harmonic motion.

Let us pause to explain the word angular before the word frequency. The units of w_0 are radians per unit time, not cycles per unit time as is the usual measure of frequency. Because we know one cycle is 2π radians, the usual frequency is given by $\frac{w_0}{2\pi}$. It is simply a matter of where we put the constant 2π, and that is a matter of taste.

The period of the motion is one over the frequency (in cycles per unit time) and hence $\frac{2\pi}{w_0}$. That is the amount of time it takes to complete one full oscillation.

Below is a video on solving a initial value problem that comes from a free undamped spring system.

Example 4.8.1

Suppose that $m = 2kg$ and $k = 8\frac{N}{m}$. The whole mass and spring setup is sitting on a truck that was traveling at $1\frac{m}{s}$. The truck crashes and hence stops. The mass was held in place 0.5 meters forward from the rest position. During the crash the mass gets loose. That is, the mass is now moving forward at $1\frac{m}{s}$, while the other end of the spring is held in place. The mass therefore starts oscillating. What is the frequency of the resulting oscillation and what is the amplitude. The units are the mks units (meters-kilograms-seconds).

The setup means that the mass was at half a meter in the positive direction during the crash and relative to the wall the spring is mounted to, the mass was moving forward (in the positive direction) at $1\frac{m}{s}$. This gives us the initial conditions.

So the equation with initial conditions is

$$2x'' + 8x = 0, \quad x(0) = 0.5, \quad x'(0) = 1$$

We can directly compute $w_0 = \sqrt{\frac{k}{m}} = \sqrt{4} = 2$. Hence the angular frequency is 2. The usual frequency in Hertz (cycles per second) is $\frac{2}{2\pi} = \frac{1}{\pi} \approx 0.318$.

The general solution is

$$x(t) = A\cos(2t) + B\sin(2t)$$

Letting $x(0) = 0.5$ means $A = 0.5$. Then $x'(t) = -2(0.5)\sin(2t) + 2B\cos(2t)$. Letting $x'(0) = 1$ we get $B = 0.5$. Therefore, the amplitude is $C = \sqrt{A^2 + B^2} = \sqrt{0.25 + 0.25} = \sqrt{0.5} \approx 0.707$. The solution is

$$x(t) = 0.5\cos(2t) + 0.5\sin(2t)$$

A plot of $x(t)$ is shown in Figure $4.8.5$.

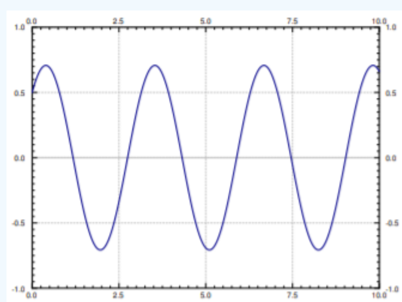

Figure $4.8.5$: Simple undamped oscillation.

In general, for free undamped motion, a solution of the form

$$x(t) = A\cos(w_0 t) + B\sin(w_0 t)$$

corresponds to the initial conditions $x(0) = A$ and $x'(0) = w_0 B$. Therefore, it is easy to figure out A and B from the initial conditions. The amplitude and the phase shift can then be computed from A and B. In the example, we have already found the amplitude C. Let us compute the phase shift. We know that $\tan\gamma = \frac{B}{A} = 1$. We take the arctangent of 1 and get approximately 0.785. We still need to check if this γ is in the correct quadrant (and add π to γ if it is not). Since both A and B are positive, then γ should be in the first quadrant, and 0.785 radians really is in the first quadrant.

Note

Many calculators and computer software do not only have the atan function for arctangent, but also what is sometimes called `atan2` . This function takes two arguments, B and A, and returns a γ in the correct quadrant for you.

Below is a video on solving a differential equation that comes from an undamped spring system.

4.8.3: Free Damped Motion

Let us now focus on damped motion. Let us rewrite the equation

$$mx'' + cx' + kx = 0$$

as

$$x'' + 2px' + w_0^2 x = 0$$

where

$$w_0 = \sqrt{\frac{k}{m}}, \quad p = \frac{c}{2m}$$

The characteristic equation is

$$r^2 + 2pr + w_0^2 = 0$$

Using the quadratic formula we get that the roots are

$$r = -p \pm \sqrt{p^2 - w_0^2}$$

The form of the solution depends on whether we get complex or real roots. We get real roots if and only if the following number is nonnegative:

$$p^2 - w_0^2 = \left(\frac{c}{2m}\right)^2 - \frac{k}{m} = \frac{c^2 - 4km}{4m^2}$$

The sign of $p^2 - w_0^2$ is the same as the sign of $c^2 - 4km$. Thus we get real roots if and only if $c^2 - 4km$ is nonnegative, or in other words if $c^2 \geq 4km$.

4.8.3.1: Overdamping

When $c^2 - 4km > 0$, we say the system is overdamped. In this case, there are two distinct real roots r_1 and r_2. Notice that both roots are negative. As $\sqrt{p^2 - w_0^2}$ is always less than P, then $-P \pm \sqrt{P^2 - w_0^2}$ is negative.

The solution is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Since r_1, r_2 are negative, $x(t) \to 0$ as $t \to \infty$. Thus the mass will tend towards the rest position as time goes to infinity. For a few sample plots for different initial conditions (Figure $4.8.6$).

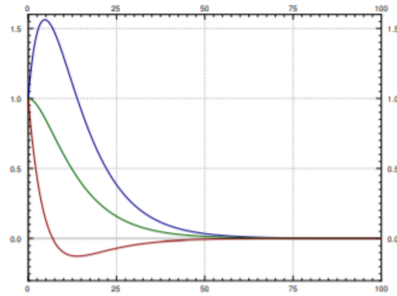

Figure 4.8.6: Overdamped motion for several different initial conditions.

Do note that no oscillation happens. In fact, the graph will cross the x axis at most once. To see why, we try to solve $0=C_1e^{r_1t}+C_2e^{r_2t}$. Therefore, $C_1e^{r_1t}=-C_2e^{r_2t}$ and using laws of exponents we obtain

$$\frac{-C_1}{C_2}=e^{(r_2-r_1)t}$$

This equation has at most one solution $t\geq 0$. For some initial conditions the graph will never cross the x axis, as is evident from the sample graphs.

Example 4.8.2

Suppose the mass is released from rest. That is $x(0)=x_0$ and $x'(0)=0$. Then

$$x(t)=\frac{x_0}{r_1-r_2}(r_1e^{r_2t}-r_2e^{r_1t})$$

It is not hard to see that this satisfies the initial conditions.

Below is a video on solving a differential equation that comes from a damped spring system.

4.8.3.2: Critical damping

When $c^2-4km=0$, we say the system is critically damped. In this case, there is one root of multiplicity 2 and this root is $-P$. Therefore, our solution is

$$x(t)=C_1e^{-pt}+C_2te^{-pt}$$

The behavior of a critically damped system is very similar to an overdamped system. After all a critically damped system is in some sense a limit of overdamped systems. Since these equations are really only an approximation to the real world, in reality we are never critically damped, it is a place we can only reach in theory. We are always a little bit underdamped or a little bit overdamped. It is better not to dwell on critical damping.

4.8.3.3: Underdamping

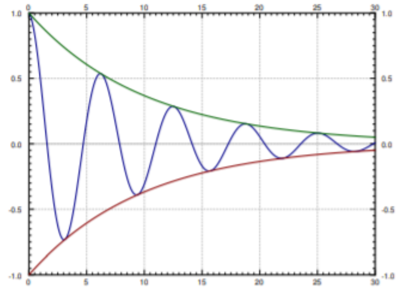

Figure 4.8.7: Underdamped motion with the envelope curves shown.

When $c^2 - 4km < 0$, we say the system is underdamped. In this case, the roots are complex.

$$\begin{aligned} r &= -p \pm \sqrt{p^2 - w_0^2} \\ &= -p \pm \sqrt{-1}\sqrt{w_0^2 - p^2} \\ &= -p \pm i w_1 \end{aligned} \tag{4.8.1}$$

where $w_1 = \sqrt{w_0^2 - p^2}$. Our solution is

$$x(t) = e^{-pt}(A\cos(w_1 t) + B\sin(w_1 t)$$

or

$$x(t) = Ce^{-pt}\cos(w_1 t - \gamma)$$

An example plot is given in Figure $4.8.7$. Note that we still have that $x(t) \to 0$ as $t \to \infty$.

In the figure we also show the *envelope curves* Ce^{-pt} and $-Ce^{pt}$. The solution is the oscillating line between the two envelope curves. The envelope curves give the maximum amplitude of the oscillation at any given point in time. For example if you are bungee jumping, you are really interested in computing the envelope curve so that you do not hit the concrete with your head.

The phase shift γ just shifts the graph left or right but within the envelope curves (the envelope curves do not change if γ changes).

Finally note that the angular pseudo-frequency[1] (we do not call it a frequency since the solution is not really a periodic function) w_1 becomes smaller when the damping c (and hence P) becomes larger. This makes sense. When we change the damping just a little bit, we do not expect the behavior of the solution to change dramatically. If we keep making c larger, then at some point the solution should start looking like the solution for critical damping or overdamping, where no oscillation happens. So if c^2 approaches $4km$, we want w_1 to approach 0.

On the other hand when c becomes smaller, w_1 approaches w_0 (w_1 is always smaller than w_0), and the solution looks more and more like the steady periodic motion of the undamped case. The envelope curves become flatter and flatter as c (and hence P) goes to 0.

4.8.4: Footnotes

[1] We do not call ω_1 a frequency since the solution is not really a periodic function.

4.9: Nonhomogeneous Equations

4.9.1: Solving Nonhomogeneous Equations

We have solved linear constant coefficient homogeneous equations. What about nonhomogeneous linear ODEs? For example, the equations for forced mechanical vibrations. That is, suppose we have an equation such as

$$y'' + 5y' + 6y = 2x + 1 \tag{4.9.1}$$

We will write $Ly = 2x + 1$ when the exact form of the operator is not important. We solve (Equation 4.9.1) in the following manner. First, we find the general solution y_c to the associated homogeneous equation

$$y'' + 5y' + 6y = 0 \tag{4.9.2}$$

We call y_c the *complementary solution*. Next, we find a single particular solution y_p to (4.9.1) in some way. Then

$$y = y_c + y_p$$

is the general solution to (4.9.1). We have $Ly_c = 0$ and $Ly_p = 2x + 1$. As L is a linear operator we verify that y is a solution, $Ly = L(y_c + y_p) = Ly_c + Ly_p = 0 + (2x + 1)$. Let us see why we obtain the general solution.

Let y_p and $\tilde{y}_p$ be two different particular solutions to (4.9.1). Write the difference as $w = y_p - \tilde{y}_p$. Then plug w into the left hand side of the equation to get

$$w'' + 5w' + 6w = (y_p'' + 5y_p' + 6y_p) - (\tilde{y}_p'' + 5\tilde{y}_p' + 6\tilde{y}_p) = (2x + 1) - (2x + 1) = 0$$

Using the operator notation the calculation becomes simpler. As L is a linear operator we write

$$Lw = L(y_p - \tilde{y}_p) = Ly_p - L\tilde{y}_p = (2x + 1) - (2x + 1) = 0$$

So $w = y_p - \tilde{y}_p$ is a solution to (4.9.2), that is $Lw = 0$. Any two solutions of (4.9.1) differ by a solution to the homogeneous equation (4.9.2). The solution $y = y_c + y_p$ includes all solutions to (4.9.1), since y_c is the general solution to the associated homogeneous equation.

Theorem 4.9.1

Let $Ly = f(x)$ be a linear ODE (not necessarily constant coefficient). Let y_c be the complementary solution (the general solution to the associated homogeneous equation $Ly = 0$) andlet y_p be any particular solution to $Ly = f(x)$. Then the general solution to $Ly = f(x)$ is

$$y = y_c + y_p.$$

The moral of the story is that we can find the particular solution in any old way. If we find a different particular solution (by a different method, or simply by guessing), then we still get the same general solution. The formula may look different, and the constants we will have to choose to satisfy the initial conditions may be different, but it is the same solution.

4.9.1.1: Undetermined Coefficients

The trick is to somehow, in a smart way, guess one particular solution to (4.9.1). Note that $2x + 1$ is a polynomial, and the left hand side of the equation will be a polynomial if we let y be a polynomial of the same degree. Let us try

$$y_p = Ax + B$$

We plug in to obtain

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= (Ax + B)'' + 5(Ax + B)' + 6(Ax + B) \\ &= 0 + 5A + 6Ax + 6B = 6Ax + (5A + 6B) \end{aligned} \tag{4.9.3}$$

So $6Ax + (5A + 6B) = 2x + 1$. Therefore, $A = \frac{1}{3}$ and $B = -\frac{1}{9}$. That means $y_p = \frac{1}{3}x - \frac{1}{9} = \frac{3x - 1}{9}$. Solving the complementary problem (exercise!) we get

$$y_c = C_1 e^{-2x} + C_2 e^{-3x}$$

Hence the general solution to (4.9.1) is

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{3x-1}{9}$$

Now suppose we are further given some initial conditions. For example, $y(0) = 0$ and $y'(0) = \frac{1}{3}$. First find $y' = -2C_1 e^{-2x} - 3C_2 e^{-3x} + \frac{1}{3}$. Then

$$0 = y(0) = C_1 + C_2 - \frac{1}{9}, \frac{1}{3} = y'(0) = -2C_1 - 3C_2 + \frac{1}{3}$$

We solve to get $C_1 = \frac{1}{3}$ and $C_2 = -\frac{2}{9}$. The particular solution we want is

$$y(x) = \frac{1}{3} e^{-2x} - \frac{2}{9} e^{-3x} + \frac{3x-1}{9} = \frac{3e^{-2x} - 2e^{-3x} + 3x - 1}{9}$$

? Exercise 4.9.1

Check that y really solves the equation (4.9.1)and the given initial conditions.

Note

A common mistake is to solve for constants using the initial conditions with y_c and only add the particular solution y_p after that. That will not work. You need to first compute $y = y_c + y_p$ and only then solve for the constants using the initial conditions.

A right hand side consisting of exponentials, sines, and cosines can be handled similarly. For example,

$$y'' + 2y' + 2y = \cos(2x)$$

Let us find some y_p. We start by guessing the solution includes some multiple of $\cos(2x)$. We may have to also add a multiple of $\sin(2x)$ to our guess since derivatives of cosine are sines. We try

$$y_p = A\cos(2x) + B\sin(2x)$$

We plug y_p into the equation and we get

$$\underbrace{-4A\cos(2x) - 4B\sin(2x)}_{y_p''} + 2\underbrace{\left(-2A\sin(2x) + 2B\cos(2x)\right)}_{y_p'}$$
$$+2\underbrace{\left(A\cos(2x) + 2B\sin(2x)\right)}_{y_p} = \cos(2x),$$

The left hand side must equal to right hand side. We group terms and we get that $-4A + 4B + 2A = 1$ and $-4B - 4A + 2B = 0$. So $-2A + 4B = 1$ and $2A + B = 0$ and hence $A = \frac{-1}{10}$ and $B = \frac{1}{5}$. So

$$y_p = A\cos(2x) + B\sin(2x) = \frac{-\cos(2x) + 2\sin(2x)}{10}$$

Similarly, if the right hand side contains exponentials we try exponentials. For example, for

$$Ly = e^{3x}$$

we will try $y = Ae^{3x}$ as our guess and try to solve for A.

When the right hand side is a multiple of sines, cosines, exponentials, and polynomials, we can use the product rule for differentiation to come up with a guess. We need to guess a form for y_p such that Ly_p is of the same form, and has all the terms needed to for the right hand side. For example,

$$Ly = (1+3x^2)e^{-x}\cos(\pi x)$$

For this equation, we will guess

$$y_p = (A+Bx+Cx^2)e^{-x}\cos(\pi x) + (D+Ex+Fx^2)e^{-x}\sin(\pi x)$$

We will plug in and then hopefully get equations that we can solve for $A, B, C, D, E$and F. As you can see this can make for a very long and tedious calculation very quickly.

There is one hiccup in all this. It could be that our guess actually solves the associated homogeneous equation. That is, suppose we have

$$y'' - 9y = e^{3x}$$

We would love to guess $y = Ae^{3x}$, but if we plug this into the left hand side of the equation we get

$$y'' - 9y = 9Ae^{3x} - 9Ae^{3x} = 0 \neq e^{3x}$$

There is no way we can choose A to make the left hand side be e^{3x}. The trick in this case is to multiply our guess by x to get rid of duplication with the complementary solution. That is first we compute y_c (solution to $Ly = 0$)

$$y_c = C_1 e^{-3x} + C_2 e^{3x}$$

and we note that the e^{3x} term is a duplicate with our desired guess. We modify our guess to $y = Axe^{3x}$ and notice there is no duplication anymore. Let us try. Note that $y' = Ae^{3x} + 3Axe^{3x}$ and $y'' = 6Ae^{3x} + 9Axe^{3x}$. So

$$y'' - 9y = 6Ae^{3x} + 9Axe^{3x} - 9Axe^{3x} = 6Ae^{3x}$$

Thus $6Ae^{3x}$ is supposed to equal e^{3x}. Hence, $6A = 1$ and so $A = \frac{1}{6}$. We can now write the general solution as

$$y = y_c + y_p = C_1 e^{-3x} + C_2 e^{3x} + \frac{1}{6} x e^{3x}$$

It is possible that multiplying by x does not get rid of all duplication. For example,

$$y'' - 6y' + 9y = e^{3x}$$

The complementary solution is $y_c = C_1 e^{3x} + C_2 x e^{3x}$. Guessing $y = Axe^{3x}$ would not get us anywhere. In this case we want to guess $y_p = Ax^2e^{3x}$. Basically, we want to multiply our guess by x until all duplication is gone. But no more! Multiplying too many times will not work.

Finally, what if the right hand side has several terms, such as

$$Ly = e^{2x} + \cos x$$

In this case we find u that solves $Lu = e^{2x}$ and v that solves $Lv = \cos x$ (that is, do each term separately). Then note that if $y = u + v$, then $Ly = e^{2x} + \cos x$. This is because L is linear; we have $Ly = L(u+v) = Lu + Lv = e^{2x} + \cos x$.

4.9.1.2: Variation of Parameters

The method of undetermined coefficients will work for many basic problems that crop up. But it does not work all the time. It only works when the right hand side of the equation $Ly = f(x)$ has only finitely many linearly independent derivatives, so that we can write a guess that consists of them all. Some equations are a bit tougher. Consider

$$y'' + y = \tan x$$

Note that each new derivative of $\tan x$ looks completely different and cannot be written as a linear combination of the previous derivatives. If we start differentiating $\tan x$, we get

$$\begin{gathered} \sec^2 x, \quad 2\sec^2 x \tan x, \quad 4\sec^2 x \tan^2 x + 2\sec^4 x, \\ 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x, \quad 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x, \quad \ldots \end{gathered}$$

This equation calls for a different method. We present the method of variation of parameters, which will handle any equation of the form $Ly = f(x)$, provided we can solve certain integrals. For simplicity, we restrict ourselves to second order constant coefficient equations, but the method works for higher order equations just as well (the computations become more tedious). The method also works for equations with nonconstant coefficients, provided we can solve the associated homogeneous equation.

Perhaps it is best to explain this method by example. Let us try to solve the equation

$$Ly = y'' + y = \tan x$$

First we find the complementary solution (solution to $Ly_c = 0$). We get $y_c = C_1 y_1 + C_2 y_2$, where $y_1 = \cos x$ and $y_2 = \sin x$. To find a particular solution to the nonhomogeneous equation we try

$$y_p = y = u_1 y_1 + u_2 y_2$$

where u_1 and u_2 are functions and not constants. We are trying to satisfy $Ly = \tan x$. That gives us one condition on the functions u_1 and u_2. Compute (note the product rule!)

$$y' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2')$$

We can still impose one more condition at our discretion to simplify computations (we have two unknown functions, so we should be allowed two conditions). We require that $(u_1' y_1 + u_2' y_2) = 0$. This makes computing the second derivative easier.

$$\begin{aligned} y' &= u_1 y_1' + u_2 y_2' \\ y'' &= (u_1' y_1' + u_2' y_2') + (u_1 y_1'' + u_2 y_2'') \end{aligned} \tag{4.9.4}$$

Since y_1 and y_2 are solutions to $y'' + y = 0$, we know that $y_1'' = -y_1$ and $y_2'' = -y_2$. (Note: If the equation was instead $y'' + p(x)y' + q(x)y = 0$ we would have $y_i'' = -p(x)y_i' - q(x)y_i$.) So

$$y'' = (u_1' y_1' + u_2' y_2') - (u_1 y_1 + u_2 y_2)$$

We have $(u_1 y_1 + u_2 y_2) = y$ and so

$$y'' = (u_1' y_1' + u_2' y_2') - y$$

and hence

$$y'' + y = Ly = u_1' y_1' + u_2' y_2'$$

For y to satisfy $Ly = f(x)$ we must have $f(x) = u_1' y_1' + u_2' y_2'$.

So what we need to solve are the two equations (conditions) we imposed on u_1 and u_2

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned} \tag{4.9.5}$$

We can now solve for u_1' and u_2' in terms of $f(x), y_1$ and y_2. We will always get these formulas for any $Ly = f(x)$, where $Ly = y'' + p(x)y' + q(x)y$. There is a general formula for the solution we can just plug into, but it is better to just repeat what we do below. In our case the two equations become

$$\begin{aligned} u_1' \cos(x) + u_2' \sin(x) &= 0 \\ -u_1' \sin(x) + u_2' \cos(x) &= \tan(x) \end{aligned} \tag{4.9.6}$$

Hence

$$\begin{aligned} u_1' \cos(x)\sin(x) + u_2' \sin^2(x) &= 0 \\ -u_1' \sin(x)\cos(x) + u_2' \cos^2(x) &= \tan(x)\cos(x) = \sin(x) \end{aligned} \tag{4.9.7}$$

And thus

$$\begin{aligned} u_2'(\sin^2(x) + \cos^2(x)) &= \sin(x) \\ u_2' &= \sin(x) \\ u_1' &= \frac{-\sin^2(x)}{\cos(x)} = -\tan(x)\sin(x) \end{aligned} \tag{4.9.8}$$

Now we need to integrate u_1' and u_2' to get u_1 and u_2.

$$\begin{aligned} u_1 &= \int u_1' dx = \int -\tan(x)\sin(x)dx = \frac{1}{2}\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right| + \sin(x) \\ u_2 &= \int u_2' dx = \int \sin(x)dx = -\cos(x) \end{aligned} \tag{4.9.9}$$

So our particular solution is

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 = \frac{1}{2}\cos(x)\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right| + \cos(x)\sin(x) - \cos(x)\sin(x) \\ &= \frac{1}{2}\cos(x)\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right| \end{aligned} \tag{4.9.10}$$

The general solution to $y'' + y = \tan x$ is, therefore,

$$y = C_1\cos(x) + C_2\sin(x) + \frac{1}{2}\cos(x)\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right|$$

4.9.2: Contributors and Attributions

- - Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

4.10: Forced Oscillations and Resonance

Let us consider to the example of a mass on a spring. We now examine the case of forced oscillations, which we did not yet handle. That is, we consider the equation

$$mx'' + cx' + kx = F(t)$$

for some nonzero $F(t)$. The setup is again: m is mass, c is friction, k is the spring constant, and $F(t)$ is an external force acting on the mass.

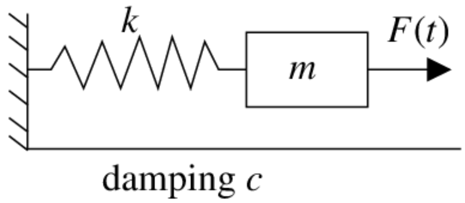

Figure 4.10.1

What we are interested in is periodic forcing, such as noncentered rotating parts, or perhaps loud sounds, or other sources of periodic force. Once we learn about Fourier series in Chapter 4, we will see that we cover all periodic functions by simply considering $F(t) = F_0 \cos(\omega t)$ (or sine instead of cosine, the calculations are essentially the same).

Below is a video on solving a forced oscillations problem.

4.10.1: Undamped Forced Motion and Resonance

First let us consider undamped $c = 0$ motion for simplicity. We have the equation

$$mx'' + kx = F_0 \cos(\omega t)$$

This equation has the complementary solution (solution to the associated homogeneous equation)

$$x_c = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the *natural frequency* (angular), which is the frequency at which the system "wants to oscillate" without external interference.

Let us suppose that $\omega_0 \neq \omega$. We try the solution $x_p = A\cos(\omega t)$ and solve for A. Note that we need not have sine in our trial solution as on the left hand side we will only get cosines anyway. If you include a sine it is fine; you will find that its coefficient will be zero.

We solve using the method of undetermined coefficients. We find that

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

We leave it as an exercise to do the algebra required.

The general solution is

$$x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

or written another way

$$x = C \cos(\omega_0 t - y) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Hence it is a superposition of two cosine waves at different frequencies.

Example 4.10.1

Take

$$0.5x'' + 8x = 10\cos(\pi t), \quad x(0) = 0, \quad x'(0) = 0$$

Let us compute. First we read off the parameters: $\omega = \pi, \omega_0 = \sqrt{\frac{8}{0.5}} = 4, F_0 = 10, m = 0.5$. The general solution is

$$x = C_1 \cos(4t) + C_2 \sin(4t) + \frac{20}{16 - \pi^2} \cos(\pi t)$$

Solve for C_1 and C_2 using the initial conditions. It is easy to see that $C_1 = \frac{-20}{16-\pi^2}$ and $C_2 = 0$. Hence

$$x = \frac{20}{16 - \pi^2} (\cos(\pi t) - \cos(4t))$$

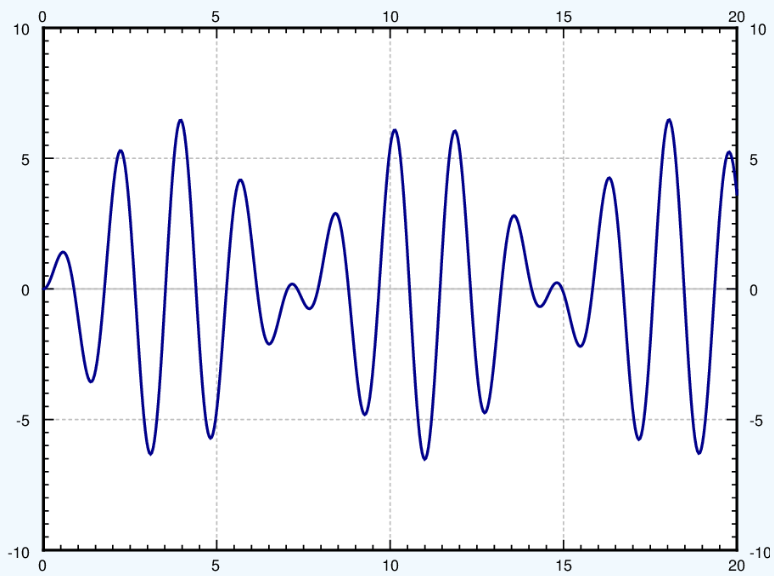

Figure 4.10.2: Graph of $\frac{20}{16-\pi^2}(\cos(\pi t) - \cos(4t))$.

Notice the "beating" behavior in Figure $4.10.2$. First use the trigonometric identity

$$2\sin(\frac{A-B}{2})\sin(\frac{A+B}{2}) = \cos B - \cos A$$

to get that

$$x = \frac{20}{16 - \pi^2}(2\sin(\frac{4-\pi}{2}t)\sin(\frac{4+\pi}{2}t))$$

Notice that x is a high frequency wave modulated by a low frequency wave.

Now suppose that $\omega_0 = \omega$. Obviously, we cannot try the solution $A\cos(\omega t)$ and then use the method of undetermined coefficients. We notice that $\cos(\omega t)$ solves the associated homogeneous equation. Therefore, we need to try $x_p = At\cos(\omega t) + Bt\sin(\omega t)$. This time we do need the sine term since the second derivative of $t\cos(\omega t)$ does contain sines. We write the equation

$$x'' + \omega^2 x = \frac{F_0}{m}\cos(\omega t)$$

Plugging x_p into the left hand side we get

$$2B\omega\cos(\omega t) - 2A\omega\sin(\omega t) = \frac{F_0}{m}\cos(\omega t)$$

Hence $A = 0$ and $B = \frac{F_0}{2m\omega}$. Our particular solution is $\frac{F_0}{2m\omega}t\sin(\omega t)$ and our general solution is

$$x = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{F_0}{2m\omega}t\sin(\omega t)$$

The important term is the last one (the particular solution we found). We can see that this term grows without bound as $t \to \infty$. In fact it oscillates between $\frac{F_0 t}{2m\omega}$ and $\frac{-F_0 t}{2m\omega}$. The first two terms only oscillate between $\pm\sqrt{C_1^2 + C_2^2}$, which becomes smaller and smaller in proportion to the oscillations of the last term as t gets larger. In Figure $4.10.3$ we see the graph with $C_1 = C_2 = 0, F_0 = 2, m = 1, \omega = \pi$.

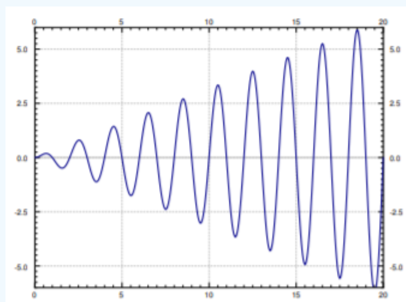

Figure 4.10.3: Graph of $\frac{1}{\pi}t\sin(\pi t)$.

By forcing the system in just the right frequency we produce very wild oscillations. This kind of behavior is called resonance or perhaps pure resonance. Sometimes resonance is desired. For example, remember when as a kid you could start swinging by just moving back and forth on the swing seat in the “correct frequency”? You were trying to achieve resonance. The force of each one of your moves was small, but after a while it produced large swings.

On the other hand resonance can be destructive. In an earthquake some buildings collapse while others may be relatively undamaged. This is due to different buildings having different resonance frequencies. So figuring out the resonance frequency can be very important.

A common (but wrong) example of destructive force of resonance is the Tacoma Narrows bridge failure. It turns out there was a different phenomenon at play.[1]

Below is a video on solving a differential equation that comes from a vibrating system with resonance.

4.10.2: Damped Forced Motion and Practical Resonance

In real life things are not as simple as they were above. There is, of course, some damping. Our equation becomes

$$mx'' + cx' + kx = F_0 \cos(\omega t), \tag{4.10.1}$$

for some $c > 0$. We have solved the homogeneous problem before. We let

$$p = \frac{c}{2m} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

We replace equation (4.10.1) with

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$$

The roots of the characteristic equation of the associated homogeneous problem are $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$. The form of the general solution of the associated homogeneous equation depends on the sign of $p^2 - \omega_0^2$, or equivalently on the sign of $c^2 - 4km$, as we have seen before. That is,

$$x_c = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t}, & \text{if } c^2 > 4km, \\ C_1 e^{pt} + C_2 t e^{-pt}, & \text{if } c^2 = 4km, \\ e^{-pt}(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)), & \text{if } c^2 < 4km, \end{cases}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. In any case, we can see that $x_c(t) \to 0$ as $t \to \infty$. Furthermore, there can be no conflicts when trying to solve for the undetermined coefficients by trying $x_p = A\cos(\omega t) + B\sin(\omega t)$. Let us plug in and solve for A and B. We get (the tedious details are left to reader)

$$((\omega_0^2 - \omega^2)B - 2\omega p A)\sin(\omega t) + ((\omega_0^2 - \omega^2)A + 2\omega p B)\cos(\omega t) = \frac{F_0}{m}\cos(\omega t)$$

We get that

$$A = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m\left(\omega_0^2 - \omega^2\right)^2}$$

$$B = \frac{2\omega p F_0}{m(2\omega p)^2 + m\left(\omega_0^2 - \omega^2\right)^2}$$

We also compute $C = \sqrt{A^2 + B^2}$ to be

$$C = \frac{F_0}{m\sqrt{(2\omega p)^2 + \left(\omega_0^2 - \omega^2\right)^2}}$$

Thus our particular solution is

$$x_P = \frac{(\omega_0^2 - \omega^2) F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \sin(\omega t)$$

Or in the alternative notation we have amplitude C and phase shift γ where (if $\omega \neq \omega_0$)

$$\tan \gamma = \frac{B}{A} = \frac{2\omega p}{\omega_0^2 - \omega^2}$$

Hence we have

$$x_p = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}} \cos(\omega t - \gamma)$$

If $\omega = \omega_0$ we see that $A = 0, B = C = \frac{F_0}{2m\omega p}$, and $\gamma = \frac{\pi}{2}$.

The exact formula is not as important as the idea. Do not memorize the above formula, you should instead remember the ideas involved. For different forcing function F, you will get a different formula for x_p. So there is no point in memorizing this specific formula. You can always recompute it later or look it up if you really need it.

For reasons we will explain in a moment, we call $x_c$the transient solution and denote it by x_{tr}. We call the x_p we found above the steady periodic solution and denote it by x_{sp}. The general solution to our problem is

$$x = x_c + x_p = x_{tr} + x_{sp}$$

We note that $x_c = x_{tr}$ goes to zero as $t \to \infty$, as all the terms involve an exponential with a negative exponent. Hence for large t, the effect of x_{tr} is negligible and we will essentially only see x_{sp}. Hence the name transient. Notice that x_{sp} involves no arbitrary constants, and the initial conditions will only affect x_{tr}. This means that the effect of the initial conditions will be negligible after some period of time. Because of this behavior, we might as well focus on the steady periodic solution and ignore the transient solution. See Figure $4.10.4$for a graph of different initial conditions.

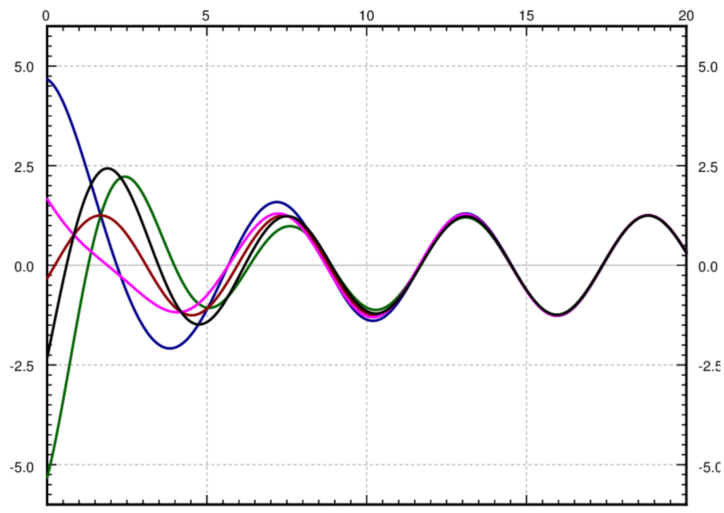

Figure 4.10.4: Solutions with different initial conditions for parameters $k = 1, m = 1, F_0 = 1, c = 0.7$, and $\omega = 1.1$.

Notice that the speed at which x_{tr} goes to zero depends on P (and hence c). The bigger P is (the bigger c is), the "faster" x_{tr} becomes negligible. So the smaller the damping, the longer the "transient region." This agrees with the observation that when $c = 0$, the initial conditions affect the behavior for all time (i.e. an infinite "transient region").

Let us describe what we mean by resonance when damping is present. Since there were no conflicts when solving with undetermined coefficient, there is no term that goes to infinity. What we will look at however is the maximum value of the amplitude of the steady periodic solution. Let C be the amplitude of x_{sp}. If we plot C as a function of ω (with all other parameters fixed) we can find its maximum. We call the ω that achieves this maximum the practical resonance frequency. We call the maximal amplitude $C(\omega)$ the practical resonance amplitude. Thus when damping is present we talk of practical resonance rather than pure resonance. A sample plot for three different values of c is given in Figure 4.10.5. As you can see the practical resonance amplitude grows as damping gets smaller, and any practical resonance can disappear when damping is large.

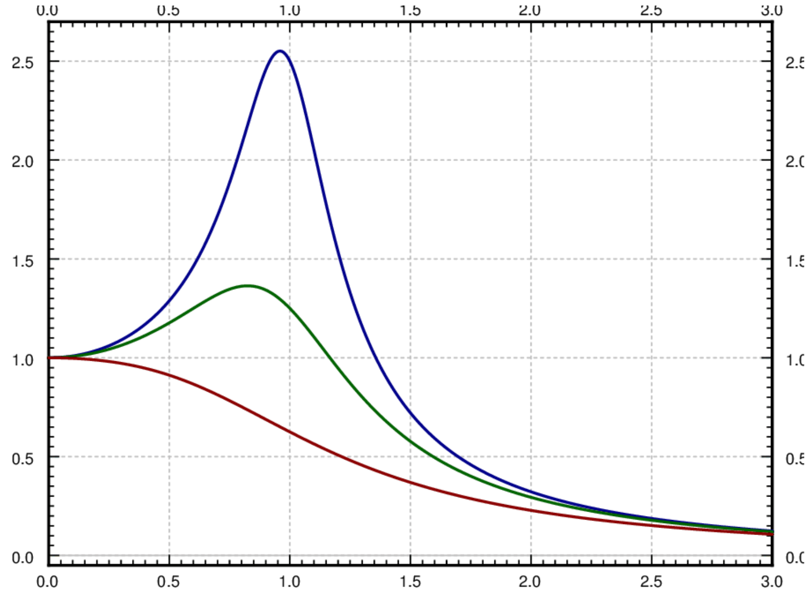

Figure 4.10.5: Graph of $C(\omega)$ showing practical resonance with parameters $k=1, m=1, F_0=1$. The top line is with $c=0.4$, the middle line with $c=0.8$, and the bottom line with $c=1.6$.

To find the maximum we need to find the derivative $C'(\omega)$. Computation shows

$$C'(\omega)=\frac{-4\omega(2p^2+\omega^2-\omega_0^2)F_0}{m\left((2\omega p)^2+(\omega_0^2-\omega^2)\right)^{3/2}}$$

This is zero either when $\omega=0$ or when $2p^2+\omega^2-\omega_0^2=0$. In other words, $C'(\omega)=0$ when

$$\omega=\sqrt{\omega_0^2-2p^2} \text{ or } \omega=0$$

It can be shown that if $\omega_0^2-2p^2$ is positive, then $\sqrt{\omega_0^2-2p^2}$ is the practical resonance frequency (that is the point where $C(\omega)$ is maximal, note that in this case $C'(\omega)>0$ for small ω). If $\omega=0$ is the maximum, then essentially there is no practical resonance since we assume that $\omega>0$ in our system. In this case the amplitude gets larger as the forcing frequency gets smaller.

If practical resonance occurs, the frequency is smaller than ω_0. As the damping c (and hence P) becomes smaller, the practical resonance frequency goes to ω_0. So when damping is very small, ω_0 is a good estimate of the resonance frequency. This behavior agrees with the observation that when $c=0$, then ω_0 is the resonance frequency.

Another interesting observation to make is that when $\omega\to\infty$, then $\omega\to 0$. This means that if the forcing frequency gets too high it does not manage to get the mass moving in the mass-spring system. This is quite reasonable intuitively. If we wiggle back and forth really fast while sitting on a swing, we will not get it moving at all, no matter how forceful. Fast vibrations just cancel each other out before the mass has any chance of responding by moving one way or the other.

The behavior is more complicated if the forcing function is not an exact cosine wave, but for example a square wave. A general periodic function will be the sum (superposition) of many cosine waves of different frequencies. The reader is encouraged to come back to this section once we have learned about the Fourier series.

4.10.3: Footnotes

[1]K. Billah and R. Scanlan, Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks, American Journal of Physics, 59(2), 1991, 118–124, http://www.ketchum.org/billah/Billah-Scanlan.pdf

4.E: Higher order linear ODEs (Exercises)

These are homework exercises to accompany Libl's "Differential Equations for Engineering" Textmap. This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Prerequisite for the course is the basic calculus sequence.

4.E.1: 2.1: Second order linear ODEs

Exercise $4.E.2.1.1$

Show that $y = e^x$ and $y = e^{2x}$ are linearly independent.

Exercise $4.E.2.1.2$

Take $y'' + 5y = 10x + 5$. Find (guess!) a solution.

Exercise $4.E.2.1.3$

Prove the superposition principle for nonhomogeneous equations. Suppose that y_1 is a solution to $Ly_1 = f(x)$ and y_2 is a solution to $Ly_2 = g(x)$ (same linear operator L). Show that $y = y_1 + y_2$ solves $Ly = f(x) + g(x)$.

Exercise $4.E.2.1.4$

For the equation $x^2y'' - xy' = 0$, find two solutions, show that they are linearly independent and find the general solution. Hint: Try $y = x'$.

Equations of the form $ax^2y'' + bxy' + cy = 0$ are called *Euler's equations* or *Cauchy-Euler equations*. They are solved by trying $y = x^r$ and solving for r (we can assume that $x \geqslant 0$ for simplicity).

Exercise $4.E.2.1.5$

Suppose that $(b-a)^2 - 4ac > 0$.

a. Find a formula for the general solution of $ax^2y'' + bxy' + cy = 0$. Hint: Try $y = x^r$ and find a formula for r.
b. What happens when $(b-a)^2 - 4ac = 0$ or $(b-a)^2 - 4ac < 0$?

We will revisit the case when $(b-a)^2 - 4ac < 0$ later.

Exercise $4.E.2.1.6$

Same equation as in Exercise $4.E.2.1.5$. Suppose $(b-a)^2 - 4ac = 0$. Find a formula for the general solution of $ax^2y'' + bxy' + cy = 0$. Hint: Try $y = x^r \ln x$ for the second solution.

Exercise $4.E.2.1.7$: reduction of order

Suppose y_1 is a solution to $y'' + p(x)y' + q(x)y = 0$. Show that

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2} dx \tag{4.E.1}$$

is also a solution.

Note: If you wish to come up with the formula for reduction of order yourself, start by trying $y_2(x) = y_1(x)v(x)$. Then plug y_2 into the equation, use the fact that y_1 is a solution, substitute $w = v'$, and you have a first order linear equation in w. Solve for w and then for v. When solving for w, make sure to include a constant of integration. Let us solve some famous equations using the method.

Exercise $4.E.2.1.8$: Chebyshev's equation of order 1

Take $(1-x^2)y''-xy'+y=0$.

a. Show that $y=x$ is a solution.
b. Use reduction of order to find a second linearly independent solution.
c. Write down the general solution.

Exercise $4.E.2.1.9$: Hermite's equation of order 2

Take $y''-2xy'+4y=0$.

a. Show that $y=1-2x^2$ is a solution.
b. Use reduction of order to find a second linearly independent solution.
c. Write down the general solution.

Exercise $4.E.2.1.10$

Are $\sin(x)$ and e^x linearly independent? Justify.

Answer

Yes. To justify try to find a constant A such that $\sin(x)=Ae^x$ for all x.

Exercise $4.E.2.1.11$

Are e^x and e^{x+2} linearly independent? Justify.

Answer

No. $e^{x+2}=e^2e^x$.

Exercise $4.E.2.1.12$

Guess a solution to $y''+y'+y=5$.

Answer

$y=5$

Exercise $4.E.2.1.13$

Find the general solution to $xy''+y'=0$. Hint: Notice that it is a first order ODE in y'.

Answer

$y=C_1\ln(x)+C_2$

Exercise $4.E.2.1.14$

Write down an equation (guess) for which we have the solutions e^x and e^{2x}. Hint: Try an equation of the form $y''+Ay'+By=0$ for constants A and B, plug in both e^x and e^{2x} and solve for A and B.

Answer

$y''-3y'+2y=0$

4.E.2: 2.2: Constant coefficient second order linear ODEs

Exercise $4.E.2.2.1$

Find the general solution of $2y'' + 2y' - 4y = 0$.

Exercise $4.E.2.2.2$

Find the general solution of $y'' + 9y' - 10y = 0$.

Exercise $4.E.2.2.3$

Solve $y'' - 8y' + 16y = 0$ for $y(0) = 2, y'(0) = 0$.

Exercise $4.E.2.2.4$

Solve $y'' + 9y' = 0$ for $y(0) = 1, y'(0) = 1$.

Exercise $4.E.2.2.5$

Find the general solution of $2y'' + 50y = 0$.

Exercise $4.E.2.2.6$

Find the general solution of $y'' + 6y' + 13y = 0$.

Exercise $4.E.2.2.7$

Find the general solution of $y'' = 0$ using the methods of this section.

Exercise $4.E.2.2.8$

The method of this section applies to equations of other orders than two. We will see higher orders later. Try to solve the first order equation $2y' + 3y = 0$ using the methods of this section.

Exercise $4.E.2.2.9$

Let us revisit Euler's equations of Exercise $4.E.1$. Suppose now that $(b-a)^2 - 4ac < 0$. Find a formula for the general solution of $ax^2y'' + bxy' + cy = 0$. Hint: Note that $x^r = e^{r \ln x}$.

Exercise $4.E.2.2.10$

Find the solution to $y'' - (2\alpha)y' + \alpha^2 y = 0$, $y(0) = a$, $y'(0) = b$, where α, a, and b are real numbers.

Exercise $4.E.2.2.11$

Construct an equation such that $y = C_1 e^{-2x} \cos(3x) + C_2 e^{-2x} \sin(3x)$ is the general solution.

Exercise $4.E.2.2.12$

Find the general solution to $y'' + 4y' + 2y = 0$.

Answer

$$y = C_1 e^{(-2+\sqrt{2})x} + C_2 e^{(-2-\sqrt{2})x}$$

Exercise $4.E.2.2.13$

Find the general solution to $y'' - 6y' + 9y = 0$.

Answer

$y = C_1 e^{3x} + C_2 x e^{3x}$

Exercise $4.E.2.2.14$

Find the solution to $2y'' + y' + y = 0, y(0) = 1, y'(0) = -2$.

Answer

$y = e^{-x/4}\cos\left((\frac{\sqrt{7}}{4})x\right) - \sqrt{7}e^{-x/4}\sin\left((\frac{\sqrt{7}}{4})x\right)$

Exercise $4.E.2.2.15$

Find the solution to $2y'' + y' - 3y = 0, y(0) = a, y'(0) = b$.

Answer

$y = \frac{2(a-b)}{5}e^{-3x/2} + \frac{3a+2b}{5}e^{x}$

Exercise $4.E.2.2.16$

Find the solution to $z''(t) = -2z'(t) - 2z(t), z(0) = 2, z'(0) = -2$.

Answer

$z(t) = 2e^{-t}\cos(t)$

Exercise $4.E.2.2.17$

Find the solution to $y'' - (\alpha + \beta)y' + \alpha\beta y = 0$, $y(0) = a$, $y'(0) = b$, where α, (\beta\), a, and b are real numbers, and $\alpha \neq \beta$.

Answer

$y = \frac{\alpha\beta - b}{\beta - \alpha}e^{\alpha x} + \frac{b - a\alpha}{\beta - \alpha}e^{\beta x}$

Exercise $4.E.2.2.18$

Construct an equation such that $y = C_1 e^{3x} + C_2 e^{-2x}$ is the general solution.

Answer

$y'' - y' - 6y = 0$

4.E.3: 2.3: Higher order linear ODEs

Exercise $4.E.2.3.1$

Find the general solution for $y''' - y'' + y' - y = 0$.

Exercise $4.E.2.3.2$

Find the general solution for $y^{(4)} - 5y''' + 6y'' = 0$.

Exercise $4.E.2.3.3$

Find the general solution for $y''' + 2y'' + 2y' = 0$.

Exercise $4.E.2.3.4$

Suppose the characteristic equation for a differential equation is $(r-1)^2(r-2)^2 = 0$.

a. Find such a differential equation.
b. Find its general solution.

Exercise $4.E.2.3.5$

Suppose that a fourth order equation has a solution $y = 2e^{4x} x \cos x$.

a. Find such an equation.
b. Find the initial conditions that the given solution satisfies.

Exercise $4.E.2.3.6$

Find the general solution for the equation of Exercise $4.E.2.3.5$.

Exercise $4.E.2.3.7$

Let $f(x) = e^x - \cos x, g(x) = e^x + \cos x$ and $h(x) = \cos x$. Are $f(x)$, $g(x)$, and $h(x)$ and linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $4.E.2.3.8$

Let $f(x) = 0, g(x) = \cos x$, and $h(x) = \sin x$. Are $f(x)$, $g(x)$, and $h(x)$ and linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $4.E.2.3.9$

Are $x, x^2, and x^4$ linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $4.E.2.3.10$

Are $e^x, xe^x, and x^2 e^x$ linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $4.E.2.3.11$

Find an equation such that $y = xe^{-2x} \sin(3x)$ is a solution.

Exercise $4.E.2.3.12$

Find the general solution of $y^{(5)} - y^{(4)} = 0$.

Answer

$y = C_1 e^x + C_2 x^3 + C_3 x^2 + C_4 x + C_5$

Exercise $4.E.2.3.13$

Suppose that the characteristic equation of a third order differential equation has roots $3 \pm 2i$.

a. What is the characteristic equation?
b. Find the corresponding differential equation.
c. Find the general solution.

Answer

a. $r^3 - 3r^2 + 4r - 12 = 0$
b. $y''' - 3y'' + 4y' - 12y = 0$
c. $y = C_1 e^{3x} + C_2 \sin(2x) + C_3 \cos(2x)$

Exercise $4.E.2.3.14$

Solve $1001y''' + 3.2y'' + \pi y' - \sqrt{4}y = 0, y(0) = 0, y'(0) = 0, y''(0) = 0$.

Answer

$y = 0$

Exercise $4.E.2.3.15$

Are $e^x, e^{x+1}, e^{2x}, \sin(x)$ linearly independent? If so, show it, if not find a linear combination that works.

Answer

No. $e^1 e^x - e^{x+1} = 0$.

Exercise $4.E.2.3.16$

Are $\sin(x), x, x\sin(x)$ linearly independent? If so, show it, if not find a linear combination that works.

Answer

Yes. (Hint: First note that $\sin(x)$ is bounded. Then note that x and $x\sin(x)$ cannot be multiples of each other.)

Exercise $4.E.2.3.17$

Find an equation such that $y = \cos(x)$, $y = \sin(x)$, $y = e^x$ are solutions.

Answer

$y''' - y'' + y' - y = 0$

4.E.4: 2.4: Mechanical Vibrations

Exercise $4.E.2.4.1$

Consider a mass and spring system with a mass $m = 2$, spring constant $k = 3$, and damping constant $c = 1$.

a. Set up and find the general solution of the system.
b. Is the system underdamped, overdamped or critically damped?
c. If the system is not critically damped, find a c that makes the system critically damped.

Exercise $4.E.2.4.2$

Do Exercise $4.E.2.4.1$ for $m = 3, k = 12, and c = 12$.

Exercise $4.E.2.4.3$

Using the mks units (meters-kilograms-seconds), suppose you have a spring with spring constant $4\frac{N}{m}$. You want to use it to weigh items. Assume no friction. You place the mass on the spring and put it in motion.

a. You count and find that the frequency is 0.8 Hz (cycles per second). What is the mass?
b. Find a formula for the mass m given the frequency w in Hz.

Exercise $4.E.2.4.4$

Suppose we add possible friction to Exercise $4.E.2.4.3$. Further, suppose you do not know the spring constant, but you have two reference weights 1 kg and 2 kg to calibrate your setup. You put each in motion on your spring and measure the frequency. For the 1 kg weight you measured 1.1 Hz, for the 2 kg weight you measured 0.8 Hz.

a. Find k (spring constant) and c (damping constant).
b. Find a formula for the mass in terms of the frequency in Hz. Note that there may be more than one possible mass for a given frequency.
c. For an unknown object you measured \(0.2\text{ Hz}\), what is the mass of the object? Suppose that you know that the mass of the unknown object is more than a kilogram.

Exercise $4.E.2.4.5$

Suppose you wish to measure the friction a mass of 0.1 kg experiences as it slides along a floor (you wish to find c). You have a spring with spring constant $k = 5\frac{N}{m}$. You take the spring, you attach it to the mass and fix it to a wall. Then you pull on the spring and let the mass go. You find that the mass oscillates with frequency 1 Hz. What is the friction?

Exercise $4.E.2.4.6$

A mass of 2 kilograms is on a spring with spring constant k newtons per meter with no damping. Suppose the system is at rest and at time $t = 0$ the mass is kicked and starts traveling at 2 meters per second. How large does k have to be to so that the mass does not go further than 3 meters from the rest position?

Answer

$k = \frac{8}{9}$ (and larger)

Exercise $4.E.2.4.7$

Suppose we have an RLC circuit with a resistor of 100 miliohms (0.1 ohms), inductor of inductance of 50 millihenries (0.05 henries), and a capacitor of 5 farads, with constant voltage.

a. Set up the ODE equation for the current I.
b. Find the general solution.

c. Solve for $I(0) = 10$ and $I'(0) = 0$.

Answer

a. $0.05I'' + 0.1I' + \left(\frac{1}{5}\right) I = 0$
b. $I = Ce^{-t} \cos(\sqrt{3}t - \gamma)$
c. $I = 10e^{-t} \cos(\sqrt{3}t) + \frac{10}{\sqrt{3}} e^{-t} \sin(\sqrt{3}t)$

Exercise $4.E.2.4.8$

A 5000 kg railcar hits a bumper (a spring) at $1\frac{m}{s}$, and the spring compresses by 0.1 m. Assume no damping.

a. Find k.
b. Find out how far does the spring compress when a 10000 kg railcar hits the spring at the same speed.
c. If the spring would break if it compresses further than 0.3 m, what is the maximum mass of a railcar that can hit it at $1\frac{m}{s}$?
d. What is the maximum mass of a railcar that can hit the spring without breaking at $2\frac{m}{s}$?

Answer

a. $k = 500000$
b. $\frac{1}{5\sqrt{2}} \approx 0.141$
c. 45000 kg
d. 11250 kg

Exercise $4.E.2.4.9$

A mass of m kg is on a spring with $k = 3\frac{\text{N}}{\text{m}}$ and $c = 2\frac{\text{Ns}}{\text{m}}$. Find the mass m_0 for which there is critical damping. If $m < m_0$, does the system oscillate or not, that is, is it underdamped or overdamped?

Answer

$m_0 = \frac{1}{2}$. If $m < m_0$, then the system is overdamped and will not oscillate.

4.E.5: 2.5: Nonhomogeneous Equations

Exercise $4.E.2.5.1$

Find a particular solution of $y'' - y' - 6y = e^{2x}$.

Exercise $4.E.2.5.2$

Find a particular solution of $y'' - 4y' + 4y = e^{2x}$.

Exercise $4.E.2.5.3$

Solve the initial value problem $y'' + 9y = \cos(3x) + \sin(3x)$ for $y(0) = 2, y'(0) = 1$.

Exercise $4.E.2.5.4$

Setup the form of the particular solution but do not solve for the coefficients for $y^{(4)} - 2y''' + y'' = e^x$.

Exercise $4.E.2.5.5$

Setup the form of the particular solution but do not solve for the coefficients for $y^{(4)} - 2y''' + y'' = e^x + x + \sin x$.

Exercise $4.E.2.5.6$

a. Using variation of parameters find a particular solution of $y'' - 2y' + y = e^x$.
b. Find a particular solution using undetermined coefficients.
c. Are the two solutions you found the same? What is going on? See also Exercise $4.E.2.5.9$

Exercise $4.E.2.5.7$

Find a particular solution of $y'' - 2y' + y = \sin(x^2)$. It is OK to leave the answer as a definite integral.

Exercise $4.E.2.5.8$

For an arbitrary constant c find a particular solution to $y'' - y = e^{cx}$. Hint: Make sure to handle every possible real c.

Exercise $4.E.2.5.9$

a. Using variation of parameters find a particular solution of $y'' - y = e^x$
b. Find a particular solution using undetermined coefficients.
c. Are the two solutions you found the same? What is going on?

Exercise $4.E.2.5.10$

Find a polynomial $P(x)$, so that $y = 2x^2 + 3x + 4$ solves $y'' + 5y' + y = P(x)$.

Exercise $4.E.2.5.11$

Find a particular solution to $y'' - y' + y = 2\sin(3x)$

Answer

$$y = \frac{-16\sin(3x) + 6\cos(3x)}{73}$$

Exercise $4.E.2.5.12$

a. Find a particular solution to $y'' + 2y = e^x + x^3$.
b. Find the general solution.

Answer

a. $y = \frac{2e^x + 3x^3 - 9x}{6}$
b. $y = C_1\cos(\sqrt{2}x) + C_2\sin(\sqrt{2}x) + \frac{2e^x + 3x^3 - 9x}{6}$

Exercise $4.E.2.5.13$

Solve $y'' + 2y' + y = x^2, y(0) = 1, y'(0) = 2$.

Answer

$$y(x) = x^2 - 4x + 6 + e^{-x}(x - 5)$$

Exercise $4.E.2.5.14$

Use variation of parameters to find a particular solution of $y'' - y = \dfrac{1}{e^x + e^{-x}}$.

Answer

$y = \frac{2xe^x - (e^x + e^{-x})\log(e^{2x}+1)}{4}$

Exercise $4.E.2.5.15$

For an arbitrary constant c find the general solution to $y'' - 2y = \sin(x+c)$.

Answer

$y = \frac{-\sin(x+c)}{3} + C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x}$

Exercise $4.E.2.5.16$

Undetermined coefficients can sometimes be used to guess a particular solution to other equations than constant coefficients. Find a polynomial $y(x)$ that solves $y' + xy = x^3 + 2x^2 + 5x + 2$.

Note: Not every right hand side will allow a polynomial solution, for example, $y' + xy = 1$ does not, but a technique based on undetermined coefficients does work, see Chapter 7.

Answer

$y = x^2 + 2x + 3$

4.E.6: 2.6: Forced Oscillations and Resonance

Exercise $4.E.2.6.1$

Derive a formula for x_{sp} if the equation is $mx'' + cx' + kx = F_0 \sin(\omega t)$. Assume $c > 0$.

Exercise $4.E.2.6.2$

Derive a formula for x_{sp} if the equation is $mx'' + cx' + kx = F_0 \cos(\omega t) + F_1 \cos(3\omega t)$. Assume $c > 0$.

Exercise $4.E.2.6.3$

Take $mx'' + cx' + kx = F_0 \cos(\omega t)$. Fix $m > 0$ and $k > 0$. Now think of the function $C(\omega)$. For what values of c (solve in terms of m, k, and F_0) will there be no practical resonance (that is, for what values of c is there no maximum of $C(\omega)$ for $\omega > 0$)?

Exercise $4.E.2.6.4$

Take $mx'' + cx' + kx = F_0 \cos(\omega t)$. Fix $c > 0$ and $k > 0$. Now think of the function $C(\omega)$. For what values of m (solve in terms of c, k, and F_0) will there be no practical resonance (that is, for what values of m is there no maximum of $C(\omega)$ for $\omega > 0$)?

Exercise $4.E.2.6.5$

Suppose a water tower in an earthquake acts as a mass-spring system. Assume that the container on top is full and the water does not move around. The container then acts as a mass and the support acts as the spring, where the induced vibrations are horizontal. Suppose that the container with water has a mass of $m = 10,000 kg$. It takes a force of 1000 newtons to displace the container 1 meter. For simplicity assume no friction. When the earthquake hits the water tower is at rest (it is not moving).

Suppose that an earthquake induces an external force $F(t) = mA\omega^2 \cos(\omega t)$.

a. What is the natural frequency of the water tower?
b. If ω is not the natural frequency, find a formula for the maximal amplitude of the resulting oscillations of the water container (the maximal deviation from the rest position). The motion will be a high frequency wave modulated by a low frequency wave, so simply find the constant in front of the sines.
c. Suppose $A = 1$ and an earthquake with frequency 0.5 cycles per second comes. What is the amplitude of the oscillations? Suppose that if the water tower moves more than 1.5 meter, the tower collapses. Will the tower collapse?

Exercise $4.E.2.6.6$

A mass of 4 kg on a spring with $k = 4$ and a damping constant $c = 1$. Suppose that $F_0 = 2$. Using forcing of $F_0 \cos(\omega t)$. Find the ω that causes practical resonance and find the amplitude.

Answer

$$\omega = \frac{\sqrt{31}}{4\sqrt{2}} \approx 0.984 \quad C(\omega) = \frac{16}{3\sqrt{7}} \approx 2.016$$

Exercise $4.E.2.6.7$

Derive a formula for x_{sp} for $mx'' + cx' + kx = F_0 \cos(\omega t) + A$ where A is some constant. Assume $c > 0$.

Answer

$$x_{sp} = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \sin(\omega t) + \frac{A}{k}$$

, where $p = \frac{c}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$

Exercise $4.E.2.6.8$

Suppose there is no damping in a mass and spring system with $m = 5$, $k = 20$, and $F_0 = 5$. Suppose that ω is chosen to be precisely the resonance frequency.

a. Find ω.
b. Find the amplitude of the oscillations at time $t = 100$.

Answer

a. $\omega = 2$
b. 25

CHAPTER OVERVIEW

5: Systems of ODEs

Contributors and Attributions

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5.1: Introduction to Systems of ODEs

Often we do not have just one dependent variable and just one differential equation, we may end up with systems of several equations and several dependent variables even if we start with a single equation.

If we have several dependent variables, suppose y_1, y_2, ..., y_n, then we can have a differential equation involving all of them and their derivatives. For example, $y_1'' = f(y_1', y_2', y_1, y_2, x)$. Usually, when we have two dependent variables we have two equations such as

$$\begin{aligned} y_1'' &= f_1(y_1', y_2', y_1, y_2, x) \\ y_2'' &= f_2(y_1', y_2', y_1, y_2, x) \end{aligned} \tag{5.1.1}$$

for some functions f_1 and f_2. We call the above a *system of differential equations*. More precisely, the above is a second order system of ODEs as second order derivatives appear. The system

$$\begin{aligned} x_1' &= g_1(x_1, x_2, x_3, t), \\ x_2' &= g_2(x_1, x_2, x_3, t), \\ x_3' &= g_3(x_1, x_2, x_3, t), \end{aligned} \tag{5.1.2}$$

is a *first order system*, where x_1, x_2, x_3 are the dependent variables, and t is the independent variable.

The terminology for systems is essentially the same as for single equations. For the system above, a *solution* is a set of three functions $x_1(t)$, $x_2(t)$, $x_3(t)$, such that

$$\begin{aligned} x_1'(t) &= g_1\left(x_1(t), x_2(t), x_3(t), t\right), \\ x_2'(t) &= g_2\left(x_1(t), x_2(t), x_3(t), t\right), \\ x_3'(t) &= g_3\left(x_1(t), x_2(t), x_3(t), t\right). \end{aligned} \tag{5.1.3}$$

We usually also have an *initial condition*. Just like for single equations we specify x_1, x_2, and x_3 for some fixed t. For example, $x_1(0) = a_1$, $x_2(0) = a_2$, $x_3(0) = a_3$. For some constants a_1, a_2, and a_3. For the second order system we would also specify the first derivatives at a point. And if we find a solution with constants in it, where by solving for the constants we find a solution for any initial condition, we call this solution the *general solution*. Best to look at a simple example.

✓ Example 5.1.1

Sometimes a system is easy to solve by solving for one variable and then for the second variable. Take the first order system

$$\begin{aligned} y_1' &= y_1, \\ y_2' &= y_1 - y_2, \end{aligned} \tag{5.1.4}$$

with initial conditions of the form $y_1(0) = 1$ and $y_2(0) = 2$.

Solution

We note that $y_1 = C_1 e^x$ is the general solution of the first equation. We then plug this y_1 into the second equation and get the equation $y_2' = C_1 e^x - y_2$, which is a linear first order equation that is easily solved for y_2. By the method of integrating factor we obtain

$$e^x y_2 = \frac{C_1}{2} e^{2x} + C_2$$

or

$$y_2 = \frac{C_1}{2} e^2 + C_2 e^{-x}.$$

The general solution to the system is, therefore,

$$y_1 = C_1 e^e, \quad \text{and} \quad y_2 = \frac{C_1}{2} e^x + C_2 e^{-x}.$$

We now solve for C_1 and C_2 given the initial conditions. We substitute $x = 0$ and find that $C_1 = 1$ and $C_2 = \frac{3}{2}$. Thus the solution is: $y_1 = e^x$, and $y_2 = \dfrac{1}{2}e^x + \dfrac{3}{2}e^{-x}$.

Generally, we will not be so lucky to be able to solve for each variable separately as in the example above, and we will have to solve for all variables at once. While we won't generally be able to solve for one variable and then the next, we will try to salvage as much as possible from this technique. It will turn out that in a certain sense we will still (try to) solve a bunch of single equations and put their solutions together. Let's not worry right now about how to solve systems yet.

We will mostly consider the *linear systems*. The example above is a so-called *linear first order system*. It is linear as none of the dependent variables or their derivatives appear in nonlinear functions or with powers higher than one (x, y, x' and y', constants, and functions of t can appear, but not xy or $(y')^2$ or x^3). Another, more complicated, example of a linear system is

$$\begin{aligned} y_1'' &= e^t y_1' + t^2 y_1 + 5y_2 + \sin(t), \\ y_2'' &= t y_1' - y_2' + 2y_1 + \cos(t). \end{aligned} \tag{5.1.5}$$

5.1.1: Applications

Let us consider some simple applications of systems and how to set up the equations.

Example 5.1.2

First, we consider salt and brine tanks, but this time water flows from one to the other and back. We again consider that the tanks are evenly mixed.

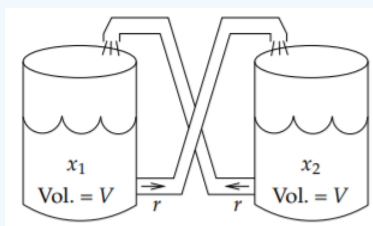


Figure 5.1.1: A closed system of two brine tanks.

Suppose we have two tanks, each containing volume V liters of salt brine. The amount of salt in the first tank is x_1 grams, and the amount of salt in the second tank is x_2 grams. The liquid is perfectly mixed and flows at the rate r liters per second out of each tank into the other. See Figure 5.1.1.

The rate of change of x_1, that is x_1', is the rate of salt coming in minus the rate going out. The rate coming in is the density of the salt in tank 2, that is $\frac{x_2}{V}$, times the rate r. The rate coming out is the density of the salt in tank 1, that is $\frac{x_1}{V}$, times the rate r. In other words it is

$$x_1' = \frac{x_2}{V}r - \frac{x_1}{V}r = \frac{r}{V}x_2 - \frac{r}{V}x_1 = \frac{r}{V}(x_2 - x_1).$$

Similarly we find the rate x_2', where the roles of x_1 and x_2 are reversed. All in all, the system of ODEs for this problem is

$$\begin{aligned} x_1' &= \frac{r}{V}(x_2 - x_1), \\ x_2' &= \frac{r}{V}(x_1 - x_2). \end{aligned} \tag{5.1.6}$$

In this system we cannot solve for x_1 or x_2 separately. We must solve for both x_1 and x_2 at once, which is intuitively clear since the amount of salt in one tank affects the amount in the other. We can't know x_1 before we know x_2, and vice versa.

We don't yet know how to find all the solutions, but intuitively we can at least find some solutions. Suppose we know that initially the tanks have the same amount of salt. That is, we have an initial condition such as $x_1(0) = x_2(0) = C$. Then clearly the amount of salt coming and out of each tank is the same, so the amounts are not changing. In other words, $x_1 = C$ and $x_2 = C$ (the constant functions) is a solution: $x_1' = x_2' = 0$, and $x_2 - x_1 = x_1 - x_2 = 0$, so the equations are satisfied.

Let us think about the setup a little bit more without solving it. Suppose the initial conditions are $x_1(0) = A$ and $x_2(0) = B$, for two different constants A and B. Since no salt is coming in or out of this closed system, the total amount of salt is constant. That is, $x_1 + x_2$ is constant, and so it equals $A + B$. Intuitively if A is bigger than B, then more salt will flow out of tank one than into it. Eventually, after a long time we would then expect the amount of salt in each tank to equalize. In other words, the solutions of both x_1 and x_2 should tend towards $\frac{A+B}{2}$. Once you know how to solve systems you will find out that this really is so.

Example 5.1.3

As an example application, let us think of mass and spring systems again.

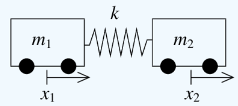


Figure 5.1.2

As an example application, let us think of mass and spring systems again. Suppose we have one spring with constant k, but two masses m_1 and m_2. We can think of the masses as carts, and we will suppose that they ride along a straight track with no friction. Let x_1 be the displacement of the first cart and x_2 be the displacement of the second cart. That is, we put the two carts somewhere with no tension on the spring, and we mark the position of the first and second cart and call those the zero positions. Then x_1 measures how far the first cart is from its zero position, and x_2 measures how far the second cart is from its zero position. The force exerted by the spring on the first cart is $k(x_2 - x_1)$, since $x_2 - x_1$ is how far the string is stretched (or compressed) from the rest position. The force exerted on the second cart is the opposite, thus the same thing with a negative sign.

Newton's second law states that force equals mass times acceleration. So the system of equations governing the setup is

$$\begin{aligned} m_1 x_1'' &= k(x_2 - x_1) \\ m_2 x_2'' &= -k(x_2 - x_1) \end{aligned} \tag{5.1.7}$$

In this system we cannot solve for the x_1 or x_2 variable separately. That we must solve for both x_1 and x_2 at once is intuitively clear, since where the first cart goes depends exactly on where the second cart goes and vice-versa.

5.1.2: Changing to First Order

Before we talk about how to handle systems, let us note that in some sense we need only consider first order systems. Let us take an n^{th} order differential equation

$$y^{(n)} = F(y^{(n-1)}, \ldots, y', y, x).$$

We define new variables $u_1, \ldots, u_n$ and write the system

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= u_3 \\ &\vdots \\ u_{n-1}' &= u_n \\ u_n' &= F(u_n, u_{n-1}, \ldots, u_2, u_1, x) \end{aligned} \tag{5.1.8}$$

We solve this system for $u_1, u_2, \ldots, u_n$. Once we have solved for the u's, we can discard u_2 through u_n and let $y = u_1$. We note that this y solves the original equation.

Example 5.1.4

Take $x''' = 2x'' + 8x' + x + t$. Letting $u_1 = x$, $u_2 = x'$, $u_3 = x''$, we find the system:

$$u_1' = u_2, \qquad u_2' = u_3, \qquad u_3' = 2u_3 + 8u_2 + u_1 + t.$$

A similar process can be followed for a system of higher order differential equations. For example, a system of k differential equations in k unknowns, all of order n, can be transformed into a first order system of $n \times k$ equations and $n \times k$ unknowns.

Example 5.1.5

Consider the system from the carts example,

$$m_1 x_1'' = k(x_2 - x_1), \qquad m_2 x_2'' = -k(x_2 - x_1).$$

Let $u_1 = x_1$, $u_2 = x_1'$, $u_3 = x_2$, $u_4 = x_2'$. The second order system becomes the first order system

$$u_1' = u_2, \qquad m_1 u_2' = k(u_3 - u_1), \qquad u_3' = u_4, \qquad m_2 u_4' = -k(u_3 - u_1).$$

Example 5.1.6

Sometimes we can use this idea in reverse as well. Let us take the system

$$x' = 2y - x, \quad y' = x,$$

where the independent variable is t. We wish to solve for the initial conditions $x(0) = 1$ and $y(0) = 0$.

If we differentiate the second equation we get $y'' = x'$. We know what x' is in terms of x and y, and we know that $x = y'$.

$$y'' = x' = 2y - x = 2y - y'.$$

We now have the equation $y'' + y' - 2y = 0$. We know how to solve this equation and we find that $y = C_1 e^{-2t} + C_2 e^t$. Once we have y we use the equation $y' = x$ to get x.

$$x = y' = -2C_1 e^{-2t} + C_2 e^t$$

We solve for the initial conditions $1 = x(0) = -2C_1 + C_2$ and $0 = y(0) = C_1 + C_2$. Hence, $C_1 = -C_2$ and $1 = 3C_2$. So $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$. Our solution is

$$x = \frac{2e^{-2t} + e^t}{3}, \quad y = \frac{-e^{-2t} + e^t}{3}$$

Exercise 5.1.1

Plug in and confirm that this really is the solution.

It is useful to go back and forth between systems and higher order equations for other reasons. For example, software for solving ODE numerically (approximation) is generally for first order systems. To use it, you take whatever ODE you want to solve and convert it to a first order system. It is not very hard to adapt computer code for the Euler or Runge–Kutta method for first order equations to handle first order systems. We simply treat the dependent variable not as a number but as a vector. In many mathematical computer languages there is almost no distinction in syntax.

5.1.3: Autonomous Systems and Vector Fields

A system where the equations do not depend on the independent variable is called an *autonomous system*. For example the system $y' = 2y - x$, $y' = x$ is autonomous as t is the independent variable but does not appear in the equations.

For autonomous systems we can draw the so-called *direction field* or *vector field*, a plot similar to a slope field, but instead of giving a slope at each point, we give a direction (and a magnitude). The previous example, $x' = 2y - x$, $y' = x$, says that at the point (x, y) the direction in which we should travel to satisfy the equations should be the direction of the vector $(2y - x, x)$ with the speed equal to the magnitude of this vector. So we draw the vector $(2y - x, x)$ at the point (x, y) and we do this for many points on the xy-plane. For example, at the point $(1, 2)$ we draw the vector $\big(2(2) - 1, 1\big) = (3, 1)$, a vector pointing to the right and a little bit up, while at the point $(2, 1)$ we draw the vector $\big(2(1) - 2, 2\big) = (0, 2)$ a vector that points straight up. When drawing the vectors, we will scale down their size to fit many of them on the same direction field. If we drew the arrows at the actual size, the diagram would be a jumbled mess once you would draw more than a couple of arrows. So we scale them all so that not even the longest one interferes with the others. We are mostly interested in their direction and relative size. See Figure 5.1.3.

We can draw a path of the solution in the plane. Suppose the solution is given by $x = f(t)$, $y = g(t)$. We pick an interval of t (say $0 \leq t \leq 2$ for our example) and plot all the points $\big(f(t), g(t)\big)$ for t in the selected range. The resulting picture is called the *phase portrait* (or phase plane portrait). The particular curve obtained is called the *trajectory* or *solution curve*. See an example plot in Figure $5.1.4$. In the figure the solution starts at $(1, 0)$ and travels along the vector field for a distance of 2 units of t. We solved this system precisely, so we compute $x(2)$ and $y(2)$ to find $x(2) \approx 2.475$ and $y(2) \approx 2.457$. This point corresponds to the top right end of the plotted solution curve in the figure.

Notice the similarity to the diagrams we drew for autonomous systems in one dimension. But note how much more complicated things become when we allow just one extra dimension.

We can draw phase portraits and trajectories in the xy-plane even if the system is not autonomous. In this case, however, we cannot draw the direction field, since the field changes as t changes. For each t we would get a different direction field.

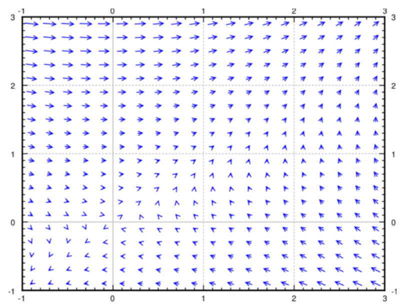


Figure $5.1.3$: The direction field for $x' = 2y - x, y' = x$.

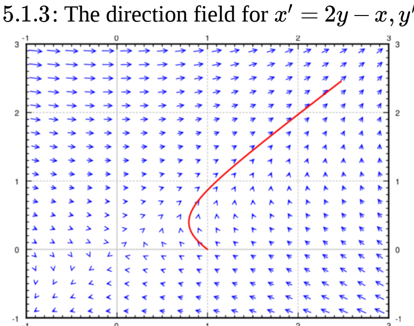


Figure $5.1.4$: The direction field for $x' = 2y - x, y' = x$ with the trajectory of the solution starting at $(1, 0)$ for $0 \leq t \leq 2$.

5.1.4: Picard's theorem

Perhaps before going further, let us mention that Picard's theorem on existence and uniqueness still holds for systems of ODE. Let us restate this theorem in the setting of systems. A general first order system is of the form

$$\begin{aligned} x_1' &= F_1(x_1, x_2, \ldots, x_n, t), \\ x_2' &= F_2(x_1, x_2, \ldots, x_n, t), \\ &\vdots \\ x_n' &= F_n(x_1, x_2, \ldots, x_n, t). \end{aligned} \tag{5.1.9}$$

Theorem $5.1.1$

Picard's Theorem on Existence and Uniqueness for Systems

If for every $j = 1, 2, \ldots, n$ and every $k = 1, 2, \ldots, n$ each F_j is continuous and the derivative $\frac{\partial F_j}{\partial x_k}$ exists and is continuous near some $(x_1^0, x_2^0, \ldots, x_n^0, t^0)$, then a solution to (3.1.21) subject to the initial condition $x_1(t^0) = x_1^0$, $x_2(t^0) = x_2^0$, …, $x_n(t^0) = x_n^0$ exists (at least for t in some small interval) and is unique.

That is, a unique solution exists for any initial condition given that the system is reasonable (F_j and its partial derivatives in the x variables are continuous). As for single equations we may not have a solution for all time t, but at least for some short period of time.

As we can change any nth order ODE into a first order system, then we notice that this theorem provides also the existence and uniqueness of solutions for higher order equations that we have until now not stated explicitly.

5.2: Matrices and linear systems

5.2.0.1: Matrices and vectors

Before we can start talking about linear systems of ODEs, we will need to talk about matrices, so let us review these briefly. A matrix is an $m \times n$ array of numbers (m rows and n columns). For example, we denote a 3×5 matrix as follows

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

The numbers a_{ij} are called *elements* or *entries*.

By a *vector* we will usually mean a *column vector*, that is an $m \times 1$ matrix. If we mean a *row vector* we will explicitly say so (a row vector is a $1 \times n$ matrix). We will usually denote matrices by upper case letters and vectors by lower case letters with an arrow such as $\vec{x}$ or $\vec{b}$. By $\vec{0}$ we will mean the vector of all zeros.

It is easy to define some operations on matrices. Note that we will want 1×1 matrices to really act like numbers, so our operations will have to be compatible with this viewpoint.

First, we can multiply by a scalar (a number). This means just multiplying each entry by the same number. For example,

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

Matrix addition is also easy. We add matrices element by element. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 7 & 10 \end{bmatrix}$$

If the sizes do not match, then addition is not defined.

If we denote by 0 the matrix of with all zero entries, by c, d scalars, and by A, B, C matrices, we have the following familiar rules.

$$\begin{aligned} A+0 &= A = 0+A \\ A+B &= B+A \\ (A+B)+C &= A+(B+C) \\ c(A+B) &= cA+cB \\ (c+d)A &= cA+dA \end{aligned} \tag{5.2.1}$$

Another useful operation for matrices is the so-called *transpose*. This operation just swaps rows and columns of a matrix. The transpose of A is denoted by A^T. Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

5.2.0.1: Matrix Multiplication

Let us now define matrix multiplication. First we define the so-called *dot product* (or *inner product*) of two vectors. Usually this will be a row vector multiplied with a column vector of the same size. For the dot product we multiply each pair of entries from the first and the second vector and we sum these products. The result is a single number. For example,

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 + a_2 b_2 + a_3 b_3 \end{bmatrix}$$

And similarly for larger (or smaller) vectors.

Armed with the dot product we can define the product of matrices. First let us denote by $\operatorname{row}_i(A)$ the i^{th} row of A and by $\operatorname{column}_j(A)$ the j^{th} column of A. For an $m \times n$ matrix A and an $n \times p$ matrix B we can define the product AB. We let AB be an $m \times p$ matrix whose ij^{th} entry is

$$\operatorname{row}_i(A) \cdot \operatorname{column}_j(B)$$

Do note how the sizes match up. Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1\cdot 1+2\cdot 1+3\cdot 1 & 1\cdot 0+2\cdot 1+3\cdot 0 & 1\cdot(-1)+2\cdot 1+3\cdot 0 \\ 4\cdot 1+5\cdot 1+6\cdot 1 & 4\cdot 0+5\cdot 1+6\cdot 0 & 4\cdot(-1)+5\cdot 1+6\cdot 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 1 \\ 15 & 5 & 1 \end{bmatrix}$$

For multiplication we want an analog of a 1. This analog is the so-called identity matrix. The identity matrix is a square matrix with 1s on the main diagonal and zeros everywhere else. It is usually denoted by I. For each size we have a different identity matrix and so sometimes we may denote the size as a subscript. For example, the I_3 would be the 3×3 identity matrix

$$I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have the following rules for matrix multiplication. Suppose that A, B, C are matrices of the correct sizes so that the following make sense. Let α denote a scalar (number).

$$\begin{aligned} A(BC) &= (AB)C \\ A(B+C) &= AB+AC \\ (B+C)A &= BA+CA \\ \alpha(AB) &= (\alpha A)B = A(\alpha B) \\ IA &= A = AI \end{aligned} \tag{5.2.2}$$

A few warnings are in order.

i. $AB \neq BA$ in general (it may be true by fluke sometimes). That is, matrices do not commute. For example take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.
ii. $AB = AC$ does not necessarily imply $B = C$, even if A is not 0.
iii. $AB = 0$ does not necessarily mean that $A = 0$ or $B = 0$. For example take $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

For the last two items to hold we would need to “divide” by a matrix. This is where the matrix inverse comes in. Suppose that A and B are $n \times n$ matrices such that

$$AB = I = BA$$

Then we call B the inverse of A and we denote B by A^{-1}. If the inverse of A exists, then we call A *invertible*. If A is not invertible we sometimes say A is *singular*.

If A is invertible, then $AB = AC$ does imply that $B = C$ (in particular the inverse of A is unique). We just multiply both sides by A^{-1} to get $A^{-1}AB = A^{-1}AC$ or $IB = IC$ or $B = C$. It is also not hard to see that $(A^{-1})^{-1} = A$.

Below is a video on determining if the product of two matrices is possible.

5.2.0.1: 3.2.3Determinant

We can now talk about determinants of square matrices. We define the determinant of a 1×1 matrix as the value of its only entry. For a 2×2 matrix we define

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \overset{\text{def}}{=} ad - bc$$

Before trying to compute the determinant for larger matrices, let us first note the meaning of the determinant. Consider an $n \times n$ matrix as a mapping of the n dimensional euclidean space $\mathbb{R}^n$ to $\mathbb{R}^n$. In particular, a 2×2 matrix A is a mapping of the plane to itself, where $\vec{x}$ gets sent to $A\vec{x}$. Then the determinant of A is the factor by which the area of objects gets changed. If we take the unit square (square of side 1) in the plane, then A takes the square to a parallelogram of area $|\det(A)|$. The sign of $\det(A)$ denotes changing of orientation (negative if the axes got flipped). For example, let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Then $\det(A) = 1 + 1 = 2$. Let us see where the square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ gets sent. Clearly $(0,0)$ gets sent to $(0,0)$.

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

So the image of the square is another square. The image square has a side of length $\sqrt{2}$ and is therefore of area 2.

If you think back to high school geometry, you may have seen a formula for computing the area of a parallelogram with vertices $(0,0)$, (a,c), (b,d) and $(a+b, c+d)$. And it is precisely

$$\left|\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right|$$

The vertical lines above mean absolute value. The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ carries the unit square to the given parallelogram.

Now we can define the determinant for larger matrices. We define A_{ij} as the matrix A with the i^{th} row and the j^{th} column deleted. To compute the determinant of a matrix, pick one row, say the i^{th} row and compute.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

For the first row we get

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \dots \begin{cases} +a_{1n}\det(A_{1n} & \text{if n is odd} \\ -a_{1n}\det(A_{1n} & \text{if n even} \end{cases}$$

We alternately add and subtract the determinants of the submatrices A_{ij} for a fixed i and all j. For a 3×3 matrix, picking the first row, we would get $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$. For example,

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) &= 1 \cdot \det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right) - 2 \cdot \det\left(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}\right) + 3 \cdot \det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\right) \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0 \end{aligned} \tag{5.2.3}$$

The numbers $(-1)^{i+j}\det(A_{ij})$ are called *cofactors* of the matrix and this way of computing the determinant is called the *cofactor expansion*. It is also possible to compute the determinant by expanding along columns (picking a column instead of a row above).

Note that a common notation for the determinant is a pair of vertical lines:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

I personally find this notation confusing as vertical lines usually mean a positive quantity, while determinants can be negative. I will not use this notation in this book.

Below is a video on evaluating the determinants of a 2x2 and a 3x3 matrix.

One of the most important properties of determinants (in the context of this course) is the following theorem.

Think of the determinants telling you the scaling of a mapping. If B doubles the sizes of geometric objects and A triples them, then AB (which applies B to an object and then A) should make size go up by a factor of 6. This is true in general:

$$\det(AB) = \det(A)\det(B).$$

This property is one of the most useful, and it is employed often to actually compute determinants. A particularly interesting consequence is to note what it means for existence of inverses. Take A and B to be inverses of each other, that is $AB = I$. Then

$$\det(A)\det(B) = \det(AB) = \det(I) = 1.$$

Neither $\det(A)$ nor $\det(B)$ can be zero. Let us state this as a theorem as it will be very important in the context of this course.

Theorem 5.2.1

An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

In fact, there is a formula for the inverse of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Notice the determinant of the matrix in the denominator of the fraction. The formula only works if the determinant is nonzero, otherwise we are dividing by zero.

5.2.0.1: Solving Linear Systems

One application of matrices we will need is to solve systems of linear equations. This is best shown by example. Suppose that we have the following system of linear equations

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2 \\ x_1 + x_2 + 3x_3 &= 5 \\ x_1 + 4x_2 + x_3 &= 10 \end{aligned} \tag{5.2.4}$$

Without changing the solution, we could swap equations in this system, we could multiply any of the equations by a nonzero number, and we could add a multiple of one equation to another equation. It turns out these operations always suffice to find a solution.

It is easier to write the system as a matrix equation. Note that the system can be written as

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$$

To solve the system we put the coefficient matrix (the matrix on the left hand side of the equation) together with the vector on the right and side and get the so-called augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array}\right]$$

We apply the following three elementary operations.

i. Swap two rows.
ii. Add a multiple of one row to another row.
iii. Multiply a row by a nonzero number.

We will keep doing these operations until we get into a state where it is easy to read off the answer, or until we get into a contradiction indicating no solution, for example if we come up with an equation such as $0 = 1$.

Let us work through the example. First multiply the first row by $\frac{1}{2}$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array}\right]$$

Now subtract the first row from the second and third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 9 \end{array}\right]$$

Multiply the last row by $\frac{1}{3}$ and the second row by $\frac{1}{2}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right]$$

Swap rows 2 and 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Subtract the last row from the first, then subtract the second row from the first.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

If we think about what equations this augmented matrix represents, we see that $x_1 = -4$, $x_2 = 3$ and $x_3 = 2$. We try this solution in the original system and, voilà, it works!

Below is a video on using matrices to solve a system of equations.

> **Exercise 5.2.1**
>
> Check that the solution above really solves the given equations.

We write this equation in matrix notation as

$$A\vec{x} = \vec{b},$$

where A is the matrix $\left[\begin{smallmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{smallmatrix}\right]$ and $\vec{b}$ is the vector $\left[\begin{smallmatrix} 2 \\ 5 \\ 10 \end{smallmatrix}\right]$. The solution can also be computed via the inverse,

$$\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}.$$

It is possible that the solution is not unique, or that no solution exists. It is easy to tell if a solution does not exist. If during the row reduction you come up with a row where all the entries except the last one are zero (the last entry in a row corresponds to the right-hand side of the equation), then the system is *inconsistent* and has no solution. For example, for a system of 3 equations and 3 unknowns, if you find a row such as $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right]$ in the augmented matrix, you know the system is inconsistent. That row corresponds to $0 = 1$.

You generally try to use row operations until the following conditions are satisfied. The first (from the left) nonzero entry in each row is called the *leading entry*.

i. The leading entry in any row is strictly to the right of the leading entry of the row above.
ii. Any zero rows are below all the nonzero rows.
iii. All leading entries are 1.
iv. All the entries above and below a leading entry are zero.

Such a matrix is said to be in *reduced row echelon form*. The variables corresponding to columns with no leading entries are said to be *free variables*. Free variables mean that we can pick those variables to be anything we want and then solve for the rest of the unknowns.

Example 5.2.1

The following augmented matrix is in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Suppose the variables are x_1, x_2 and x_3. Then x_2 is the free variable, $x_1 = 3 - 2x_2$, and $x_3 = 1$.

On the other hand if during the row reduction process you come up with the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 13 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

there is no need to go further. The last row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 3$, which is preposterous. Hence, no solution exists.

5.2.1: Computing the Inverse

If the coefficient matrix is square and there exists a unique solution $\vec{x}$ to $A\vec{x} = \vec{b}$ for any $\vec{b}$, then A is invertible. In fact by multiplying both sides by A^{-1} you can see that $\vec{x} = A^{-1}\vec{b}$. So it is useful to compute the inverse if you want to solve the equation for many different right hand sides $\vec{b}$.

The 2×2 inverse can be given by a formula, but it is also not hard to compute inverses of larger matrices. While we will not have too much occasion to compute inverses for larger matrices than 2×2 by hand, let us touch on how to do it. Finding the inverse of A is actually just solving a bunch of linear equations. If we can solve $A\vec{x}_k = \vec{e}_k$ where $\vec{e}_k$ is the vector with all zeros except a 1 at the k^{th} position, then the inverse is the matrix with the columns $\vec{x}_k$ for $k = 1, \ldots, n$ (exercise: why?). Therefore, to find the inverse we can write a larger $n \times 2n$ augmented matrix $[A \mid I]$, where I is the identity. We then perform row reduction. The reduced row echelon form of $[A \mid I]$ will be of the form $[I \mid A^{-1}]$ if and only if A is invertible. We can then just read off the inverse A^{-1}.

5.3: Linear systems of ODEs

First let us talk about matrix or vector valued functions. Such a function is just a matrix whose entries depend on some variable. If t is the independent variable, we write a *vector valued function* $\vec{x}(t)$ as

$$\vec{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Similarly a matrix valued function $A(t)$ is

$$A(t)=\begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

We can talk about the derivative $A'(t)$ or $\frac{dA}{dt}$. This is just the matrix valued function whose ij^{th} entry is $a'_{ij}(t)$.

Rules of differentiation of matrix valued functions are similar to rules for normal functions. Let $A(t)$ and $B(t)$ be matrix valued functions. Let c be a scalar and let C be a constant matrix. Then

$$\begin{aligned} (A(t)+B(t))' &= A'(t)+B'(t) \\ (A(t)B(t))' &= A'(t)B(t)+A(t)B'(t) \\ (cA(t))' &= cA'(t) \\ (CA(t))' &= CA'(t) \\ (A(t)C)' &= A'(t)C \end{aligned} \tag{5.3.1}$$

Note the order of the multiplication in the last two expressions.

A *first order linear system of ODEs* is a system that can be written as the vector equation

$$\vec{x}(t)=P(t)\vec{x}(t)+\vec{f}(t)$$

where $P(t)$ is a matrix valued function, and $\vec{x}(t)$ and $\vec{f}(t)$ are vector valued functions. We will often suppress the dependence on t and only write $\vec{x}=P\vec{x}+\vec{f}$. A solution of the system is a vector valued function $\vec{x}$ satisfying the vector equation.

For example, the equations

$$\begin{aligned} x_1' &= 2tx_1+e^tx_2+t^2 \\ x_2' &= \frac{x_1}{t}-x_2+e^t \end{aligned} \tag{5.3.2}$$

can be written as

$$\overrightarrow{x'}=\begin{bmatrix} 2t & e^t \\ \frac{1}{t} & -1 \end{bmatrix}\overrightarrow{x'}+\begin{bmatrix} t^2 \\ e^t \end{bmatrix}$$

We will mostly concentrate on equations that are not just linear, but are in fact *constant coefficient* equations. That is, the matrix P will be constant; it will not depend on t.

When $\vec{f}=\vec{0}$ (the zero vector), then we say the system is *homogeneous*. For homogeneous linear systems we have the principle of superposition, just like for single homogeneous equations.

Theorem 5.3.1

Superposition

Let $\overrightarrow{x'}=P\overrightarrow{x'}$ be a linear homogeneous system of ODEs. Suppose that $\vec{x}_1,\dots,\vec{x}_n$ are n solutions of the equation, then

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$$

is also a solution. Furthermore, if this is a system of n equations (P is $\mathrm{n \times n}$), and $\vec{x}_1, \ldots, \vec{x}_n$ are linearly independent, then every solution can be written as (5.3.3).

Linear independence for vector valued functions is the same idea as for normal functions. The vector valued functions $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent when

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = \vec{0} \tag{5.3.3}$$

has only the solution $c_1 = c_2 = \cdots = c_n = 0$, where the equation must hold for all t.

Example 3.3.1

$\vec{x}_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1+t \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} -t^2 \\ 1 \end{bmatrix}$ are linearly depdendent because $\vec{x}_1 + \vec{x}_3 = \vec{x}_2$, and this holds for all t. So $c_1 = 1, c_2 = -1$ and $c_3 = 1$ above will work.

On the other hand if we change the example just slightly $\vec{x}_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} -t^2 \\ 1 \end{bmatrix}$, then the functions are linearly independent. First write $c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \vec{0}$ and note that it has to hold for all t. We get that

$$c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \begin{bmatrix} c_1t^2 - c_3t^3 \\ c_1t + c_2t + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In other words $c_1t^2 - c_3t^3 = 0$ and $c_1t + c_2t + c_3 = 0$. If we set $t = 0$, then the second equation becomes $c_3 = 0$. However, the first equation becomes $c_1t^2 = 0$ for all t and so $c_1 = 0$. Thus the second equation is just $c_2t = 0$, which means $c_2 = 0$. So $c_1 = c_2 = c_3 = 0$ is the only solution and $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$ are linearly independent.

The linear combination $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$ could always be written as

$$X(t)\vec{c}$$

where $X(t)$ is the matrix with columns $\vec{x}_1, \ldots, \vec{x}_n$, and $\vec{c}$ is the column vector with entries $c_1, \ldots, c_n$. The matrix valued function $X(t)$ is called the *fundamental matrix*, or the *fundamental matrix solution*.

To solve nonhomogeneous first order linear systems, we use the same technique as we applied to solve single linear nonhomogeneous equations.

Theorem 5.3.2

Let $\vec{x}' = P\vec{x} + \vec{f}$ be a linear system of ODEs. Suppose $\vec{x}_p$ is one particular solution. Then every solution can be written as

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

where $\vec{x}_c$ is a solution to the associated homogeneous equation $(\vec{x} = P\vec{x})$.

So the procedure will be the same as for single equations. We find a particular solution to the nonhomogeneous equation, then we find the general solution to the associated homogeneous equation, and finally we add the two together.

Alright, suppose you have found the general solution $\vec{x}' = P\vec{x} + \vec{f}$. Now you are given an initial condition of the form

$$\vec{x}t_0 = \vec{b}$$

for some constant vector $\vec{b}$. Suppose that $X(t)$ is the fundamental matrix solution of the associated homogeneous equation (i.e. columns of $X(t)$ are solutions). The general solution can be written as

$$\vec{x}(t) = X(t)\vec{c} + \vec{x}_p(t)$$

We are seeking a vector $\vec{c}$ such that

$$\vec{b} = \vec{x}(t_0) = X(t_0)\vec{c} + \vec{x}_p(t_0)$$

In other words, we are solving for $\vec{c}$ the nonhomogeneous system of linear equations

$$X(t_0)\vec{c} = \vec{b} - \vec{x}_p(t_0)$$

Example 3.3.2

In Section 3.1 we solved the system

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= x_1 - x_2 \end{aligned} \tag{5.3.4}$$

with initial conditions $x_1(0) = 1, x_2(0) = 2$.

Solution

This is a homogeneous system, so $\vec{f}(t) = \vec{0}$. We write the system and the initial conditions as

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We found the general solution was $x_1 = C_1 e^t$ and $x_2 = \frac{c_1}{2} e^t + c_2 e^{-t}$. Letting $C_1 = 1$ and $C_2 = 0$, we obtain the solution $\begin{bmatrix} e^t \\ \frac{1}{2} e^t \end{bmatrix}$. Letting $C_1 = 0$ and $C_2 = 1$, we obtain $\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$. These two solutions are linearly independent, as can be seen by setting $t = 0$, and noting that the resulting constant vectors are linearly independent. In matrix notation, the fundamental matrix solution is, therefore,

$$X(t) = \begin{bmatrix} e^t & 0 \\ \frac{1}{2} e^t & e^{-t} \end{bmatrix}$$

Hence to solve the initial problem we solve the equation

$$X(0)\vec{(c)} = \vec{b}$$

or in other words,

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\vec{x}(t) = X(t)\vec{c} = \begin{bmatrix} e^t & 0 \\ \frac{1}{2} e^t & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} e^t \\ \frac{1}{2} e^t + \frac{3}{2} e^{-t} \end{bmatrix}$$

This agrees with our previous solution from Section 3.1.

5.4: Eigenvalue Method

In this section we will learn how to solve linear homogeneous constant coefficient systems of ODEs by the eigenvalue method. Suppose we have such a system

$$\vec{x}' = P\vec{x},$$

where P is a constant square matrix. We wish to adapt the method for the single constant coefficient equation by trying the function $e^{\lambda t}$. However, $\vec{x}$ is a vector. So we try $\vec{x} = \vec{v}e^{\lambda t}$, where $\vec{v}$ is an arbitrary constant vector. We plug this $\vec{x}$ into the equation to get

$$\underbrace{\lambda\vec{v}e^{\lambda t}}_{\vec{x}'} = \underbrace{P\vec{v}e^{\lambda t}}_{P\vec{x}}.$$

We divide by $e^{\lambda t}$ and notice that we are looking for a scalar λ and a vector $\vec{x}$ that satisfy the equation

$$\lambda\vec{v} = P\vec{v}.$$

To solve this equation we need a little bit more linear algebra, which we now review.

5.4.1: Eigenvalues and Eigenvectors of a Matrix

Let A be a constant square matrix. Suppose there is a scalar λ and a nonzero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

We then call λ an *eigenvalue* of A and $\vec{x}$ is said to be a corresponding *eigenvector*.

Example 5.4.1

The matrix $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ has an eigenvalue of $\lambda = 2$ with a corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let us see how to compute the eigenvalues for any matrix. We rewrite the equation for an eigenvalue as

$$(A - \lambda I)\vec{v} = \vec{0}.$$

We notice that this equation has a nonzero solution $\vec{v}$ only if $A - \lambda I$ is not invertible. Were it invertible, we could write $(A - \lambda I)^{-1}(A - \lambda I)\vec{v} = (A - \lambda I)^{-1}\vec{0}, \quad$ which implies $\vec{v} = \vec{0}$. Therefore, A has the eigenvalue λ if and only if λ solves the equation

$$det(A - \lambda I) = 0.$$

Consequently, we will be able to find an eigenvalue of A without finding a corresponding eigenvector. An eigenvector will have to be found later, once λ is known.

Example 5.4.2

Find all eigenvalues of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution

We write

$$det\left(\begin{bmatrix}2&1&1\\1&2&0\\0&0&2\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=det\left(\begin{bmatrix}2-\lambda&1&1\\1&2-\lambda&0\\0&0&2-\lambda\end{bmatrix}\right) \\ =(2-\lambda)\left((2-\lambda)^2-1\right)=-(\lambda-1)(\lambda-2)(\lambda-3). \tag{5.4.1}$$

So the eigenvalues are $\lambda=1, \lambda=2,$ and $\lambda=3$.

Note that for an $n\times n$ matrix, the polynomial we get by computing $det(A-\lambda I)$ will be of degree n, and hence we will in general have n eigenvalues. Some may be repeated, some may be complex.

To find an eigenvector corresponding to an eigenvalue λ, we write

$$(A-\lambda I)\vec{v}=\vec{0},$$

and solve for a nontrivial (nonzero) vector $\vec{v}$. If λ is an eigenvalue, there will be at least one free variable, and so for each distinct eigenvalue λ, we can always find an eigenvector

Example 5.4.3

Find an eigenvector of $\begin{bmatrix}2&1&1\\1&2&0\\0&0&2\end{bmatrix}$ corresponding to the eigenvalue $\lambda=3$.

Solution

We write

$$(A-\lambda I)\vec{v}=\left(\begin{bmatrix}2&1&1\\1&2&0\\0&0&2\end{bmatrix}-3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}-1&0&0\\1&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\vec{0}.$$

It is easy to solve this system of linear equations. We write down the augmented matrix

$$\left[\begin{array}{ccc|c}-1&0&0&0\\1&-1&0&0\\0&0&-1&0\end{array}\right],$$

and perform row operations (exercise: which ones?) until we get:

$$\left[\begin{array}{ccc|c}1&-1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right].$$

The entries of $\vec{v}$ have to satisfy the equations $v_1-v_2=0, v_3=0$ and v_2 is a free variable. We can pick v_2 to be arbitrary (but nonzero), let $v_1=v_2$, and of course $v_3=0$. For example, if we pick $v_2=1$, then $\vec{v}=\begin{bmatrix}1\\1\\0\end{bmatrix}$. Let us verify that $\vec{v}$ really is an eigenvector corresponding to $\lambda=3$:

$$\begin{bmatrix}2&1&1\\1&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}3\\3\\0\end{bmatrix}=3\begin{bmatrix}1\\1\\0\end{bmatrix}.$$

Yay! It worked.

Exercise 5.4.1: (easy)

Are eigenvectors unique? Can you find a different eigenvector for $\lambda = 3$ in the example above? How are the two eigenvectors related?

Exercise 5.4.2

Note that when the matrix is 2×2 you do not need to write down the augmented matrix and do row operations when computing eigenvectors (if you have computed the eigenvalues correctly). Can you see why? Try it for the matrix v_2.

5.4.2: 3.4.2Eigenvalue Method with Distinct Real Eigenvalues

We have the system of equations

$$\vec{x}' = P\vec{x}.$$

We find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix P, and corresponding eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$. Now we notice that the functions $\vec{v}_1 e^{\lambda_1 t}, \vec{v}_2 e^{\lambda_2 t}, \ldots, \vec{v}_n e^{\lambda_n t}$ are solutions of the system of equations and hence $\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t}$ is a solution.

Theorem 5.4.1

Take $\vec{x}' = P\vec{x}$. If P is an $n \times n$ constant matrix that has n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then there exist n linearly independent corresponding eigenvectors$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and the general solution to $\vec{x}' = P\vec{x}$ can be written as

$$\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t}.$$

The corresponding fundamental matrix solution is

$$X(t) = [\vec{v}_1 e^{\lambda_1 t} \quad \vec{v}_2 e^{\lambda_2 t} \quad \cdots \quad \vec{v}_n e^{\lambda_n t}].$$

That is, $X(t)$ is the matrix whose j^{th} column is $\vec{v}_j e^{\lambda_j t}$.

Below is a video on solving a system of differential equations and matrices..

Example 5.4.4

Consider the system

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}.$$

Find the general solution.

Solution

Earlier, we found the eigenvalues are $1, 2, 3$. We found the eigenvector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ for the eigenvalue 3. Similarly we find the eigenvector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for the eigenvalue 1, and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ for the eigenvalue 2 (exercise: check). Hence our general solution is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t} = \begin{bmatrix} c_1 e^t + c_3 e^{3t} \\ -c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ -c_2 e^{2t} \end{bmatrix}.$$

In terms of a fundamental matrix solution

$$\vec{x} = X(t)\vec{c} = \begin{bmatrix} e^t & 0 & e^{3t} \\ -e^t & e^{2t} & e^{3t} \\ 0 & -e^{2t} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

? Exercise 5.4.3

Check that this $\vec{x}$ really solves the system.

Note: If we write a homogeneous linear constant coefficient n^{th} order equation as a first order system (as we did in Section 3.1), then the eigenvalue equation

$$det(P - \lambda I) = 0$$

is essentially the same as the characteristic equation we got in Section 2.2 and Section 2.3.

Below is a video on solving a system differential equations where the associated matrix has distince real eigenvalues.

5.4.3: Complex Eigenvalues

A matrix might very well have complex eigenvalues even if all the entries are real. For example, suppose that we have the system

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}.$$

Let us compute the eigenvalues of the matrix $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

$$det(P - \lambda I) = det\left(\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} \right) = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0.$$

Thus $\lambda = 1 \pm i$. The corresponding eigenvectors are also complex. First take $\lambda = 1 - i$,

$$\begin{aligned} (P-(1-i)I)\vec{v} &= \vec{0}, \\ \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \vec{v} &= \vec{0}. \end{aligned} \tag{5.4.2}$$

The equations $iv_1 + v_2 = 0$ and $-v_1 + iv_2 = 0$ are multiples of each other. So we only need to consider one of them. After picking $v_2 = 1$, for example, we have an eigenvector $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$. In similar fashion we find that $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $1+i$.

We could write the solution as

$$\vec{x} = c_1 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1-i)t} + c_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} c_1 i e^{(1-i)t} - c_2 i e^{(1+i)t} \\ c_1 e^{(1-i)t} + c_2 e^{(1+i)t} \end{bmatrix}.$$

We would then need to look for complex values c_1 and c_2 to solve any initial conditions. It is perhaps not completely clear that we get a real solution. We could use Euler's formula and do the whole song and dance we did before, but we will not. We will do something a bit smarter first.

We claim that we did not have to look for a second eigenvector (nor for the second eigenvalue). All complex eigenvalues come in pairs (because the matrix P is real).

First a small side note. The real part of a complex number z can be computed as $\frac{z+\bar{z}}{2}$, where the bar above z means $\overline{a+ib} = a - ib$. This operation is called the complex conjugate. If a is a real number, then $\bar{a} = a$. Similarly we can bar whole vectors or matrices by taking the complex conjugate of every entry. If a matrix P is real, then $\bar{P} = P$. We note that $\overline{P\vec{x}} = \bar{P}\bar{\vec{x}} = P\bar{\vec{x}}$.. Also the complex conjugate of 0 is still 0, therefore,

$$\vec{0} = \bar{\vec{0}} = \overline{(P-\lambda I)\vec{v}} = (P-\bar{\lambda}I)\bar{\vec{v}}.$$

So if $\vec{v}$ is an eigenvector corresponding to the eigenvalue $\lambda = a + ib$, then $\bar{\vec{v}}$ is an eigenvector corresponding to the eigenvalue $\bar{\lambda} = a - ib$.

Suppose that $a+ib$ is a complex eigenvalue of P, and $\vec{v}$ is a corresponding eigenvector. Then

$$\vec{x}_1 = \vec{v} e^{(a+ib)t}$$

is a solution (complex valued) of $\vec{x}' = P\vec{x}$. Euler's formula shows that $\overline{e^{a+ib}} = e^{a-ib}$, and so

$$\vec{x}_2 = \overline{\vec{x}_1} = \bar{\vec{v}} e^{(a+ib)t}$$

is also a solution. As $\vec{x_1}$ and $\vec{x_2}$ are solutions, the function

$$\vec{x}_3 = \operatorname{Re} \vec{x}_1 = \operatorname{Re} \vec{v} e^{(a+ib)t} = \frac{\vec{x}_1 + \overline{\vec{x}_1}}{2} = \frac{\vec{x}_1 + \vec{x_2}}{2} = \frac{1}{2}\vec{x_1} + \frac{1}{2}\vec{x_2}$$

is also a solution. And $\vec{x}_3$ is real-valued! Similarly as $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$ is the imaginary part, we find that

$$\vec{x}_4 = \operatorname{Im} \vec{x}_1 = \frac{\vec{x}_1 - \overline{\vec{x}_1}}{2i} = \frac{\vec{x}_1 - \vec{x_2}}{2i}$$

is also a real-valued solution. It turns out that $\vec{x}_3$ and $\vec{x}_4$ are linearly independent. We will use Euler's formula to separate out the real and imaginary part.

Returning to our problem,

$$\vec{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1-i)t} = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(e^t \cos t - i e^t \sin t\right) = \begin{bmatrix} i e^t \cos t + e^t \sin t \\ e^t \cos t - i e^t \sin t \end{bmatrix} = \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + i \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix}.$$

Then

$$\text{Re}\,\vec{x}_1 = \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix}, \quad \text{and} \quad \text{Im}\,\vec{x}_1 = \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix},$$

are the two real-valued linearly independent solutions we seek.

Below is a video on solving a of system differential equation where the associated matrix has complet eigenvalues.

? Exercise 5.4.4

Check that these really are solutions.

The general solution is

$$\vec{x} = c_1 \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + c_2 \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix} = \begin{bmatrix} c_1 e^t \sin t + c_2 e^t \cos t \\ c_1 e^t \cos t - c_2 e^t \sin t \end{bmatrix}.$$

This solution is real-valued for real c_1 and c_2. At this point, we would solve for any initial conditions we may have to find c_1 and c_2.

Let us summarize the discussion as a theorem.

Theorem 5.4.2

Let P be a real-valued constant matrix. If P has a complex eigenvalue $a+ib$ and a corresponding eigenvector $\vec{v}$, then P also has a complex eigenvalue $a-ib$ with a corresponding eigenvector $\bar{\vec{v}}$. Furthermore, $\vec{x}' = P\vec{x}$ has two linearly independent real-valued solutions

$$\vec{x}_1 = \text{Re}\,\vec{v}e^{(a+ib)t}, \quad \textit{and} \quad \vec{x}_2 = \text{Im}\,\vec{v}e^{(a+ib)t}.$$

For each pair of complex eigenvalues $a+ib$ and $a-ib$, we get two real-valued linearly independent solutions. We then go on to the next eigenvalue, which is either a real eigenvalue or another complex eigenvalue pair. If we have n distinct eigenvalues (real or complex), then we end up with n linearly independent solutions. If we had only two equations $(n=2)$ as in the example above, then once we found two solutions we are finished, and our general solution is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1\left(\text{Re}\,\vec{v}e^{(a+ib)t}\right) + c_2\left(\text{Im}\,\vec{v}e^{(a+ib)t}\right).$$

We can now find a real-valued general solution to any homogeneous system where the matrix has distinct eigenvalues. When we have repeated eigenvalues, matters get a bit more complicated and we will look at that situation in Section 3.7.

5.5: Two dimensional systems and their vector fields

Let us take a moment to talk about constant coefficient linear homogeneous systems in the plane. Much intuition can be obtained by studying this simple case. Suppose we use coordinates (x, y) for the plane as usual, and suppose $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix . Consider the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = P \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{5.5.1}$$

The system is *autonomous* (compare this section to Section 1.6) and so we can draw a **vector field** (see end of Section 3.1). We will be able to visually tell what the vector field looks like and how the solutions behave, once we find the eigenvalues and eigenvectors of the matrix P. For this section, we assume that P has two eigenvalues and two corresponding eigenvectors.

5.5.0.1: 1

Suppose that the eigenvalues of P are real and positive. We find two corresponding eigenvectors and plot them in the plane. For example, take the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. The eigenvalues are 1 and 2 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. See Figure $5.5.1$.

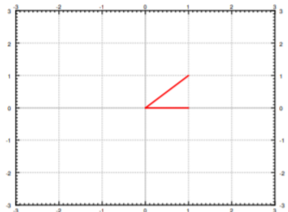


Figure $5.5.1$: Eigenvectors of P.

Now suppose that x and y are on the line determined by an eigenvector $\vec{v}$ for an eigenvalue λ. That is, $\begin{bmatrix} x \\ y \end{bmatrix} = a\vec{v}$ for some scalar a. Then

$$\begin{bmatrix} x \\ y \end{bmatrix}' = P \begin{bmatrix} x \\ y \end{bmatrix} = P(a\vec{v}) = a(P\vec{v}) = a\lambda\vec{v}$$

The derivative is a multiple of $\vec{v}$ and hence points along the line determined by $\vec{v}$. As $\lambda > 0$, the derivative points in the direction of $vecv$ when α is positive and in the opposite direction when α is negative. Let us draw the lines determined by the eigenvectors, and let us draw arrows on the lines to indicate the directions. See Figure $5.5.2$.

We fill in the rest of the arrows for the vector field and we also draw a few solutions. See Figure $5.5.3$. Notice that the picture looks like a source with arrows coming out from the origin. Hence we call this type of picture a source or sometimes an *unstable node*.

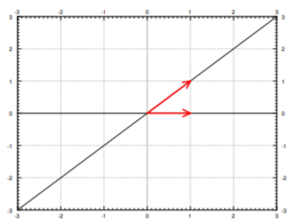


Figure $5.5.2$: Eigenvectors of P with directions.

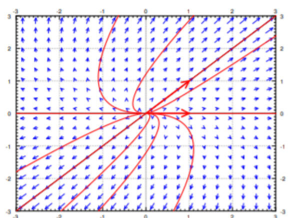


Figure $5.5.3$: Example source vector field with eigenvectors and solutions.

5.5.0.1: 2

Suppose both eigenvalues were negative. For example, take the negation of the matrix in case 1, $\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$. The eigenvalues are -1 and -2 and corresponding eigenvectors are the same, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The calculation and the picture are almost the same. The only difference is that the eigenvalues are negative and hence all arrows are reversed. We get the picture in Figure $5.5.4$. We call this kind of picture a *sink* or sometimes a *stable node*.

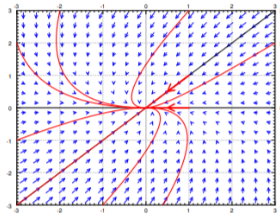


Figure $5.5.4$: Example sink vector field with eigenvectors and solutions.

Below is a video on phase portraits.

5.5.0.1: 3

Suppose one eigenvalue is positive and one is negative. For example the matrix $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$. The eigenvalues are 1 and -2 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

We reverse the arrows on one line (corresponding to the negative eigenvalue) and we obtain the picture in Figure $5.5.5$. We call this picture a saddle point.

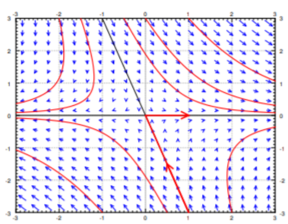


Figure $5.5.5$: Example saddle vector field with eigenvectors and solutions.

Below is a video on a phase portrait of a saddle point.

For the next three cases we will assume the eigenvalues are complex. In this case the eigenvectors are also complex and we cannot just plot them in the plane.

5.5.0.1: 4

Suppose the eigenvalues are purely imaginary. That is, suppose the eigenvalues are $\pm ib$. For example, let $P = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$. The eigenvalues turn out to be $\pm 2i$ and eigenvectors are $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$. Consider the eigenvalue $2i$ and its eigenvector $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$. The real and imaginary parts of $\vec{v}e^{i2t}$ are

$$Re \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{i2t} = \begin{bmatrix} cos(2t) \\ -2sin(2t) \end{bmatrix}, \quad Im \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{i2t} = \begin{bmatrix} sin(2t) \\ 2cos(2t) \end{bmatrix}$$

We can take any linear combination of them to get other solutions, which one we take depends on the initial conditions. Now note that the real part is a parametric equation for an ellipse. Same with the imaginary part and in fact any linear combination of the two. This is what happens in general when the eigenvalues are purely imaginary. So when the eigenvalues are purely imaginary, we get *ellipses* for the solutions. This type of picture is sometimes called a *center*. See Figure $5.5.6$.

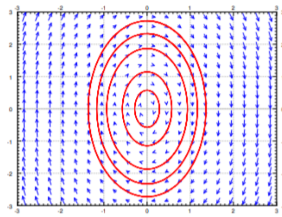


Figure $5.5.6$: Example center vector field.

5.5.0.1: 5

Now suppose the complex eigenvalues have a positive real part. That is, suppose the eigenvalues are $a \pm ib$ for some $a > 0$. For example, let $P = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$. The eigenvalues turn out to be $1 \pm 2i$ and eigenvectors are $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$. We take $1 \pm 2i$ and its eigenvector $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ and find the real and imaginary of $\vec{v}e^{(1+2i)t}$ are

$$Re \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} = e^t \begin{bmatrix} cos(2t) \\ -2sin(2t) \end{bmatrix} \quad Im \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} = e^t \begin{bmatrix} sin(2t) \\ 2cos(2t) \end{bmatrix}$$

Note the e^t in front of the solutions. This means that the solutions grow in magnitude while spinning around the origin. Hence we get a *spiral source*. See Figure $5.5.7$.

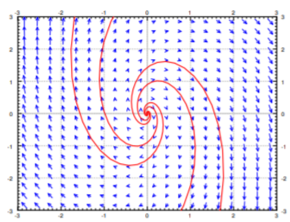


Figure $5.5.7$: Example spiral source vector field.

Below is a video on a phase portrait of a spiral point.

5.5.0.1: 6

Finally suppose the complex eigenvalues have a negative real part. That is, suppose the eigenvalues are $-a \pm ib$ for some $a > 0$. For example, let $P = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$. The eigenvalues turn out to be $-1 \pm 2i$ and eigenvectors are $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$. We take $-1 - 2i$ and its eigenvector $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ and find the real and imaginary of $\vec{v}e^{(-1-2i)t}$ are

$$Re \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(-1-2i)t} = e^{-t} \begin{bmatrix} cos(2t) \\ 2sin(2t) \end{bmatrix} \quad Im \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(-1-2i)t} = e^{-t} \begin{bmatrix} -sin(2t) \\ 2cos(2t) \end{bmatrix}$$

Note the e^{-t} in front of the solutions. This means that the solutions shrink in magnitude while spinning around the origin. Hence we get a spiral sink. See Figure $5.5.8$.

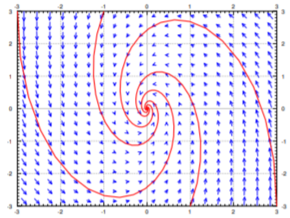


Figure 5.5.8: Example spiral sink vector field.

We summarize the behavior of linear homogeneous two dimensional systems given by a nonsingular matrix in Table 5.5.1. Systems where one of the eigenvalues is zero (the matrix is singular) come up in practice from time to time, see Example 3.1.2, and the pictures are somewhat different (simpler in a way). See the exercises.

Table 5.5.1: Summary of behavior of linear homogeneous two dimensional systems.

Eigenvalues	Behavior
real and both positive	source / unstable node
real and both negative	sink / stable node
real and opposite signs	saddle
purely imaginary	center point / ellipses
complex with positive real part	spiral source
complex with negative real part	spiral sink

Below is a video on phase portraits of linear systems.

5.5.1:

5.5.2: Contributors and Attributions

- Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

5.6: Second order systems and applications

5.6.1: Undamped Mass-Spring Systems

While we did say that we will usually only look at first order systems, it is sometimes more convenient to study the system in the way it arises naturally. For example, suppose we have 3 masses connected by springs between two walls. We could pick any higher number, and the math would be essentially the same, but for simplicity we pick 3 right now. Let us also assume no friction, that is, the system is undamped. The masses are m_1, m_2, and m_3 and the spring constants are k_1, k_2, k_3, and k_4. Let x_1 be the displacement from rest position of the first mass, and x_2 and x_3 the displacement of the second and third mass. We will make, as usual, positive values go right (as x_1 grows, the first mass is moving right). See Figure $5.6.1$.

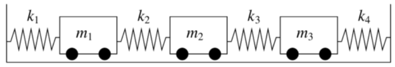


Figure $5.6.1$: System of masses and springs.

This simple system turns up in unexpected places. For example, our world really consists of many small particles of matter interacting together. When we try the above system with many more masses, we obtain a good approximation to how an elastic material behaves. By somehow taking a limit of the number of masses going to infinity, we obtain the continuous one dimensional wave equation (that we study in Section 4.7). But we digress.

Let us set up the equations for the three mass system. By Hooke's law we have that the force acting on the mass equals the spring compression times the spring constant. By Newton's second law we have that force is mass times acceleration. So if we sum the forces acting on each mass and put the right sign in front of each term, depending on the direction in which it is acting, we end up with the desired system of equations.

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) & &= -(k_1+k_2)x_1 + k_2 x_2, \\ m_2 x_2'' &= -k_2(x_2 - x_1) + k_3(x_3 - x_2) & &= k_2 x_1 - (k_2+k_3)x_2 + k_3 x_3, \\ m_3 x_3'' &= -k_3(x_3 - x_2) - k_4 x_3 & &= k_3 x_2 - (k_3+k_4)x_3. \end{aligned}$$

We define the matrices

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -(k_3+k_4) \end{bmatrix}.$$

We write the equation simply as

$$M\vec{x}'' = K\vec{x}.$$

At this point we could introduce 3 new variables and write out a system of 6 first order equations. We claim this simple setup is easier to handle as a second order system. We call $\vec{x}$ the displacement vector, M the mass matrix, and K the stiffness matrix.

? Exercise 5.6.1

Repeat this setup for 4 masses (find the matrices M and K). Do it for 5 masses. Can you find a prescription to do it for n masses?

As with a single equation we want to "divide by M." This means computing the inverse of M. The masses are all nonzero and $M$is a diagonal matrix, so comping the inverse is easy:

$$M^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix}.$$

This fact follows readily by how we multiply diagonal matrices. As an exercise, you should verify that $MM^{-1} = M^{-1}M = I$.

Let $A = M^{-1}K$. We look at the system $\vec{x}'' = M^{-1}K\vec{x}$, or

$$\vec{x}'' = A\vec{x}.$$

Many real world systems can be modeled by this equation. For simplicity, we will only talk about the given masses-and-springs problem. We try a solution of the form

$$\vec{x} = \vec{v}e^{\alpha t}.$$

We compute that for this guess, $\vec{x}'' = \alpha^2 \vec{v} e^{\alpha t}$. We plug our guess into the equation and get

$$\alpha^2 \vec{v} e^{\alpha t} = A\vec{v}e^{\alpha t}.$$

We divide by $e^{\alpha t}$ to arrive at $\alpha^2\vec{v} = A\vec{v}$. Hence if α^2 is an eigenvalue of A and $\vec{v}$ is a corresponding eigenvector, we have found a solution.

In our example, and in other common applications, A has only real negative eigenvalues (and possibly a zero eigenvalue). So we study only this case. When an eigenvalue λ is negative, it means that $\alpha^2 = \lambda$ is negative. Hence there is some real number ω such that $-\omega^2 = \lambda$. Then $\alpha = \pm i\omega$. The solution we guessed was

$$\vec{x} = \vec{v}(\cos(\omega t) + i\sin(\omega t)).$$

By taking the real and imaginary parts (note that $\vec{v}$ is real), we find that $\vec{v}\cos(\omega t)$ and $\vec{v}\sin(\omega t)$ are linearly independent solutions.

If an eigenvalue is zero, it turns out that both $\vec{v}$ and $\vec{v}t$ are solutions, where $\vec{v}$ is an eigenvector corresponding to the eigenvalue 0.

? Exercise 5.6.2

Show that if A has a zero eigenvalue and $\vec{v}$ is a corresponding eigenvector, then $\vec{x} = \vec{v}(a+bt)$ is a solution of $\vec{x}'' = A\vec{x}$ for arbitrary constants a and b.

Theorem 5.6.1

Let A be an $n \times n$ matrix with n distinct real negative eigenvalues we denote by $-\omega_1^2 > -\omega_2^2 > \cdots > -\omega_n^2$, and corresponding eigenvectors by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. If A is invertible (that is, if $\omega_1 > 0$), then

$$\vec{x}(t)=\sum_{i=1}^{n}\vec{v}_i\left(a_i\cos(\omega_i t)+b_i\sin(\omega_i t)\right),$$

is the general solution of

$$\vec{x}''=A\vec{x},$$

for some arbitrary constants a_i and b_i. If A has a zero eigenvalue, that is $\omega_1=0$, and all other eigenvalues are distinct and negative, then the general solution can be written as

$$\vec{x}(t)=\vec{v}_1(a_1+b_1t)+\sum_{i=2}^{n}\vec{v}_i\left(a_i\cos(\omega_i t)+b_i\sin(\omega_i t)\right).$$

We use this solution and the setup from the introduction of this section even when some of the masses and springs are missing. For example, when there are only 2 masses and only 2 springs, simply take only the equations for the two masses and set all the spring constants for the springs that are missing to zero.

Example 5.6.1:

Suppose we have the system in Figure 5.6.2, with $m_1=2, m_2=1, k_1=4,$ and $k_2=2$.

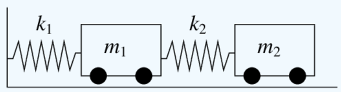


Figure 5.6.2: System of masses and springs.

The equations we write down are

$$\begin{bmatrix}2&0\\0&1\end{bmatrix}\vec{x}''=\begin{bmatrix}-(4+2)&2\\2&-2\end{bmatrix}\vec{x},$$

or

$$\vec{x}''=\begin{bmatrix}-3&1\\2&-2\end{bmatrix}\vec{x}.$$

We find the eigenvalues of A to be $\lambda=-1,-4$ (exercise). We find corresponding eigenvectors to be $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ respectively (exercise).

We check the theorem and note that $\omega_1=1$ and $\omega_2=2$. Hence the general solution is

$$\vec{x}=\begin{bmatrix}1\\2\end{bmatrix}\left(a_1\cos(t)+b_1\sin(t)\right)+\begin{bmatrix}1\\-1\end{bmatrix}\left(a_2\cos(2t)+b_2\sin(2t)\right).$$

The two terms in the solution represent the two so-called natural or normal modes of oscillation. And the two (angular) frequencies are the natural frequencies. The first natural frequency is 1, and second natural frequency is 2. The two modes are plotted in Figure 5.6.3.

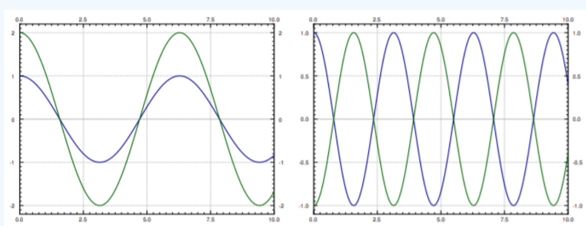


Figure 5.6.3: The two modes of the mass-spring system. In the left plot the masses are moving in unison and in the right plot are masses moving in the opposite direction.

Let us write the solution as

$$\vec{x}=\begin{bmatrix}1\\2\end{bmatrix}c_1\cos(t-\alpha_1)+\begin{bmatrix}1\\-1\end{bmatrix}c_2\cos(2t-\alpha_2).$$

The first term,

$$\begin{bmatrix}1\\2\end{bmatrix}c_1\cos(t-\alpha_1)=\begin{bmatrix}c_1\cos(t-\alpha_1)\\2c_1\cos(t-\alpha_1)\end{bmatrix},$$

corresponds to the mode where the masses move synchronously in the same direction.

The second term,

$$\begin{bmatrix}1\\-1\end{bmatrix}c_2\cos(2t-\alpha_2)=\begin{bmatrix}c_2\cos(2t-\alpha_2)\\-c_2\cos(2t-\alpha_2)\end{bmatrix},$$

corresponds to the mode where the masses move synchronously but in opposite directions.

The general solution is a combination of the two modes. That is, the initial conditions determine the amplitude and phase shift of each mode. As an example, suppose we have initial conditions

$$\vec{x}(0)=\begin{bmatrix}1\\-1\end{bmatrix},\qquad \vec{x}'(0)=\begin{bmatrix}0\\6\end{bmatrix}.$$

We use the a_j, b_j constants to solve for initial conditions. First

$$\begin{bmatrix}1\\-1\end{bmatrix}=\vec{x}(0)=\begin{bmatrix}1\\2\end{bmatrix}a_1+\begin{bmatrix}1\\-1\end{bmatrix}a_2=\begin{bmatrix}a_1+a_2\\2a_1-a_2\end{bmatrix}.$$

We solve (exercise) to find $a_1=0$, $a_2=1$. To find the b_1 and b_2, we differentiate first:

$$\vec{x}' = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(-a_1 \sin(t) + b_1 \cos(t)\right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(-2a_2 \sin(2t) + 2b_2 \cos(2t)\right).$$

Now we solve:

$$\begin{bmatrix} 0 \\ 6 \end{bmatrix} = \vec{x}'(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} b_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} 2b_2 = \begin{bmatrix} b_1 + 2b_2 \\ 2b_1 - 2b_2 \end{bmatrix}.$$

Again solve (exercise) to find $b_1 = 2$, $b_2 = -1$. So our solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2\sin(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(\cos(2t) - \sin(2t)\right) = \begin{bmatrix} 2\sin(t) + \cos(2t) - \sin(2t) \\ 4\sin(t) - \cos(2t) + \sin(2t) \end{bmatrix}.$$

The graphs of the two displacements, x_1 and x_2 of the two carts is in Figure 5.6.4.

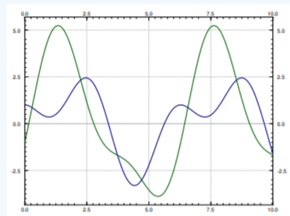


Figure 5.6.4: Superposition of the two modes given the initial conditions.

Below is a video on coupled oscillators.

Example 5.6.2

We have two toy rail cars. Car 1 of mass 2 kg is traveling at 3 $\frac{\text{m}}{\text{s}}$ towards the second rail car of mass 1 kg. There is a bumper on the second rail car that engages at the moment the cars hit (it connects to two cars) and does not let go. The bumper acts like a spring of spring constant $k = 2\ \frac{\text{N}}{\text{m}}$. The second car is 10 meters from a wall. See Figure 5.6.5.

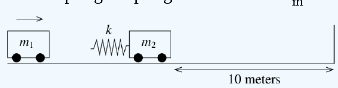


Figure 5.6.5: The crash of two rail cars.

We want to ask several questions. At what time after the cars link does impact with the wall happen? What is the speed of car 2 when it hits the wall?

OK, let us first set the system up. Let $t = 0$ be the time when the two cars link up. Let x_1 be the displacement of the first car from the position at $t = 0$, and let x_2 be the displacement of the second car from its original location. Then the time when $x_2(t) = 10$ is exactly the time when impact with wall occurs. For this t, $x_2'(t)$ is the speed at impact. This system acts just like the system of the previous example but without k_1. Hence the equation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

or

$$\vec{x}'' = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

We compute the eigenvalues of A. It is not hard to see that the eigenvalues are 0 and -3 (exercise). Furthermore, eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively (exercise). Then $\omega_2 = \sqrt{3}$ and by the second part of the theorem we find our general solution to be

$$\begin{aligned} \vec{x} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} (a_1 + b_1 t) + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \left(a_2 \cos(\sqrt{3}t) + b_2 \sin(\sqrt{3}t)\right) \\ &= \begin{bmatrix} a_1 + b_1 t + a_2 \cos(\sqrt{3}t) + b_2 \sin(\sqrt{3}t) \\ a_1 + b_1 t - 2a_2 \cos(\sqrt{3}t) - 2b_2 \sin(\sqrt{3}t) \end{bmatrix} \end{aligned} \tag{5.6.1}$$

We now apply the initial conditions. First the cars start at position 0 so $x_1(0) = 0$ and $x_2(0) = 0$. The first car is traveling at 3 $\frac{\text{m}}{\text{s}}$, so $x_1'(0) = 3$ and the second car starts at rest, so $x_2'(0) = 0$. The first conditions says

$$\vec{0} = \vec{x}(0) = \begin{bmatrix} a_1 + a_2 \\ a_1 - 2a_2 \end{bmatrix}.$$

It is not hard to see that $a_1 = a_2 = 0$. We set $a_1 = 0$ and $a_2 = 0$ in $\vec{x}(t)$ and differentiate to get

$$\vec{x}'(t) = \begin{bmatrix} b_1 + \sqrt{3}b_2 \cos(\sqrt{3}t) \\ b_1 - 2\sqrt{3}b_2 \cos(\sqrt{3}t) \end{bmatrix}.$$

So

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \vec{x}'(0) = \begin{bmatrix} b_1 + \sqrt{3}b_2 \\ b_1 - 2\sqrt{3}b_2 \end{bmatrix}.$$

Solving these two equations we find $b_1 = 2$ and $b_2 = \frac{1}{\sqrt{3}}$. Hence the position of our cars is (until the impact with the wall)

$$\vec{x} = \begin{bmatrix} 2t + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) \\ 2t - \frac{2}{\sqrt{3}}\sin(\sqrt{3}t) \end{bmatrix}.$$

Note how the presence of the zero eigenvalue resulted in a term containing t. This means that the carts will be traveling in the positive direction as time grows, which is what we expect.

What we are really interested in is the second expression, the one for x_2. We have $x_2(t) = 2t - \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)$. See Figure $5.6.6$ for the plot of x_2 versus time.

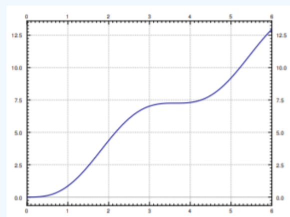


Figure $5.6.6$: Position of the second car in time (ignoring the wall).

Just from the graph we can see that time of impact will be a little more than 5 seconds from time zero. For this we have to solve the equation $10 = x_2(t) = 2t - \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)$. Using a computer (or even a graphing calculator) we find that $t_{\text{impact}} \approx 5.22$ seconds.

As for the speed we note that $x_2' = 2 - 2\cos(\sqrt{3}t)$. At time of impact (5.22 seconds from $t = 0$) we get that $x_2'(t_{\text{impact}}) \approx 3.85$.

The maximum speed is the maximum of $2 - 2\cos(\sqrt{3}t)$, which is 4. We are traveling at almost the maximum speed when we hit the wall.

Suppose that Tiana is a tiny person sitting on car 2. Tiana has a Martini in her hand and would like not to spill it. Let us suppose Tiana would not spill her Martini when the first car links up with car 2, but if car 2 hits the wall at any speed greater than zero, Tiana will spill her drink. Suppose Tiana can move car 2 a few meters towards or away from the wall (he cannot go all the way to the wall, nor can she get out of the way of the first car). Is there a "safe" distance for her to be at? A distance such that the impact with the wall is at zero speed?

The answer is yes. Looking at Figure $5.6.6$, we note the "plateau" between $t = 3$ and $t = 4$. There is a point where the speed is zero. To find it we need to solve $x_2'(t) = 0$. This is when $\cos(\sqrt{3}t) = 1$ or in other words when $t = \frac{2\pi}{\sqrt{3}}, \frac{4\pi}{\sqrt{3}}, \ldots$ and so on. We plug in the first value to obtain $x_2\left(\frac{2\pi}{\sqrt{3}}\right) = \frac{4\pi}{\sqrt{3}} \approx 7.26$. So a "safe" distance is about 7 and a quarter meters from the wall.

Alternatively Tiana could move away from the wall towards the incoming car 2 where another safe distance is $\frac{8\pi}{\sqrt{3}} \approx 14.51$ and so on, using all the different t such that $x_2'(t) = 0$. Of course $t = 0$ is always a solution here, corresponding to $x_2 = 0$, but that means standing right at the wall.

Below is a video on normal modes.

Below is another video on normal modes.

5.6.2: Forced Oscillations

Finally we move to forced oscillations. Suppose that now our system is

$$\vec{x}'' = A\vec{x} + \vec{F}\cos(\omega t). \tag{5.6.2}$$

That is, we are adding periodic forcing to the system in the direction of the vector $\vec{F}$.

As before, this system just requires us to find one particular solution $\vec{x}_p$, add it to the general solution of the associated homogeneous system $\vec{x}_c$, and we will have the general solution to (5.6.2). Let us suppose that ω is not one of the natural frequencies of $\vec{x}'' = A\vec{x}$, then we can guess

$$\vec{x}_p = \vec{c}\cos(\omega t),$$

where $\vec{c}$ is an unknown constant vector. Note that we do not need to use sine since there are only second derivatives. We solve for $\vec{c}$ to find $\vec{x}_p$. This is really just the method of undetermined coefficients for systems. Let us differentiate $\vec{x}_p$ twice to get

$$\vec{x}_p'' = -\omega^2\vec{c}\cos(\omega t).$$

Plug $\vec{x}_p$ and $\vec{x}_p''$ into the equation (5.6.2):

$$\overbrace{-\omega^2\vec{c}\cos(\omega t)}^{\vec{x}_p''} = \overbrace{A\vec{c}\cos(\omega t)}^{A\vec{x}_p} + \vec{F}\cos(\omega t).$$

We cancel out the cosine and rearrange the equation to obtain

$$(A + \omega^2 I)\vec{c} = -\vec{F}.$$

So

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}).$$

Of course this is possible only if $(A + \omega^2 I) = (A - (-\omega^2)I)$ is invertible. That matrix is invertible if and only if $-\omega^2$ is not an eigenvalue of A. That is true if and only if ω is not a natural frequency of the system.

We simplified things a little bit. If we wish to have the forcing term to be in the units of force, say Newtons, then we must write

$$M\vec{x}'' = K\vec{x} + \vec{G}\cos(\omega t).$$

If we then write things in terms of $A = M^{-1}K$, we have

$$\vec{x}'' = M^{-1}K\vec{x} + M^{-1}\vec{G}\cos(\omega t) \qquad \text{or} \qquad \vec{x}'' = A\vec{x} + \vec{F}\cos(\omega t),$$

where $\vec{F} = M^{-1}\vec{G}$.

Example 5.6.3

Let us take the example in Figure 5.6.2 with the same parameters as before: $m_1 = 2, m_2 = 1, k_1 = 4,$ and $k_2 = 2$. Now suppose that there is a force $2\cos(3t)$ acting on the second cart.

The equation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}'' = \begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\cos(3t) \qquad \text{or} \qquad \vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\cos(3t).$$

We solved the associated homogeneous equation before and found the complementary solution to be

$$\vec{x}_c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}(a_1\cos(t) + b_1\sin(t)) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}(a_2\cos(2t) + b_2\sin(2t)).$$

The natural frequencies are 1 and 2. Hence as 3 is not a natural frequency, we can try $\vec{c}\cos(3t)$. We invert $(A + 3^2 I)$:

$$\left(\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} + 3^2 I\right)^{-1} = \begin{bmatrix} 6 & 1 \\ 2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{20} & \frac{3}{20} \end{bmatrix}.$$

Hence,

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}) = \begin{bmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{20} & \frac{3}{20} \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{bmatrix}.$$

Combining with what we know the general solution of the associated homogeneous problem to be, we get that the general solution to $\vec{x}'' = A\vec{x} + \vec{F}\cos(\omega t)$ is

$$\vec{x} = \vec{x}_c + \vec{x}_p = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (a_1 \cos(t) + b_1 \sin(t)) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_2 \cos(2t) + b_2 \sin(2t)) + \begin{bmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{bmatrix} \cos(3t).$$

The constants a_1, a_2, b_1, and b_2 must then be solved for given any initial conditions.

Note that given force $\vec{f}$, we write the equation as $M\vec{x}'' = K\vec{x} + \vec{f}$ to get the units right. Then we write $\vec{x}'' = M^{-1}K\vec{x} + M^{-1}\vec{f}$. The term $\vec{g} = M^{-1}\vec{f}$ in $\vec{x}'' = A\vec{x} + \vec{g}$ is in units of force per unit mass.

If ω is a natural frequency of the system resonance occurs because we will have to try a particular solution of the form

$$\vec{x}_p = \vec{c}t \sin(\omega t) + \vec{d}\, t \cos(\omega t).$$

That is assuming that the eigenvalues of the coefficient matrix are distinct. Next, note that the amplitude of this solution grows without bound as t grows.

5.7: Multiple Eigenvalues

It may very well happen that a matrix has some "repeated" eigenvalues. That is, the characteristic equation $\det(A - \lambda I) = 0$ may have repeated roots. As we have said before, this is actually unlikely to happen for a random matrix. If we take a small perturbation of A (we change the entries of A slightly), then we will get a matrix with distinct eigenvalues. As any system we will want to solve in practice is an approximation to reality anyway, it is not indispensable to know how to solve these corner cases. On the other hand, these cases do come up in applications from time to time. Furthermore, if we have distinct but very close eigenvalues, the behavior is similar to that of repeated eigenvalues, and so understanding that case will give us insight into what is going on.

5.7.1: Geometric Multiplicity

Take the diagonal matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

A has an eigenvalue 3 of multiplicity 2. We call the multiplicity of the eigenvalue in the characteristic equation the algebraic multiplicity. In this case, there also exist 2 linearly independent eigenvectors, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue 3. This means that the so-called geometric multiplicity of this eigenvalue is also 2.

In all the theorems where we required a matrix to have n distinct eigenvalues, we only really needed to have n linearly independent eigenvectors. For example, $\vec{x} = A\vec{x}$ has the general solution

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}.$$

Let us restate the theorem about real eigenvalues. In the following theorem we will repeat eigenvalues according to (algebraic) multiplicity. So for the above matrix A, we would say that it has eigenvalues 3 and 3.

Theorem 5.7.1

Take $\vec{x} = P\vec{x}$. Suppose the matrix P is $n \times n$, has n real eigenvalues (not necessarily distinct), $\lambda_1, \cdots, \lambda_n$and there are n linearly independent corresponding eigenvectors $\overrightarrow{v_1}, \cdots, \overrightarrow{v_n}$. Then the general solution to $\vec{x}' = P\vec{x}$ can be written as:

$$\vec{x} = c_1 \overrightarrow{v_1} e^{\lambda_1 t} + c_2 \overrightarrow{v_2} e^{\lambda_2 t} + \cdot + c_n \overrightarrow{v_n} e^{\lambda_n t}$$

The *geometric multiplicity* of an eigenvalue of algebraic multiplicity n is equal to the number of corresponding linearly independent eigenvectors. The geometric multiplicity is always less than or equal to the algebraic multiplicity. We have handled the case when these two multiplicities are equal. If the geometric multiplicity is equal to the algebraic multiplicity, then we say the eigenvalue is *complete*.

In other words, the hypothesis of the theorem could be stated as saying that if all the eigenvalues of P are complete, then there are n linearly independent eigenvectors and thus we have the given general solution.

If the geometric multiplicity of an eigenvalue is 2 or greater, then the set of linearly independent eigenvectors is not unique up to multiples as it was before. For example, for the diagonal matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ we could also pick eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, or in fact any pair of two linearly independent vectors. The number of linearly independent eigenvectors corresponding to λ is the number of free variables we obtain when solving $A\vec{v} = \lambda\vec{v}$. We pick specific values for those free variables to obtain eigenvectors. If you pick different values, you may get different eigenvectors.

5.7.2: Defective Eigenvalues

If an $n \times n$ matrix has less than n linearly independent eigenvectors, it is said to be deficient. Then there is at least one eigenvalue with an algebraic multiplicity that is higher than its geometric multiplicity. We call this eigenvalue *defective* and the difference between the two multiplicities we call the *defect*.

Example 5.7.1

The matrix

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

has an eigenvalue 3 of algebraic multiplicity 2. Let us try to compute eigenvectors.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$$

Solution

We must have that $v_2 = 0$. Hence any eigenvector is of the form $\begin{bmatrix} v_1 \\ 0 \end{bmatrix}$. Any two such vectors are linearly dependent, and hence the geometric multiplicity of the eigenvalue is 1. Therefore, the defect is 1, and we can no longer apply the eigenvalue method directly to a system of ODEs with such a coefficient matrix.

Roughly, the key observation is that if λ is an eigenvalue of A of algebraic multiplicity m, then we can find certain m linearly independent vectors solving $(A - \lambda I)^k \vec{v} = \vec{0}$ for various powers k. We will call these *generalized eigenvectors*.

Let us continue with the example $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and the equation $\vec{x} = A\vec{x}$. We have an eigenvalue $\lambda = 3$ of (algebraic) multiplicity 2 and defect 1. We have found one eigenvector $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We have the solution

$$\vec{x_1} = \vec{v} e^{3t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}$$

We are now stuck, we get no other solutions from standard eigenvectors. But we need two linearly independent solutions to find the general solution of the equation.

In this case, let us try (in the spirit of repeated roots of the characteristic equation for a single equation) another solution of the form

$$\vec{x_2} = (\vec{v_2} + \vec{v_1} t) e^{3t}$$

We differentiate to get

$$\vec{x}_2{}' = \vec{v}_1 e^{3t} + 3(\vec{v}_2 + \vec{v}_1 t)\, e^{3t} = (3\vec{v}_2 + \vec{v}_1)\, e^{3t} + 3\vec{v}_1 t e^{3t}.$$

As we are assuming that $\vec{x}_2$ is a solution, $\vec{x}_2{}'$ must equal $A\vec{x}_2$. So let's compute $A\vec{x}_2$:

$$A\vec{x}_2 = A(\vec{v}_2 + \vec{v}_1 t)\, e^{3t} = A\vec{v}_2 e^{3t} + A\vec{v}_1 t e^{3t}.$$

By looking at the coefficients of e^{3t} and te^{3t} we see $3\vec{v}_2 + \vec{v}_1 = A\vec{v}_2$ and $3\vec{v}_1 = A\vec{v}_1$. This means that

$$(A - 3I)\vec{v}_2 = \vec{v}_1, \qquad \text{and} \qquad (A - 3I)\vec{v}_1 = \vec{0}.$$

Therefore, $\vec{x}_2$ is a solution if these two equations are satisfied. The second equation is satisfied if $\vec{v}_1$ is an eigenvector, and we found the eigenvector above, so let $\vec{v}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. So, if we can find a $\vec{v}_2$ that solves $(A - 3I)\vec{v}_2 = \vec{v}_1$, then we are done. This is just a bunch of linear equations to solve and we are by now very good at that. Let us solve $(A - 3I)\vec{v}_2 = \vec{v}_1$. Write

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

By inspection we see that letting $a = 0$ (a could be anything in fact) and $b = 1$ does the job. Hence we can take $\vec{v}_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Our general solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \right) e^{3t} = \begin{bmatrix} c_1 e^{3t} + c_2 t e^{3t} \\ c_2 e^{3t} \end{bmatrix}.$$

Let us check that we really do have the solution. First $x_1' = c_1 3e^{3t} + c_2 e^{3t} + 3c_2 te^{3t} = 3x_1 + x_2$. Good. Now $x_2' = 3c_2 e^{3t} = 3x_2$. Good.

Below is a video on solving a defective system of differential equations.

Note that the system $\vec{x}' = A\vec{x}$ has a simpler solution since A is a so-called upper triangular matrix, that is every entry below the diagonal is zero. In particular, the equation for x_2 does not depend on x_1. Mind you, not every defective matrix is triangular.

? Exercise 3.7.1

Solve $\vec{x}' = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \vec{x}$ by first solving for x_2 and then for x_1 independently. Check that you got the same solution as we did above.

Let us describe the general algorithm. Suppose that λ is an eigenvalue of multiplicity 2, defect 1. First find an eigenvector $\vec{v_1}$ of λ. Then, find a vector $\vec{v_2}$ such that

$$(A - \lambda I)\vec{v_2} = \vec{v_1}$$

This gives us two linearly independent solutions

$$\begin{aligned} \vec{x_1} &= \vec{v_1} e^{\lambda t} \\ \vec{x_2} &= (\vec{v_2} + \vec{v_1} t)e^{\lambda t} \end{aligned} \tag{5.7.1}$$

✓ Example 5.7.2

Consider the system

$$\vec{x}' = \begin{bmatrix} 2 & -5 & 0 \\ 0 & 2 & 0 \\ -1 & 4 & 1 \end{bmatrix} \vec{x}.$$

Compute the eigenvalues,

Solution

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & -5 & 0 \\ 0 & 2-\lambda & 0 \\ -1 & 4 & 1-\lambda \end{bmatrix} \right) = (2-\lambda)^2(1-\lambda).$$

The eigenvalues are 1 and 2, where 2 has multiplicity 2. We leave it to the reader to find that $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 1$.

Let's focus on $\lambda = 2$. We compute eigenvectors:

$$\vec{0} = (A - 2I)\vec{v} = \begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ -1 & 4 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The first equation says that $v_2 = 0$, so the last equation is $-v_1 - v_3 = 0$. Let v_3 be the free variable to find that $v_1 = -v_3$. Perhaps let $v_3 = -1$ to find an eigenvector $\left[\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right]$. Problem is that setting v_3 to anything else just gets multiples of this vector and so we have a defect of 1. Let $\vec{v}_1$ be the eigenvector and let's look for a generalized eigenvector $\vec{v}_2$:

$$(A - 2I)\vec{v}_2 = \vec{v}_1,$$

or

$$\begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ -1 & 4 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

where we used a, b, c as components of $\vec{v}_2$ for simplicity. The first equation says $-5b = 1$ so $b = \frac{-1}{5}$. The second equation says nothing. The last equation is $-a + 4b - c = -1$, or $a + \frac{4}{5} + c = 1$, or $a + c = \frac{1}{5}$. We let c be the free variable and we choose $c = 0$. We find $\vec{v}_2 = \left[\begin{smallmatrix} \frac{1}{5} \\ \frac{-1}{5} \\ 0 \end{smallmatrix}\right]$.

The general solution is therefore,

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} \frac{1}{5} \\ \frac{-1}{5} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} t \right) e^{2t}.$$

This machinery can also be generalized to higher multiplicities and higher defects. We will not go over this method in detail, but let us just sketch the ideas. Suppose that A has an eigenvalue λ of multiplicity m. We find vectors such that

$$(A - \lambda I)^k \vec{(v)} = \vec{(0)}, \quad \text{but} \quad (A - \lambda I)^{k-1}\vec{v} \neq \vec{0}$$

Such vectors are called *generalized eigenvectors* (then $\overrightarrow{v_1} = (A - \lambda I)^{k-1}\vec{v}$ is an eigenvector). For every eigenvector $\vec{v}_1$ we find a chain of generalized eigenvectors $\vec{v}_2$ through $vecv_k$ such that:

$$\begin{aligned} (A - \lambda I)\overrightarrow{v_1} &= \vec{0}, \\ (A - \lambda I)\overrightarrow{v_2} &= \overrightarrow{v_1}, \\ &\vdots \\ (A - \lambda I)\overrightarrow{v_k} &= \overrightarrow{v_{k-1}}. \end{aligned} \tag{5.7.2}$$

Really once you find the $\vec{v}_k$ such that $(A - \lambda I)^k \vec{v}_k = \vec{0}$ but $(A - \lambda I)^{k-1}\vec{v}_k \neq \vec{0}$, you find the entire chain since you can compute the rest, $\vec{v}_{k-1} = (A - \lambda I)\vec{v}_k$, $\vec{v}_{k-2} = (A - \lambda I)\vec{v}_{k-1}$, etc. We form the linearly independent solutions

$$\begin{aligned} \overrightarrow{x_1} &= \overrightarrow{v_1} e^{\lambda t} \\ \overrightarrow{x_2} &= (\overrightarrow{v_2} + \overrightarrow{v_1} t) e^{\lambda t} \\ &\vdots \\ \overrightarrow{x_k} &= \left(\overrightarrow{v_k} + \vec{v}_{k-1} t + \vec{v}_{k-2} \frac{t^2}{2} + \cdots + \vec{v}_2 \frac{t^{k-2}}{(k-2)!} + \vec{v}_1 \frac{t^{k-1}}{(k-1)!}) e^{\lambda t} \right) \end{aligned} \tag{5.7.3}$$

Recall that $k! = 1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k$ is the factorial. If you have an eigenvalue of geometric multiplicity ℓ, you will have to find ℓ such chains (some of them might be short: just the single eigenvector equation). We go until we form m linearly independent solutions where m is the algebraic multiplicity. We don't quite know which specific eigenvectors go with which chain, so start by finding $\vec{v}_k$ first for the longest possible chain and go from there.

For example, if λ is an eigenvalue of A of algebraic multiplicity 3 and defect 2, then solve

$$(A-\lambda I)\vec{v}_1=\vec{0}, \qquad (A-\lambda I)\vec{v}_2=\vec{v}_1, \qquad (A-\lambda I)\vec{v}_3=\vec{v}_2.$$

That is, find $\vec{v}_3$ such that $(A-\lambda I)^3\vec{v}_3=\vec{0}$, but $(A-\lambda I)^2\vec{v}_3\neq\vec{0}$. Then you are done as $\vec{v}_2=(A-\lambda I)\vec{v}_3$ and $\vec{v}_1=(A-\lambda I)\vec{v}_2$. The 3 linearly independent solutions are

$$\vec{x}_1=\vec{v}_1e^{\lambda t}, \qquad \vec{x}_2=(\vec{v}_2+\vec{v}_1t)\,e^{\lambda t}, \qquad \vec{x}_3=\left(\vec{v}_3+\vec{v}_2t+\vec{v}_1\frac{t^2}{2}\right)e^{\lambda t}.$$

If on the other hand A has an eigenvalue λ of algebraic multiplicity 3 and defect 1, then solve

$$(A-\lambda I)\vec{v}_1=\vec{0}, \qquad (A-\lambda I)\vec{v}_2=\vec{0}, \qquad (A-\lambda I)\vec{v}_3=\vec{v}_2.$$

Here $\vec{v}_1$ and $\vec{v}_2$ are actual honest eigenvectors, and $\vec{v}_3$ is a generalized eigenvector. So there are two chains. To solve, first find a $\vec{v}_3$ such that $(A-\lambda I)^2\vec{v}_3=\vec{0}$, but $(A-\lambda I)\vec{v}_3\neq\vec{0}$. Then $\vec{v}_2=(A-\lambda I)\vec{v}_3$ is going to be an eigenvector. Then solve for an eigenvector $\vec{v}_1$ that is linearly independent from $\vec{v}_2$. You get 3 linearly independent solutions

$$\vec{x}_1=\vec{v}_1e^{\lambda t}, \qquad \vec{x}_2=\vec{v}_2e^{\lambda t}, \qquad \vec{x}_3=(\vec{v}_3+\vec{v}_2t)\,e^{\lambda t}.$$

5.8: Matrix exponentials

5.8.1: Definition

In this section we present a different way of finding the fundamental matrix solution of a system. Suppose that we have the constant coefficient equation

$$\vec{x}' = P\vec{x}$$

as usual. Now suppose that this was one equation (P is a number or a 1×1 matrix). Then the solution to this would be

$$\vec{x} = e^{Pt}.$$

That doesn't make sense if P is a larger matrix, but essentially the same computation that led to the above works for matrices when we define e^{Pt} properly. First let us write down the Taylor series for e^{at} for some number a.

$$e^{at} = 1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \frac{(at)^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}$$

Recall $k! = 1\cdot 2\cdot 3\cdots k$ is the factorial, and $0! = 1$. We differentiate this series term by term

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{at}\right) = a + a^2t + \frac{a^3t^2}{2} + \frac{a^4t^3}{6} + \cdots = a\left(1 + at\frac{(at)^2}{2} + \frac{(at)^3}{6} + \cdots\right) = ae^{at}.$$

Maybe we can try the same trick with matrices. Suppose that for an $n\times n$ matrix A we define the *matrix exponential* as

$$e^A \stackrel{\text{def}}{=} I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{k!}A^k + \cdots$$

Let us not worry about convergence. The series really does always converge. We usually write Pt as tP by convention when P is a matrix. With this small change and by the exact same calculation as above we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tP}\right) = Pe^{tP}.$$

Now P and hence e^{tP} is an $n\times n$ matrix. What we are looking for is a vector. We note that in the 1×1 case we would at this point multiply by an arbitrary constant to get the general solution. In the matrix case we multiply by a column vector $\vec{c}$.

Theorem 5.8.1

Let P be an $n\times n$ matrix. Then the general solution to $\vec{x}' = P\vec{x}$ is

$$\vec{x} = e^{tP}\vec{c},$$

where $\vec{c}$ is an arbitrary constant vector. In fact $\vec{x}(0) = \vec{c}$.

Let us check.

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tP}\vec{c}\right) = Pe^{tP}\vec{c} = P\vec{x}.$$

Hence e^{tP} is the *fundamental matrix solution* of the homogeneous system. If we find a way to compute the matrix exponential, we will have another method of solving constant coefficient homogeneous systems. It also makes it easy to solve for initial conditions. To solve $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{b}$ we take the solution

$$\vec{x} = e^{tA}\vec{b}$$

This equation follows because $e^{0A} = I$, so $\vec{x}(0) = e^{0A}\vec{b} = \vec{b}$.

We mention a drawback of matrix exponentials. In general $e^{A+B} \neq e^Ae^B$. The trouble is that matrices do not commute, that is, in general $AB \neq BA$. If you try to prove $e^{A+B} \neq e^Ae^B$ using the Taylor series, you will see why the lack of commutativity becomes a problem. However, it is still true that if $AB = BA$, that is, if A and B *commute*, then $e^{A+B} = e^A e^B$. We will find this fact useful. Let us restate this as a theorem to make a point.

Theorem 5.8.2

If $AB = BA$, then $e^{A+B} = e^A e^B$. Otherwise $e^{A+B} \neq e^A e^B$ in general.

5.8.2: Simple cases

In some instances it may work to just plug into the series definition. Suppose the matrix is diagonal. For example, $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then

$$D^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

and

$$\begin{aligned} e^D &= I + D + \frac{1}{2}D^2 + \frac{1}{6}D^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix} + \cdots = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix} \end{aligned} \tag{5.8.1}$$

So by this rationale we have that

$$e^I = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} \quad \text{and} \quad e^{aI} = \begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix}$$

This makes exponentials of certain other matrices easy to compute. Notice for example that the matrix $A = \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}$ can be written as $3I + B$ where $B = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$. Notice that $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So $B^k = 0$ for all $k \geq 2$. Therefore, $e^B = I + B$. Suppose we actually want to compute e^{tA}. The matrices $3tI$ and tB commute (exercise: check this) and $e^{tB} = I + tB$, since $(tB)^2 = t^2B^2 = 0$. We write

$$\begin{aligned} e^{tA} &= e^{3tI+tB} = e^{3tI}e^{tB} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}(I + tB) \\ &= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix} = \begin{bmatrix} (1+2t)\,e^{3t} & 4te^{3t} \\ -te^{3t} & (1-2t)\,e^{3t} \end{bmatrix} \end{aligned} \tag{5.8.2}$$

So we have found the fundamental matrix solution for the system $\vec{x}' = A\vec{x}$. Note that this matrix has a repeated eigenvalue with a defect; there is only one eigenvector for the eigenvalue 3. So we have found a perhaps easier way to handle this case. In fact, if a matrix A is 2×2 and has an eigenvalue λ of multiplicity 2, then either A is diagonal, or $A = \lambda I + B$ where $B^2 = 0$. This is a good exercise.

Exercise 5.8.1

Suppose that A is 2×2 and λ is the only eigenvalue. Then show that $(A - \lambda I)^2 = 0$. Then we can write $A = \lambda I + B$, where $B^2 = 0$. Hint: First write down what does it mean for the eigenvalue to be of multiplicity 2. You will get an equation for the entries. Now compute the square of B.

Matrices B such that $B^k = 0$ for some k are called *nilpotent*. Computation of the matrix exponential for nilpotent matrices is easy by just writing down the first k terms of the Taylor series.

5.8.3: General Matrices

In general, the exponential is not as easy to compute as above. We usually cannot write a matrix as a sum of commuting matrices where the exponential is simple for each one. But fear not, it is still not too difficult provided we can find enough eigenvectors.

First we need the following interesting result about matrix exponentials. For two square matrices A and B, with B *invertible*, we have

$$e^{BAB^{-1}} = Be^{A}B^{-1}.$$

This can be seen by writing down the Taylor series. First note that

$$\left(BAB^{-1}\right)^2 = BAB^{-1}BAB^{-1} = BAIAB^{-1} = BA^2B^{-1}$$

And hence by the same reasoning $\left(BAB^{-1}\right)^k = BA^kB^{-1}$. Now write down the Taylor series for $e^{BAB^{-1}}$.

$$\begin{aligned} e^{BAB^{-1}} &= I + BAB^{-1} + \frac{1}{2}\left(BAB^{-1}\right)^2 + \frac{1}{6}\left(BAB^{-1}\right)^3 + \cdots \\ &= BB^{-1} + BAB^{-1} + \frac{1}{2}BA^2B^{-1} + \frac{1}{6}BA^3B^{-1} + \cdots \\ &= B\left(I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots\right)B^{-1} \\ &= Be^{A}B^{-1}. \end{aligned} \tag{5.8.3}$$

Given a square matrix A, we can sometimes write $A = EDE^{-1}$, where D is diagonal and E invertible. This procedure is called *diagonalization*. If we can do that, the computation of the exponential becomes easy. Adding t into the mix we see that we can then easily compute the exponential

$$e^{tA} = Ee^{tD}E^{-1}.$$

To diagonalize A we will need n *linearly independent* eigenvectors of A. Otherwise this method of computing the exponential does not work and we need to be trickier, but we will not get into such details. We let E be the matrix with the eigenvectors as columns. Let $\lambda_1,\ \lambda_2, \cdots, \lambda_n$ be the eigenvalues and let $\vec{v}_1,\ \overrightarrow{v_2}, \cdots, \vec{v}_n$ be the eigenvectors, then $E = [\,\vec{v}_1\ \ \vec{v}_2\ \ \cdots\ \ \vec{v}_n\,]$. Let D be the diagonal matrix with the eigenvalues on the main diagonal. That is

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix}$$

We compute

$$\begin{aligned} AE &= A\,[\,\vec{v}_1\quad \vec{v}_2\quad \cdots\quad \vec{v}_n\,] \\ &= [\,A\vec{v}_1\quad A\vec{v}_2\quad \cdots\quad A\vec{v}_3\,] \\ &= [\,\lambda_1\vec{v}_1\quad \lambda_2\vec{v}_2\quad \cdots\quad \lambda_n\vec{v}_n\,] \\ &= [\,\vec{v}_1\quad \vec{v}_2\quad \cdots\quad \vec{v}_n\,]\,D \\ &= ED. \end{aligned} \tag{5.8.4}$$

The columns of E are linearly independent as these are linearly independent eigenvectors of A. Hence E is *invertible*. Since $AE = ED$, we right multiply by E^{-1} and we get

$$A = EDE^{-1}.$$

This means that . $e^{A} = Ee^{D}E^{-1}$ Multiplying the matrix by t we obtain

$$e^{tA} = Ee^{tD}E^{-1} = E\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}E^{-1} \tag{5.8.5}$$

The formula (5.8.5), therefore, gives the formula for computing the fundamental matrix solution e^{tA} for the system $\vec{x}\,' = A\vec{x}$, in the case where we have n linearly independent eigenvectors.

Notice that this computation still works when the eigenvalues and eigenvectors are complex, though then you will have to compute with complex numbers. It is clear from the definition that if A is real, then e^{tA} is real. So you will only need complex numbers in the computation and you may need to apply Euler's formula to simplify the result. If simplified properly the final matrix will not have any complex numbers in it.

Example 5.8.1

Compute the fundamental matrix solution using the matrix exponentials for the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then compute the particular solution for the initial conditions $x(0) = 4$ and $y(0) = 2$.

Let A be the coefficient matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. We first compute (exercise) that the eigenvalues are 3 and -1 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence the diagonalization of A is

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}_{E^{-1}}.$$

We write

$$\begin{aligned} e^{tA} = Ee^{tD}E^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{-1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} -e^{3t} - e^{-t} & -e^{3t} + e^{-t} \\ -e^{3t} + e^{-t} & -e^{3t} - e^{-t} \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{2} \\ \frac{e^{3t}-e^{-t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix}. \end{aligned} \tag{5.8.6}$$

The initial conditions are $x(0) = 4$ and $y(0) = 2$. Hence, by the property that $e^{0A} = I$ we find that the particular solution we are looking for is $e^{tA}\vec{b}$ where $\vec{b}$ is $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Then the particular solution we are looking for is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{2} \\ \frac{e^{3t}-e^{-t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2e^{3t} + 2e^{-t} + e^{3t} - e^{-t} \\ 2e^{3t} - 2e^{-t} + e^{3t} + e^{-t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} + e^{-t} \\ 3e^{3t} - e^{-t} \end{bmatrix}$$

Below is a video on using the matrix exponential to solve a differential equation.

5.8.4: Fundamental Matrix Solutions

We note that if you can compute the fundamental matrix solution in a different way, you can use this to find the matrix exponential e^{tA}. The fundamental matrix solution of a system of ODEs is not unique. The exponential is the fundamental matrix solution with the property that for $t = 0$ we get the identity matrix. So we must find the right fundamental matrix solution. Let X be any fundamental matrix solution to $\vec{x}' = A\vec{x}$. Then we claim

$$e^{tA} = X(t)[X(0)]^{-1}.$$

Clearly, if we plug $t = 0$ into $X(t)[X(0)]^{-1}$ we get the identity. We can multiply a fundamental matrix solution on the right by any constant invertible matrix and we still get a fundamental matrix solution. All we are doing is changing what the arbitrary constants are in the general solution $\vec{x}(t) = X(t)\vec{c}$.

5.8.5: Approximations

If you think about it, the computation of any fundamental matrix solution X using the eigenvalue method is just as difficult as the computation of e^{tA}. So perhaps we did not gain much by this new tool. However, the Taylor series expansion actually gives us a very easy way to approximate solutions, which the eigenvalue method did not.

The simplest thing we can do is to just compute the series up to a certain number of terms. There are better ways to approximate the exponential[1]. In many cases however, few terms of the Taylor series give a reasonable approximation for the exponential and may suffice for the application. For example, let us compute the first 4 terms of the series for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

$$\begin{aligned} e^{tA} \approx I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 = I + t\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + t^2\begin{bmatrix} \frac{5}{2} & 2 \\ 2 & \frac{5}{2} \end{bmatrix} + t^3\begin{bmatrix} \frac{13}{6} & \frac{7}{3} \\ \frac{7}{3} & \frac{13}{6} \end{bmatrix} = \\ = \begin{bmatrix} 1 + t + \frac{5}{2}t^2 + \frac{13}{6}t^3 & 2t + 2t^2 + \frac{7}{3}t^3 \\ 2t + 2t^2 + \frac{7}{3}t^3 & 1 + t + \frac{5}{2}t^2 + \frac{13}{6}t^3 \end{bmatrix}. \end{aligned}$$

Just like the scalar version of the Taylor series approximation, the approximation will be better for small t and worse for larger t. For larger t, we will generally have to compute more terms. Let us see how we stack up against the real solution with $t = 0.1$. The approximate solution is approximately (rounded to 8 decimal places)

$$e^{0.1A} \approx I + 0.1A + \frac{0.1^2}{2} + \frac{0.1^3}{6}A^3 = \begin{bmatrix} 1.12716667 & 0.22233333 \\ 0.22233333 & 1.12716667 \end{bmatrix}$$

And plugging $t = 0.1$ into the real solution (rounded to 8 decimal places) we get

$$e^{0.1A} = \begin{bmatrix} 1.12734811 & 0.22251069 \\ 0.22251069 & 1.12734811 \end{bmatrix}$$

Not bad at all! Although if we take the same approximation for $t = 1$ we get

$$I+A+\frac{1}{2}A^2+\frac{1}{6}A^3=\begin{bmatrix}6.66666667 & 6.33333333\\ 6.33333333 & 6.66666667\end{bmatrix}$$

while the real value is (again rounded to 8 decimal places)

$$e^A=\begin{bmatrix}10.22670818 & 9.85882874\\ 9.85882874 & 10.22670818\end{bmatrix}$$

So the approximation is not very good once we get up to $t=1$. To get a good approximation at $t=1$ (say up to 2 decimal places) we would need to go up to the 11^{th} power (exercise).

5.8.6: Footnotes

[1] C. Moler and C.F. Van Loan, Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later, SIAM Review 45 (1), 2003, 3–49

5.9: Nonhomogeneous systems

5.9.1: First Order Constant Coefficient

5.9.1.1: Integrating factor

Let us first focus on the nonhomogeneous first order equation

$$\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t),$$

where A is a constant matrix. The first method we will look at is the integrating factor method. For simplicity we rewrite the equation as

$$\vec{x}'(t) + P\vec{x}(t) = \vec{f}(t),$$

where $P = -A$. We multiply both sides of the equation by e^{tP} (being mindful that we are dealing with matrices that may not commute) to obtain

$$e^{tP}\vec{x}(t) + e^{tP}P\vec{x}(t) = e^{tP}\vec{f}(t).$$

We notice that $Pe^{tP} = e^{tP}P$. This fact follows by writing down the series definition of e^{tP},

$$\begin{aligned} Pe^{tP} &= P\left(I + I + tP + \frac{1}{2}(tP)^2 + \cdots\right) = P + tP^2 + \frac{1}{2}t^2P^3 + \cdots \\ &= \left(I + I + tP + \frac{1}{2}(tP)^2 + \cdots\right)P = Pe^{tP} \end{aligned} \tag{5.9.1}$$

We have already seen that $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{tP}\right) = Pe^{tP} = e^{tP}P$. The product rule says,

$$\frac{d}{dt}\left(e^{tP}\vec{x}(t)\right) = e^{tP}\vec{x}'(t) + e^{tP}P\vec{x}(t),$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tP}\vec{x}(t)\right) = e^{tP}\vec{f}(t).$$

We can now integrate. That is, we integrate each component of the vector separately

$$e^{tP}\vec{x}(t) = \int e^{tP}\vec{f}(t)dt + \vec{c}.$$

Recall from Exercise 3.8.6 that $\left(e^{tP}\right)^{-1} = e^{-tP}$. Therefore, we obtain

$$\vec{x}(t) = e^{-tP}\int e^{tP}\vec{f}(t)dt + e^{-tP}\vec{c}.$$

Perhaps it is better understood as a definite integral. In this case it will be easy to also solve for the initial conditions as well. Suppose we have the equation with initial conditions

$$\vec{x}'(t) + P\vec{x}(t) = \vec{f}(t), \quad \vec{x}(0) = \vec{b}.$$

The solution can then be written as

$$\vec{x}(t) = e^{-tP}\int_0^t e^{sP}\vec{f}(s)ds + e^{-tP}\vec{b}. \tag{5.9.2}$$

Again, the integration means that each component of the vector $e^{sP}\vec{f}(s)$ is integrated separately. It is not hard to see that (5.9.2) really does satisfy the initial condition $\vec{x}(0) = \vec{b}$

$$\vec{x}(0) = e^{-0P}\int_0^0 e^{sP}\vec{f}ds + e^{-0P}\vec{b} = I\vec{b} = \vec{b}.$$

Example 5.9.1

Suppose that we have the system

$$\begin{aligned} x_1' + 5x_1 - 3x_2 &= e^t, \\ x_2' + 3x_1 - x_2 &= 0, \end{aligned} \tag{5.9.3}$$

with initial conditions $x_1(0) = 1, x_2(0) = 0$.

Let us write the system as

$$\vec{x}' + \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We have previously computed e^{tP} for $P = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$. We immediately have e^{-tP}, simply by negating t.

$$e^{tP} = \begin{bmatrix} (1+3t)\, e^{2t} & -3te^{2t} \\ 3te^{2t} & (1-3t)\, e^{2t} \end{bmatrix}, \quad e^{-tP} = \begin{bmatrix} (1-3t)\, e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)\, e^{-2t} \end{bmatrix}.$$

Instead of computing the whole formula at once. Let us do it in stages. First

$$\begin{aligned} \int_0^t e^{sP} \vec{f}(s) ds &= \int_0^t \begin{bmatrix} (1+3s)\, e^{2s} & -3se^{2s} \\ 3se^{2s} & (1-3s)\, e^{2s} \end{bmatrix} \begin{bmatrix} e^s \\ 0 \end{bmatrix} ds \\ &= \int_0^t \begin{bmatrix} (1+3s)\, e^{3s} \\ 3se^{3s} \end{bmatrix} ds \\ &= \begin{bmatrix} \int_0^t (1+3s) e^{3s} ds \\ \int_0^t 3se^{3s} ds \end{bmatrix} \\ &= \begin{bmatrix} te^{3t} \\ \frac{(3t-1)e^{3t}+1}{3} \end{bmatrix}. \quad \text{(used integration by parts).} \end{aligned} \tag{5.9.4}$$

Then

$$\begin{aligned} \vec{x}(t) &= e^{-tP} \int_0^t e^{sP} \vec{f}(s)\, ds + e^{-tP} \vec{b} \\ &= \begin{bmatrix} (1-3t)\, e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)\, e^{-2t} \end{bmatrix} \begin{bmatrix} te^{3t} \\ \frac{(3t-1)\, e^{3t}+1}{3} \end{bmatrix} + \begin{bmatrix} (1-3t)\, e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)\, e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} te^{-2t} \\ -\frac{e^t}{3} + \left(\frac{1}{3} + t\right)\, e^{-2t} \end{bmatrix} + \begin{bmatrix} (1-3t)\, e^{-2t} \\ -3te^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} (1-2t)\, e^{-2t} \\ -\frac{e^t}{3} + \left(\frac{1}{3} - 2t\right)\, e^{-2t} \end{bmatrix}. \end{aligned} \tag{5.9.5}$$

Phew!

Let us confirm that this really works.

$$x_1' + 5x_1 - 3x_2 = \left(4te^{-2t} - 4e^{-2t}\right) + 5\,(1-2t)\, e^{-2t} + e^t - (1-6t)\, e^{-2t} = e^t.$$

Similarly (exercise) $x_2' + 3x_1 - x_2 = 0$. The initial conditions are also satisfied as well (exercise).

For systems, the integrating factor method only works if P does not depend on t, that is, P is constant. The problem is that in general

$$\frac{\mathrm{d}}{\mathrm{dt}}[e^{\int P(t) dt}] \neq P(t) e^{\int P(t) dt},$$

because matrix multiplication is not commutative.

5.9.1.2: Eigenvector Decomposition

For the next method, we note that eigenvectors of a matrix give the directions in which the matrix acts like a scalar. If we solve our system along these directions these solutions would be simpler as we can treat the matrix as a scalar. We can put those solutions together to get the general solution.

Take the equation

$$\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t) \tag{5.9.6}$$

Assume that A has n linearly independent eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$. Let us write

$$\vec{x}(t) = \overrightarrow{v_1}\xi_1(t) + \overrightarrow{v_2}\xi_2 t + \cdots + \overrightarrow{v_n}\xi_n(t) \tag{5.9.7}$$

That is, we wish to write our solution as a linear combination of eigenvectors of A. If we can solve for the scalar functions ξ_1 through ξ_n we have our solution $\vec{x}$. Let us decompose $\vec{f}$ in terms of the eigenvectors as well. We wish to write

$$\vec{f}(t) = \overrightarrow{v_1}g_1(t) + \overrightarrow{v_2}g_2 t + \cdots + \overrightarrow{v_n}g_n(t) \tag{5.9.8}$$

That is, we wish to find g_1 through g_n that satisfy (5.9.8). We note that since all the eigenvectors are independent, the matrix $E = \begin{bmatrix} \overrightarrow{v_1} & \overrightarrow{v_2} & \cdots & \overrightarrow{v_n} \end{bmatrix}$ is invertible. We see that (5.9.8) can be written as $\vec{f} = E\vec{g}$, where the components of $\vec{g}$ are the functions g_1 through g_n. Then $\vec{g} = E^{-1}\vec{f}$. Hence it is always possible to find $\vec{g}$ when there are n linearly independent eigenvectors.

We plug (5.9.7) into (5.9.6), and note that $A\vec{v}_k = \lambda_k \vec{v}_k$.

$$\begin{aligned} \overbrace{\vec{v}_1\xi_1' + \vec{v}_2\xi_2' + \cdots + \vec{v}_n\xi_n'}^{\vec{x}'} &= \overbrace{A\left(\vec{v}_1\xi_1 + \vec{v}_2\xi_2 + \cdots + \vec{v}_n\xi_n\right)}^{A\vec{x}} + \overbrace{\vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n}^{\vec{f}} \\ &= A\overrightarrow{v_1}\xi_1 + A\overrightarrow{v_2}\xi_2 + \cdots + A\overrightarrow{v_n}\xi_n + \overrightarrow{v_1}g_1 + \overrightarrow{v_2}g_2 + \cdots + \overrightarrow{v_n}g_n \\ &= \overrightarrow{v_1}\lambda_1\xi_1 + \overrightarrow{v_2}\lambda_2\xi_2 + \cdots + \overrightarrow{v_n}\lambda_n\xi_n + \overrightarrow{v_1}g_1 + \overrightarrow{v_2}g_2 + \cdots + \overrightarrow{v_n}g_n \\ &= \overrightarrow{v_1}\left(\lambda_1\xi_1 + g_1\right) + \overrightarrow{v_2}\left(\lambda_2\xi_2 + g_2\right) + \cdots + \overrightarrow{v_n}\left(\lambda_n\xi_n + g_n\right). \end{aligned} \tag{5.9.9}$$

If we identify the coefficients of the vectors $\vec{v}_1$ through $\vec{v}_n$ we get the equations

$$\begin{aligned} \xi_1' &= \lambda_1\xi_1 + g_1, \\ \xi_2' &= \lambda_2\xi_2 + g_2, \\ &\vdots \\ \xi_n' &= \lambda_n\xi_n + g_n. \end{aligned} \tag{5.9.10}$$

Each one of these equations is independent of the others. They are all linear first order equations and can easily be solved by the standard integrating factor method for single equations. That is, for example for the k^{th} equation we write

$$\xi_k'(t) - \lambda_k\xi_k(t) = g_k(t).$$

We use the integrating factor $e^{-\lambda_k t}$ to find that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\xi_k(t)e^{-\lambda_k t}\right] = e^{-\lambda_k t}g_k(t).$$

Now we integrate and solve for ξ_k to get

$$\xi_k(t) = e^{\lambda_k t}\int e^{-\lambda_k t}g_k(t)dt + C_k e^{\lambda_k t}.$$

If we are looking for just any particular solution, we can set C_k to be zero. If we leave these constants in, we get the general solution. Write $\vec{x}(t) = \vec{v}_1\xi_1(t) + \vec{v}_2\xi_2(t) + \cdots + \vec{v}_n\xi_n(t)$, and we are done.

Again, as always, it is perhaps better to write these integrals as definite integrals. Suppose that we have an initial condition $\vec{x}(0) = \vec{b}$. We take $\vec{c} = E^{-1}\vec{b}$ and note $\vec{b} = \overrightarrow{v_1}a_1 + \cdots + \overrightarrow{v_n}a_n$, just like before. Then if we write

$$\xi_k(t) = e^{\lambda_k(t)}\int_0^t e^{-\lambda_k s}g_k(s)dt + a_k e^{\lambda_k t},$$

we will actually get the particular solution $\vec{x}(t) = \vec{v}_1\xi_1 + \vec{x}_2\xi_2 + \cdots + \vec{v}_n\xi_n$ satisfying $\vec{x}(0) = \vec{b}$, because $\xi_k(0) = a_k$.

Let us remark that the technique we just outlined is the eigenvalue method applied to nonhomogeneous systems. If a system is homogeneous, that is, if $\vec{f} = \vec{0}$, then the equations we get are $\xi_k' = \lambda_k\xi_k$, and so $\xi_k = C_k e^{\lambda_k t}$ are the solutions and that's precisely what we got in Section 3.4.

Example 5.9.2

Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. Solve $\vec{x}' = A\vec{x} + \vec{f}$ where $\vec{f}(t) = \begin{bmatrix} 2e^t \\ 2t \end{bmatrix}$ for $\vec{x}(0) = \begin{bmatrix} \frac{3}{16} \\ \frac{-5}{16} \end{bmatrix}$.

The eigenvalues of A are -2 and 4 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively. This calculation is left as an exercise. We write down the matrix E of the eigenvectors and compute its inverse (using the inverse formula for 2×2 matrices)

$$E = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad E^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

We are looking for a solution of the form $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\xi_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\xi_2$. We also wish to write $\vec{f}$ in terms of the eigenvectors. That is we wish to write $\vec{f} = \begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} g_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} g_2$. Thus

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1}\begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \begin{bmatrix} e^t - t \\ e^t + t \end{bmatrix}.$$

So $g_1 = e^t - t$ and $g_2 = e^t + t$.

We further want to write $\vec{x}(0)$ in terms of the eigenvectors. That is, we wish to write $\vec{x}(0) = \begin{bmatrix} \frac{3}{16} \\ \frac{-5}{16} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_2$. Hence

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = E^{-1}\begin{bmatrix} \frac{3}{16} \\ \frac{-5}{16} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{-1}{16} \end{bmatrix}.$$

So $a_1 = \frac{1}{4}$ and $a_2 = \frac{-1}{16}$. We plug our $\vec{x}$ into the equation and get that

$$\begin{aligned} \overbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\xi_1' + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\xi_2'}^{\vec{x}'} &= \overbrace{A\begin{bmatrix} 1 \\ -1 \end{bmatrix}\xi_1 + A\begin{bmatrix} 1 \\ 1 \end{bmatrix}\xi_2}^{A\vec{x}} + \overbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix} g_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} g_2}^{\vec{f}} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}(-2\xi_1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 4\xi_2 + \begin{bmatrix} 1 \\ -1 \end{bmatrix}(e^t - t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}(e^t + t). \end{aligned} \tag{5.9.11}$$

We get the two equations

$$\begin{aligned} \xi_1' &= -2\xi_1 + e^t - t, \qquad && \text{where } \xi_1(0) = a_1 = \frac{1}{4}, \\ \xi_2' &= 4\xi_2 + e^t + t, && \text{where } \xi_2(0) = a_2 = \frac{-1}{16}. \end{aligned} \tag{5.9.12}$$

We solve with integrating factor. Computation of the integral is left as an exercise to the student. Note that we will need integration by parts.

$$\xi_1 = e^{-2t}\int e^{2t}\left(e^t - t\right)dt + C_1 e^{-2t} = \frac{e^t}{3} - \frac{t}{2} + \frac{1}{4} + C_1 e^{-2t}.$$

C_1 is the constant of integration. As $\xi_1(0) = \frac{1}{4}$, then $\frac{1}{4} = \frac{1}{3} + \frac{1}{4} + C_1$ and hence $C_1 = \frac{-1}{3}$. Similarly

$$\xi_2 = e^{4t} \int e^{-4t} \left(e^t + t\right) dt + C_2 e^{4t} = -\frac{e^t}{3} - \frac{t}{4} - \frac{1}{16} + C_2 e^{4t}.$$

As $\xi_2(0) = \frac{1}{16}$ we have that $\frac{-1}{16} = \frac{-1}{3} - \frac{1}{16} + C_2$ and hence $C_2 = \frac{1}{3}$. The solution is

$$\vec{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(\frac{e^t - e^{-2t}}{3} + \frac{1-2t}{4} \right) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{e^{4t} - e^t}{3} - \frac{4t+1}{16} \right) = \begin{bmatrix} \frac{e^{4t} - e^{-2t}}{3} + \frac{3-12t}{16} \\ \frac{e^{-2t} + e^{4t} + 2e^t}{3} + \frac{4t-5}{16} \end{bmatrix}.$$

That is, $x_1 = \frac{e^{4t} - e^{-2t}}{3} + \frac{3-12t}{16}$ and $x_2 = \frac{e^{-2t} + e^{4t} + 2e^t}{3} + \frac{4t-5}{16}$.

Exercise 5.9.1

Check that x_1 and x_2 solve the problem. Check both that they satisfy the differential equation and that they satisfy the initial conditions.

5.9.1.3: Undetermined Coefficients

We also have the method of undetermined coefficients for systems. The only difference here is that we will have to take unknown vectors rather than just numbers. Same caveats apply to undetermined coefficients for systems as for single equations. This method does not always work. Furthermore if the right hand side is complicated, we will have to solve for lots of variables. Each element of an unknown vector is an unknown number. So in system of 3 equations if we have say 4 unknown vectors (this would not be uncommon), then we already have 12 unknown numbers that we need to solve for. The method can turn into a lot of tedious work. As this method is essentially the same as it is for single equations, let us just do an example.

Example 5.9.3

Let $A = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$. Find a particular solution of $\vec{x}' = A\vec{x} + \vec{f}$ where $\vec{f} = \begin{bmatrix} e^t \\ t \end{bmatrix}$.

Note that we can solve this system in an easier way (can you see how?), but for the purposes of the example, let us use the eigenvalue method plus undetermined coefficients.

The eigenvalues of A are -1 and 1 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively. Hence our complementary solution is

$$\vec{x}_c = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t,$$

for some arbitrary constants α_1 and α_2.

We would want to guess a particular solution of

$$\vec{x} = \vec{a}e^t + \vec{b}t + \vec{c}.$$

However, something of the form $\vec{a}e^t$ appears in the complementary solution. Because we do not yet know if the vector $\vec{a}$ is a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we do not know if a conflict arises. It is possible that there is no conflict, but to be safe we should also try $\vec{b}te^t$. Here we find the crux of the difference for systems. We try both terms $\vec{a}e^t$ and $\vec{b}te^t$ in the solution, not just the term $\vec{b}te^t$. Therefore, we try

$$\vec{x} = \vec{a}e^t + \vec{b}te^t + \vec{c}t + \vec{d}.$$

Thus we have 8 unknowns. We write $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. We plug $\vec{x}$ into the equation. First let us compute $\vec{x}'$.

$$\vec{x}' = (\vec{a} + \vec{b})e^t + \vec{b}te^t + \vec{c} = \begin{bmatrix} \vec{a}_1 + \vec{b}_1 \\ \vec{a}_2 + \vec{b}_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Now $\vec{x}'$ must equal $A\vec{x} + \vec{f}$, which is

$$\begin{aligned} A\vec{x} + \vec{f} &= A\vec{a}e^t + A\vec{b}te^t + A\vec{c}t + A\vec{d} + \vec{f} \\ &= \begin{bmatrix} -\vec{a}_1 \\ -2\vec{a}_1 + \vec{a}_2 \end{bmatrix} e^t + \begin{bmatrix} -\vec{b}_1 \\ -2\vec{b}_1 + \vec{b}_2 \end{bmatrix} te^t + \begin{bmatrix} -\vec{c}_1 \\ -2\vec{c}_1 + \vec{c}_2 \end{bmatrix} t + \begin{bmatrix} -\vec{d}_1 \\ -2\vec{d}_1 + \vec{d}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t. \\ &= \begin{bmatrix} -a_1 + 1 \\ -2a_1 + a_2 \end{bmatrix} e^t + \begin{bmatrix} -b_1 \\ -2b_1 + b_2 \end{bmatrix} te^t + \begin{bmatrix} -c_1 \\ -2c_1 + c_2 + 1 \end{bmatrix} t + \begin{bmatrix} -d_1 \\ -2d_1 + d_2 \end{bmatrix}. \end{aligned} \tag{5.9.13}$$

We identify the coefficients of e^t, te^t, t and any constant vectors in $\vec{x}'$ and in $A\vec{x} + \vec{f}$ to find the equations:

$$\begin{aligned} a_1 + b_1 &= -a_1 + 1, & 0 &= -c_1, \\ a_2 + b_2 &= -2a_1 + a_2, & 0 &= -2c_1 + c_2 + 1, \\ b_1 &= -b_1, & c_1 &= -d_1, \\ b_2 &= -2b_1 + b_2, & c_2 &= -2d_1 + d_2. \end{aligned} \tag{5.9.14}$$

We could write the 8×9 augmented matrix and start row reduction, but it is easier to just solve the equations in an ad hoc manner. Immediately we see that $b_1 = 0, c_1 = 0, d_1 = 0$. Plugging these back in, we get that $c_2 = -1$ and $d_2 = -1$. The remaining equations that tell us something are

$$\begin{aligned} a_1 &= -a_1 + 1, \\ a_2 + b_2 &= -2a_1 + a_2. \end{aligned} \tag{5.9.15}$$

So $a_1 = \frac{1}{2}$ and $b_2 = -1$. Finally, a_2 can be arbitrary and still satisfy the equations. We are looking for just a single solution so presumably the simplest one is when $a_2 = 0$. Therefore,

$$\vec{x} = \vec{a}e^t + \vec{b}te^t + \vec{c}t + \vec{d} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^t \\ -te^t - t - 1 \end{bmatrix}.$$

That is, $x_1 = \frac{1}{2}e^t, x_2 = -te^t - t - 1$. We would add this to the complementary solution to get the general solution of the problem. Notice also that both $\vec{a}e^t$ and $\vec{b}te^t$ were really needed.

? Exercise 5.9.2

Check that x_1 and x_2 solve the problem. Also try setting $a_2 = 1$ and again check these solutions. What is the difference between the two solutions we obtained (one with $a_2 = 0$ and one with $a_2 = 1$)?

As you can see, other than the handling of conflicts, undetermined coefficients works exactly the same as it did for single equations. However, the computations can get out of hand pretty quickly for systems. The equation we had done was very simple.

5.9.2: First Order Variable Coefficient

5.9.2.1: Variation of Parameters

Just as for a single equation, there is the method of variation of parameters. In fact for constant coefficient systems, this is essentially the same thing as the integrating factor method we discussed earlier. However, this method will work for any linear system, even if it is not constant coefficient, provided we can somehow solve the associated homogeneous problem.

Suppose we have the equation

$$\vec{x}' = A(t)\vec{x} + \vec{f}(t). \tag{5.9.16}$$

Further, suppose we have solved the associated homogeneous equation $\vec{x}' = A(t)\vec{x}$ and found the fundamental matrix solution $X(t)$. The general solution to the associated homogeneous equation is $X(t)\vec{c}$ for a constant vector $\vec{c}$. Just like for variation of parameters for single equation we try the solution to the nonhomogeneous equation of the form

$$\vec{x}_p = X(t)\vec{u}(t),$$

where $\vec{u}(t)$ is a vector valued function instead of a constant. Now we substitute into (5.9.16) to obtain

$$\vec{x}_p{}'(t) = \underbrace{X'(t)\,\vec{u}(t) + X(t)\,\vec{u}'(t)}_{\vec{x}_p{}'(t)} = \underbrace{A(t)\,X(t)\,\vec{u}(t)}_{A(t)\vec{x}_p(t)} + \vec{f}(t).$$

But $X(t)$ is a fundamental matrix solution to the homogeneous problem. So $X'(t) = A(t)X(t)$, and

$$X'(t)\,\cancel{\vec{u}(t)} + X(t)\,\vec{u}'(t) = X'(t)\,\cancel{\vec{u}(t)} + \vec{f}(t).$$

Hence $X(t)\vec{u}'(t) = \vec{f}(t)$. If we compute $[X(t)]^{-1}$, then $\vec{u}'(t) = [X(t)]^{-1}\vec{f}(t)$. We integrate to obtain $\vec{u}$ and we have the particular solution $\vec{x}_p = X(t)\vec{u}(t)$. Let us write this as a formula

$$\vec{x}_p = X(t)\int [X(t)]^{-1}\vec{f}(t)dt.$$

Note that if A is constant and we let $X(t) = e^{tA}$, then $[X(t)]^{-1} = e^{-tA}$ and hence we get a solution $\vec{x}_p = e^{tA}\int e^{-tA}\vec{f}(t)dt$, which is precisely what we got using the integrating factor method.

✔ Example 5.9.4

Find a particular solution to

$$\vec{x}' = \frac{1}{t^2+1}\begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}\vec{x} + \begin{bmatrix} t \\ 1 \end{bmatrix}(t^2+1). \tag{5.9.17}$$

Here $A = \frac{1}{t^2+1}\begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}$ is most definitely not constant. Perhaps by a lucky guess, we find that $X = \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}$ solves $X'(t) = A(t)X(t)$. Once we know the complementary solution we can easily find a solution to (5.9.17). First we find

$$[X(t)]^{-1} = \frac{1}{t^2+1}\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}.$$

Next we know a particular solution to (5.9.17) is

$$\begin{aligned} \vec{x}_p &= X(t)\int [X(t)]^{-1}\vec{f}(t)dt \\ &= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}\int \frac{1}{t^2+1}\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\begin{bmatrix} t \\ 1 \end{bmatrix}(t^2+1)dt \\ &= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}\int \begin{bmatrix} 2t \\ -t^2+1 \end{bmatrix}dt \\ &= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}\begin{bmatrix} t^2 \\ -\frac{1}{3}t^3+t \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3}t^4 \\ \frac{2}{3}t^3+t \end{bmatrix}. \end{aligned} \tag{5.9.18}$$

Adding the complementary solution we have that the general solution to (5.9.17).

$$\vec{x} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}t^4 \\ \frac{2}{3}t^3+t \end{bmatrix} = \begin{bmatrix} c_1 - c_2t + \frac{1}{3}t^4 \\ c_2 + (c_1+1)t + \frac{2}{3}t^3 + \end{bmatrix}.$$

? Exercise 5.9.3:

Check that $x_1 = \frac{1}{3}t^4$ and $x_2 = \frac{2}{3}t^3 + t$ really solve (5.9.17).

In the variation of parameters, just like in the integrating factor method we can obtain the general solution by adding in constants of integration. That is, we will add $X(t)\vec{c}$ for a vector of arbitrary constants. But that is precisely the complementary solution.

5.9.3: Second Order Constant Coefficients

5.9.3.1: Undetermined Coefficients

We have already seen a simple example of the method of undetermined coefficients for second order systems in Section 3.6. This method is essentially the same as undetermined coefficients for first order systems. There are some simplifications that we can make, as we did in Section 3.6. Let the equation be

$$\vec{x}'' = A\vec{x} + \vec{f}(t),$$

where A is a constant matrix. If $\vec{F}(t)$ is of the form $\vec{F}_0 \cos(\omega t)$, then as two derivatives of cosine is again cosine we can try a solution of the form

$$\vec{x}_p = \vec{c}\cos(\omega t),$$

and we do not need to introduce sines.

If the $\vec{F}$ is a sum of cosines, note that we still have the superposition principle. If $\vec{F}(t) = \vec{F}_0 \cos(\omega_0 t) + \vec{F}_1 \cos(\omega_1 t)$, then we would try $\vec{a}\cos(\omega_0 t)$ for the problem $\vec{x}'' = A\vec{x} + \vec{F}_0 \cos(\omega_0 t)$, and we would try $\vec{b}\cos(\omega_1 t)$ for the problem $\vec{x}'' = A\vec{x} + \vec{F}_0 \cos(\omega_1 t)$. Then we sum the solutions.

However, if there is duplication with the complementary solution, or the equation is of the form $\vec{x}'' = A\vec{x}' + B\vec{x} + \vec{F}(t)$, then we need to do the same thing as we do for first order systems.

You will never go wrong with putting in more terms than needed into your guess. You will find that the extra coefficients will turn out to be zero. But it is useful to save some time and effort.

5.9.3.2: Eigenvector Decomposition

If we have the system

$$\vec{x}'' = A\vec{x} + \vec{F}(t,)$$

we can do *eigenvector decomposition*, just like for first order systems.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues and $\vec{v}_1, \ldots, \vec{v}_n$ be eigenvectors. Again form the matrix $E = [\vec{v}_1 \cdots \vec{v}_n]$. We write

$$\vec{x}(t) = \vec{v}_1\xi_1(t) + \vec{v}_2\xi_2(t) + \cdots + \vec{v}_n\xi_n(t).$$

We decompose $\vec{F}$ in terms of the eigenvectors

$$\vec{f}(t) = \vec{v}_1 g_1(t) + \vec{v}_2 g_2(t) + \cdots + \vec{v}_n g_n(t).$$

And again $\vec{g} = E^{-1}\vec{F}$.

Now we plug in and doing the same thing as before we obtain

$$\begin{aligned} \overbrace{\vec{v}_1\xi_1'' + \vec{v}_2\xi_2'' + \cdots \vec{v}_n\xi_n''}^{\vec{x}''} &= \overbrace{A(\vec{v}_1\xi_1 + \vec{v}_2\xi_2 + \cdots \vec{v}_n\xi_n)}^{A\vec{x}} + \overbrace{\vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n}^{\vec{f}} \\ &= A\vec{v}_1\xi_1 + A\vec{v}_2\xi_2 + \cdots A\vec{v}_n\xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\ &= \vec{v}_1\lambda_1\xi_1 + \vec{v}_2\lambda_2\xi_2 + \cdots \vec{v}_n\lambda_n\xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\ &= \vec{v}_1(\lambda_1\xi_1 + g_1) + \vec{v}_2(\lambda_2\xi_2 + g_2) + \cdots + \vec{v}_n(\lambda_n\xi_n + g_n). \end{aligned} \tag{5.9.19}$$

We identify the coefficients of the eigenvectors to get the equations

$$\begin{aligned} \xi_1'' &= \lambda_1\xi_1 + g_1, \\ \xi_2'' &= \lambda_2\xi_2 + g_2, \\ &\vdots \\ \xi_n'' &= \lambda_n\xi_n + g_n. \end{aligned} \tag{5.9.20}$$

Each one of these equations is independent of the others. We solve each equation using the methods of Chapter 2. We write $\vec{x}(t) = \vec{v}_1\xi_1(t) + \cdots + \vec{v}_n\xi_n(t)$, and we are done; we have a particular solution. If we have found the general solution for ξ_1

through ξ_2, then again $\vec{x}(t) = \vec{v}_1\xi_1(t) + \cdots + \vec{v}_n\xi_n(t)$ is the general solution (and not just a particular solution).

Example 5.9.5

Let us do the example from Section 3.6 using this method. The equation is

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\cos(3t).$$

The eigenvalues were -1 and -4, with eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore $E = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ and $E^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Therefore,

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1}\vec{F}(t) = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 2\cos(3t) \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\cos(3t) \\ \frac{-2}{3}\cos(3t) \end{bmatrix}.$$

So after the whole song and dance of plugging in, the equations we get are

$$\xi_1'' = -\xi_1 + \frac{2}{3}\cos(3t), \quad \xi_2'' = -4\xi_2 - \frac{2}{3}\cos(3t).$$

For each equation we use the method of undetermined coefficients. We try $C_1\cos(3t)$ for the first equation and $C_2\cos(3t)$ for the second equation. We plug in to get

$$\begin{aligned} -9C_1\cos(3t) &= -C_1\cos(3t) + \frac{2}{3}\cos(3t), \\ -9C_2\cos(3t) &= -4C_2\cos(3t) - \frac{2}{3}\cos(3t). \end{aligned} \tag{5.9.21}$$

We solve each of these equations separately. We get $-9C_1 = -C_1 + \frac{2}{3}$ and $-9C_2 = -4C_2 - \frac{2}{3}$. And hence $C_1 = \frac{-1}{12}$ and $C_2 = \frac{2}{12}$. So our particular solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\left(\frac{-1}{12}\cos(3t)\right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\left(\frac{2}{15}\cos(3t)\right) = \begin{bmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{bmatrix}\cos(3t).$$

This solution matches what we got previously in Section 3.6.

5.E: Systems of ODEs (Exercises)

These are homework exercises to accompany Libl's "Differential Equations for Engineering" Textmap. This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Prerequisite for the course is the basic calculus sequence.

5.E.1: 3.1 Introduction to Systems of ODEs

Exercise $5.E.3.1.1$

Find the general solution of $x_1' = x_2 - x_1 + t, x_2' = x_2$.

Exercise $5.E.3.1.2$

Find the general solution of $x_1' = 3x_1 - x_2 + e^t, x_2' = x_1$.

Exercise $5.E.3.1.3$

Write $ay'' + by' + cy = f(x)$ as a first order system of ODEs.

Exercise $5.E.3.1.4$

Write $x'' + y^2y' - x^3 = \sin(t), y'' + (x' + y')^2 - x = 0$ as a first order system of ODEs.

Exercise $5.E.3.1.5$

Suppose two masses on carts on frictionless surface are at displacements x_1 and x_2 as in Example 3.1.3. Suppose that a rocket applies force F in the positive direction on cart x_1. Set up the system of equations.

Example $5.E.3.1.6$

Suppose the tanks are as in Example 3.1.2, starting both at volume V, but now the rate of flow from tank 1 to tank 2 is r_1, and rate of flow from tank 2 to tank one is r_2. Notice that the volumes are now not constant. Set up the system of equations.

Exercise $5.E.3.1.7$

Find the general solution to $y_1' = 3y_1, y_2' = y_1 + y_2, y_3' = y_1 + y_3$.

Answer

$$y_1 = C_1 e^{3x}, \; y_2 = y(x) = C_2 e^x + \frac{C_1}{2} e^{3x}, \; y_3 = y(x) = C_3 e^x + \frac{C_1}{2} e^{3x}$$

Exercise $5.E.3.1.8$

Solve $y' = 2x, x' = x + y, x(0) = 1, y(0) = 3$.

Answer

$$x = \frac{5}{3} e^{2t} - \frac{2}{3} e^{-t}, \; y = \frac{5}{3} e^{2t} + \frac{4}{3} e^{-t}$$

Exercise $5.E.3.1.9$

Write $x''' = x + t$ as a first order system.

Answer

$$x_1' = x_2, \; x_2' = x_3, \; x_3' = x_1 + t$$

Exercise $5.E.3.1.10$

Write $y_1'' + y_1 + y_2 = t y_2'' + y_1 - y_2 = t^2$ as a first order system.

Answer

$y_3' + y_1 + y_2 = t$, $y_4' + y_1 - y_2 = t^2$, $y_1' = y_3$, $y_2' = y_4$

Exercise $5.E.3.1.11$

Suppose two masses on carts on frictionless surface are at displacements x_1 and x_2 as in Example 3.1.3. Suppose initial displacement is $x_1(0) = x_2(0) = 0$, and initial velocity is $x_1'(0) = x_2'(0) = a$ for some number a. Use your intuition to solve the system, explain your reasoning.

Answer

$x_1 = x_2 = at$. Explanation of the intuition is left to reader.

Exercise $5.E.3.1.12$

Suppose the tanks are as in Example 3.1.2 except that clean water flows in at the rate s liters per second into tank 1, and brine flows out of tank 2 and into the sewer also at the rate of s liters per second.

a. Draw the picture.
b. Set up the system of equations.
c. Intuitively, what happens as t goes to infinity, explain.

Answer

a. Left to reader
b. $x_1' = \frac{r}{V}(x_2 - x_1)$, $x_2' = \frac{r}{V} x_1 - \frac{r-s}{V} x_2$.
c. As t goes to infinity, both x_1 and x_2 go to zero, explanation is left to reader.

5.E.2: 3.2: Matrices and linear systems

Exercise $5.E.3.2.1$

Solve $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ by using matrix inverse.

Exercise $5.E.3.2.2$

Compute determinant of $\begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix}$.

Exercise $5.E.3.2.3$

Compute determinant of $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 0 & 5 & 0 \\ 6 & 0 & 7 & 0 \\ 8 & 0 & 10 & 1 \end{bmatrix}$. Hint: Expand along the proper row or column to make the calculations simpler.

Exercise $5.E.3.2.4$

Compute inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Exercise $5.E.3.2.5$

For which h is $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & h \end{bmatrix}$ not invertible? Is there only one such h? Are there several? Infinitely many?

Exercise $5.E.3.2.6$

For which h is $\begin{bmatrix} h & 1 & 1 \\ 0 & h & 0 \\ 1 & 1 & h \end{bmatrix}$ not invertible? Find all such h.

Exercise $5.E.3.2.7$

Solve $\begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Exercise $5.E.3.2.8$

Solve $\begin{bmatrix} 5 & 3 & 7 \\ 8 & 4 & 4 \\ 6 & 3 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

Exercise $5.E.3.2.9$

Solve $\begin{bmatrix} 3 & 2 & 3 & 0 \\ 3 & 3 & 3 & 3 \\ 0 & 2 & 4 & 2 \\ 2 & 3 & 4 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$.

Exercise $5.E.3.2.10$

Find 3 nonzero 2×2 matrices A, B, and C such that $AB = AC$ but $B \neq C$.

Exercise $5.E.3.2.11$

Compute determinant of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -5 \\ 1 & -1 & 0 \end{bmatrix}$

Answer

-15

Exercise $5.E.3.2.12$

Find t such that $\begin{bmatrix} 1 & t \\ -1 & 2 \end{bmatrix}$ is not invertible.

Answer

-2

? Exercise $5.E.3.2.12$

Solve $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$.

Answer

$\vec{x} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$

? Exercise $5.E.3.2.12$

Suppose a, b, c are nonzero numbers. Let $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $N = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

a. Compute M^{-1}.
b. Compute N^{-1}.

Answer

1. $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$
2. $\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

5.E.3: 3.3: Linear systems of ODEs

? Exercise $5.E.3.3.1$

Write the system $x_1' = 2x_1 - 3tx_2 + \sin t$ and $x_2' = e^t x_1 - 3x_2 + \cos t$ in the form $\vec{x}' = P(t)\vec{x} + \vec{f}(t)$.

? Exercise $5.E.3.3.2$

a. Verify that the system $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \vec{x}$ has the two solutions $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$.
b. Write down the general solution.
c. Write down the general solution in the form $x_1 = ?, x_2 = ?$ (i.e. write down a formula for each element of the solution).

? Exercise $5.E.3.3.3$

Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$ are linearly independent. Hint: Just plug in $t = 0$.

? Exercise $5.E.3.3.4$

Verify that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^t$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^t$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$ are linearly independent. Hint: You must be a bit more tricky than in the previous exercise.

? Exercise $5.E.3.3.5$

Verify that $\begin{bmatrix} t \\ t^2 \end{bmatrix}$ and $\begin{bmatrix} t^3 \\ t^4 \end{bmatrix}$ are linearly independent.

Exercise $5.E.3.3.6$

Take the system $x_1' + x_2' = x_1$, $x_1' - x_2' = x_2$.

a. Write it in the form $A\vec{x}' = B\vec{x}$ for matrices A and B.
b. Compute A^{-1} and use that to write the system in the form $\vec{x}' = P\vec{x}$.

Exercise $5.E.3.3.7$

Are $\begin{bmatrix} e^{2t} \\ e^t \end{bmatrix}$ and $\begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}$ linearly independent? Justify.

Answer

Yes.

Exercise $5.E.3.3.8$

Are $\begin{bmatrix} cosh(t) \\ 1 \end{bmatrix}$, $\begin{bmatrix} e^t \\ 1 \end{bmatrix}$ and $\begin{bmatrix} e^{-t} \\ 1 \end{bmatrix}$ linearly independent? Justify.

Answer

No. $2\,[\,\cosh(t) 1\,] - \begin{bmatrix} e^t \\ 1 \end{bmatrix} - \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix} = \vec{0}$

Exercise $5.E.3.3.9$

Write $x' = 3x - y + e^t$ and $y' = tx$ in matrix notation.

Answer

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ t & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

Exercise $5.E.3.3.10$

a. Write $x_1' = 2t\,x_2$ and $x_2' = 2t\,x_2$ in matrix notation.
b. Solve and write the solution in matrix notation.Add exercises text here.

Answer

a. $\vec{x}' = \begin{bmatrix} 0 & 2t \\ 0 & 2t \end{bmatrix} \vec{x}$
b. $\vec{x} = \begin{bmatrix} C_2 e^{t^2} + C_1 \\ C_2 e^{t^2} \end{bmatrix}$

5.E.4: 3.4: Eigenvalue Method

Exercise $5.E.3.4.1$: (easy)

Let A be a 3×3 matrix with an eigenvalue of 3 and a corresponding eigenvector $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. Find $A\vec{v}$.

Exercise $5.E.3.4.2$

a. Find the general solution of $x_1' = 2x_1$, $x_2' = 3x_2$ using the eigenvalue method (first write the system in the form $\vec{x}' = A\vec{x}$).
b. Solve the system by solving each equation separately and verify you get the same general solution.

Exercise 5.E.3.4.3

Find the general solution of $x_1' = 3x_1 + x_2, x_2' = 2x_1 + 4x_2$ using the eigenvalue method.

Exercise 5.E.3.4.4

Find the general solution of $x_1' = x_1 - 2x_2, x_2' = 2x_1x_2$ using the eigenvalue method. Do not use complex exponentials in your solution.

Exercise 5.E.3.4.5

a. Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix}$.

b. Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise 5.E.3.4.6

Compute eigenvalues and eigenvectors of $\begin{bmatrix} -2 & -1 & -1 \\ 3 & 2 & 1 \\ -3 & -1 & 0 \end{bmatrix}$.

Exercise 5.E.3.4.7

Let a, b, c, d, e, f be numbers. Find the eigenvalues of $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

Exercise 5.E.3.4.8

a. Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}$.

b. Solve the system $\vec{x}' = A\vec{x}$.

Answer

a. Eigenvalues: $4,\ 0,\ -1$ Eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$

b. $\vec{x} = C_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{4t} + C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$

Exercise 5.E.3.4.9

a. Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$.

b. Solve the system $\vec{x}' = A\vec{x}$.

Answer

a. Eigenvalues: $\frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$, Eigenvectors: $\begin{bmatrix} -2 \\ 1-\sqrt{3}i \end{bmatrix}, \begin{bmatrix} -2 \\ 1+\sqrt{3}i \end{bmatrix}$

b. $\vec{x} = C_1 e^{t/2} \begin{bmatrix} -2\cos\left(\frac{\sqrt{3}t}{2}\right) \\ \cos\left(\frac{\sqrt{3}t}{2}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}t}{2}\right) \end{bmatrix} + C_2 e^{t/2} \begin{bmatrix} -2\sin\left(\frac{\sqrt{3}t}{2}\right) \\ \sin\left(\frac{\sqrt{3}t}{2}\right) - \sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right) \end{bmatrix}$

Exercise $5.E.3.4.10$

Solve $x_1' = x_2, x_2' = x_1$ using the eigenvalue method.

Answer

$$\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Exercise $5.E.3.4.11$

Solve $x_1' = x_2, x_2' = -x_1$ using the eigenvalue method.

Answer

$$\vec{x} = C_1 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

5.E.5: 3.5: Two dimensional systems and their vector fields

Exercise $5.E.3.5.1$

Take the equation $mx'' + cx' + kx = 0$, with $m > 0, c \geq 0, k > 0$ for the mass-spring system.

a. Convert this to a system of first order equations.
b. Classify for what m, c, k do you get which behavior.
c. Can you explain from physical intuition why you do not get all the different kinds of behavior here?

Exercise $5.E.3.5.2$

Can you find what happens in the case when $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$? In this case the eigenvalue is repeated and there is only one eigenvector. What picture does this look like?

Exercise $5.E.3.5.3$

Can you find what happens in the case when $P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$? Does this look like any of the pictures we have drawn?

Exercise $5.E.3.5.4$

Which behaviors are possible if P is diagonal, that is $P = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$? You can assume that a and b are not zero.

Exercise $5.E.3.5.5$

Take the system from Example 3.1.2, $x_1' = \frac{r}{V}(x_2 - x_1)$, $x_2' = \frac{r}{V}(x_1 - x_2)$. As we said, one of the eigenvalues is zero. What is the other eigenvalue, how does the picture look like and what happens when t goes to infinity.

Exercise $5.E.3.5.6$

Describe the behavior of the following systems without solving:

a. $x' = x + y, \quad y' = x - y$
b. $x_1' = x_1 + x_2, \quad x_2' = 2x_2$
c. $x_1' = -2x_2, \quad x_2' = 2x_1$
d. $x' = x + 3y, \quad y' = -2x - 4y$
e. $x' = x - 4y, \quad y' = -4x + y$

Answer

a. Two eigenvalues: $\pm\sqrt{2}$ so the behavior is a saddle.
b. Two eigenvalues: 1 and 2, so the behavior is a source.
c. Two eigenvalues: $\pm 2i$, so the behavior is a center (ellipses).
d. Two eigenvalues: -1 and -2, so the behavior is a sink.
e. Two eigenvalues: 5 and -3, so the behavior is a saddle.

Exercise $5.E.3.5.7$

Suppose that $\vec{x} = A\vec{x}$ where A is a 2×2 matrix with eigenvalues $2\pm i$. Describe the behavior.

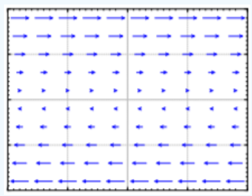
Answer

Spiral source.

Exercise $5.E.3.5.8$

Take $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Draw the vector field and describe the behavior. Is it one of the behaviours that we have seen before?

Answer



The solution does not move anywhere if $y = 0$. When y is positive, the solution moves (with constant speed) in the positive x direction. When y is negative, the solution moves (with constant speed) in the negative x direction. It is not one of the behaviors we saw. Note that the matrix has a double eigenvalue 0 and the general solution is $x = C_1 t + C_2$ and $y = C_1$, which agrees with the description

5.E.6: 3.6: Second order systems and applications

Exercise $5.E.3.6.1$

Find a particular solution to

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(2t). \tag{5.E.1}$$

Exercise $5.E.3.6.2$: challenging

Let us take the example in Figure 3.6.3 with the same parameters as before: $m_1 = 2, k_1 = 4,$ and $k_2 = 2,$ except for m_2, which is unknown. Suppose that there is a force $\cos(5t)$ acting on the first mass. Find an m_2 such that there exists a particular solution where the first mass does not move.

Note

This idea is called dynamic damping. In practice there will be a small amount of damping and so any transient solution will disappear and after long enough time, the first mass will always come to a stop.

Example $5.E.3.6.3$

Let us take the Example 3.6.2, but that at time of impact, cart 2 is moving to the left at the speed of 3 $\frac{\text{m}}{\text{s}}$.

a. Find the behavior of the system after linkup.
b. Will the second car hit the wall, or will it be moving away from the wall as time goes on?

c. At what speed would the first car have to be traveling for the system to essentially stay in place after linkup?

Exercise $5.E.3.6.4$

Let us take the example in Figure 3.6.2 with parameters $m_1 = m_2 = 1, k_1 = k_2 = 1$. Does there exist a set of initial conditions for which the first cart moves but the second cart does not? If so, find those conditions. If not, argue why not.

Exercise $5.E.3.6.5$

Find the general solution to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -3 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & 6 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} \cos(2t) \\ 0 \\ 0 \end{bmatrix}. \tag{5.E.2}$$

Answer

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \left(a_1 \cos(\sqrt{3}t) + b_1 \sin(\sqrt{3}t)\right) + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \left(a_2 \cos(\sqrt{2}t) + b_2 \sin(\sqrt{2}t)\right) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(a_3 \cos(t) + b_3 \sin(t)\right) + \begin{bmatrix} -1 \\ 1/2 \\ 2/3 \end{bmatrix} \cos(2t)$$

Exercise $5.E.3.6.6$

Suppose there are three carts of equal mass m and connected by two springs of constant k (and no connections to walls). Set up the system and find its general solution.

Answer

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \vec{x}'' = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \vec{x}\,.$$

Solution:

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \left(a_1 \cos(\sqrt{3k/m}t) + b_1 \sin(\sqrt{3k/m}t)\right) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \left(a_2 \cos(\sqrt{k/m}t) + b_2 \sin(\sqrt{k/m}t)\right) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (a_3 t + b_3) \quad .$$

Exercise $5.E.3.6.7$

Suppose a cart of mass 2 kg is attached by a spring of constant $k = 1$ to a cart of mass 3 kg, which is attached to the wall by a spring also of constant $k = 1$. Suppose that the initial position of the first cart is 1 meter in the positive direction from the rest position, and the second mass starts at the rest position. The masses are not moving and are let go. Find the position of the second mass as a function of time.

Answer

$$x_2 = \left(\frac{2}{5}\right) \cos\left(\sqrt{\frac{1}{6}}t\right) - \left(\frac{2}{5}\right) \cos(t)$$

5.E.7: 3.7: Multiple Eigenvalues

Exercise $5.E.3.7.1$

Let $A = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$. Find the general solution of $\vec{x}' = A\vec{x}$.

? Exercise $5.E.3.7.2$

Let $A = \begin{bmatrix} 5 & -4 & 4 \\ 0 & 3 & 0 \\ -2 & 4 & -1 \end{bmatrix}$.

a. What are the eigenvalues?
b. What is/are the defect(s) of the eigenvalue(s)?
c. Find the general solution of $\vec{x}' = A\vec{x}$.

? Exercise $5.E.3.7.3$

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

a. What are the eigenvalues?
b. What is/are the defect(s) of the eigenvalue(s)?
c. Find the general solution of $\vec{x}' = A\vec{x}$ in two different ways and verify you get the same answer.

? Exercise $5.E.3.7.4$

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -2 & -2 \\ -4 & 4 & 7 \end{bmatrix}$.

a. What are the eigenvalues?
b. What is/are the defect(s) of the eigenvalue(s)?
c. Find the general solution of $\vec{x}' = A\vec{x}$.

? Exercise $5.E.3.7.5$

Let $A = \begin{bmatrix} 0 & 4 & -2 \\ -1 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

a. What are the eigenvalues?
b. What is/are the defect(s) of the eigenvalue(s)?
c. Find the general solution of $\vec{x}' = A\vec{x}$.

? Exercise $5.E.3.7.6$

Let $\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix}$.

a. What are the eigenvalues?
b. What is/are the defect(s) of the eigenvalue(s)?
c. Find the general solution of $\vec{x}' = A\vec{x}$.

? Exercise $5.E.3.7.7$

Suppose that A is a 2×2 matrix with a repeated eigenvalue λ. Suppose that there are two linearly independent eigenvectors. Show that $A = \lambda I$.

? Exercise $5.E.3.7.8$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

a. What are the eigenvalues?
b. What is/are the defect(s) of the eigenvalue(s)?
c. Find the general solution of $\vec{x}' = A\vec{x}$.

Answer

a. $3, 0, 0$
b. No defects.
c. $\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

? Exercise $5.E.3.7.9$

Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$.

a. What are the eigenvalues?
b. What is/are the defect(s) of the eigenvalue(s)?
c. Find the general solution of $\vec{x}' = A\vec{x}$.

Answer

a. $1, 1, 2$
b. Eigenvalue 1 has a defect of 1
c. $\vec{x} = C_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{t} + C_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) e^{t} + C_3 \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} e^{2t}$

? Exercise $5.E.3.7.10$

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 9 \\ 0 & -1 & 5 \end{bmatrix}$.

a. What are the eigenvalues?
b. What is/are the defect(s) of the eigenvalue(s)?
c. Find the general solution of $\vec{x}' = A\vec{x}$.

Answer

a. $2, 2, 2$
b. Eigenvalue 2 has a defect of 2
c. $\vec{x} = C_1 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} e^{2t} + C_2 \left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right) e^{2t} + C_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right) e^{2t}$

? Exercise $5.E.3.7.11$

Let $A = \begin{bmatrix} a & a \\ b & c \end{bmatrix}$, where a, b, and c are unknowns. Suppose that 5 is a doubled eigenvalue of defect 1, and suppose that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the eigenvector. Find A and show that there is only one solution.

Answer

$$A = \begin{bmatrix} 5 & 5 \\ 0 & 5 \end{bmatrix}$$

5.E.8: 3.8: Matrix Exponentials

Exercise $5.E.3.8.1$

Using the matrix exponential, find a fundamental matrix solution for the system , $x' = 3x + y, y' = x + 3y$.

Exercise $5.E.3.8.2$

Find e^{tA} for the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

Exercise $5.E.3.8.3$

Find a fundamental matrix solution for the system , $x_1' = 7x_1 + 4x_2 + 12x_3, \quad x_2' = x_1 + 2x_2 + x_3, \quad x_3' = -3x_1 - 2x_2 - 5x_3$. Then find the solution that satisfies $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

Exercise $5.E.3.8.4$

Compute the matrix exponential e^A for $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$.

Exercise $5.E.3.8.5$: (challenging)

Suppose $AB = BA$. Show that under this assumption, $e^{A+B} = e^A e^B$.

Exercise $5.E.3.8.6$

Use Exercise $5.E.3.8.5$to show that $(e^A)^{-1} = e^{-A}$. In particular this means that e^A is invertible even if A is not.

Exercise $5.E.3.8.7$

Suppose A is a matrix with eigenvalues -1, 1, and corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

a. Find matrix A with these properties.
b. Find the fundamental matrix solution to $\vec{x}' = A\vec{x}$.
c. Solve the system in with initial conditions $\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Exercise $5.E.3.8.8$

Suppose that A is an $n \times n$ matrix with a repeated eigenvalue λ of multiplicity n. Suppose that there are n linearly independent eigenvectors. Show that the matrix is diagonal, in particular $A = \lambda I$. Hint: Use diagonalization and the fact that the identity matrix commutes with every other matrix.

Exercise $5.E.3.8.9$

Let $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$.

a. Find e^{tA}.
b. Solve $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Exercise 5.E.3.8.10

Let $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$. Approximate e^{tA} by expanding the power series up to the third order.

Exercise 5.E.3.8.11

For any positive integer n, find a formula (or a recipe) for A^n for the following matrices:

a. $\begin{bmatrix}3&0\\0&9\end{bmatrix}$

b. $\begin{bmatrix}5&2\\4&7\end{bmatrix}$

c. $\begin{bmatrix}0&1\\0&0\end{bmatrix}$

d. $\begin{bmatrix}2&1\\0&2\end{bmatrix}$

Exercise 5.E.3.8.12

Compute e^{tA} where $A=\begin{bmatrix}1&-2\\-2&1\end{bmatrix}$.

Answer

$$e^{tA}=\begin{bmatrix}\frac{e^{3t}+e^{-t}}{2}&\frac{e^{-t}-e^{3t}}{2}\\\frac{e^{-t}-e^{3t}}{2}&\frac{e^{3t}+e^{-t}}{2}\end{bmatrix}$$

Exercise 5.E.3.8.13

Compute e^{tA} where $A=\begin{bmatrix}1&-3&2\\-2&1&2\\-1&-3&4\end{bmatrix}$.

Answer

$$e^{tA}=\begin{bmatrix}2e^{3t}-4e^{2t}+3e^{t}&\frac{3e^{t}}{2}-\frac{3e^{3t}}{2}&-e^{3t}+4e^{2t}-3e^{t}\\2e^{t}-2e^{2t}&e^{t}&2e^{2t}-2e^{t}\\2e^{3t}-5e^{2t}+3e^{t}&\frac{3e^{t}}{2}-\frac{3e^{3t}}{2}&-e^{3t}+5e^{2t}-3e^{t}\end{bmatrix}$$

Exercise 5.E.3.8.14

a. Compute e^{tA} where $A=\begin{bmatrix}3&-1\\1&1\end{bmatrix}$.

b. Solve $\vec{x}'=A\vec{x}$ for $\vec{x}(0)=\begin{bmatrix}1\\2\end{bmatrix}$.

Answer

a. $e^{tA}=\begin{bmatrix}(t+1)e^{2t}&-te^{2t}\\te^{2t}&(1-t)e^{2t}\end{bmatrix}$

b. $\vec{x}=\begin{bmatrix}(1-t)e^{2t}\\(2-t)e^{2t}\end{bmatrix}$

Exercise $5.E.3.8.15$

Compute the first 3 terms (up to the second degree) of the Taylor expansion of e^{tA} where $A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$ (Write as a single matrix). Then use it to approximate $e^{0.1A}$.

Answer

$$\begin{bmatrix} 1+2t+5t^2 & 3t+6t^2 \\ 2t+4t^2 & 1+2t+5t^2 \end{bmatrix} \quad e^{0.1A} \approx \begin{bmatrix} 1.25 & 0.36 \\ 0.24 & 1.25 \end{bmatrix}$$

Exercise $5.E.3.8.16$

For any positive integer n, find a formula (or a recipe) for A^n for the following matrices:

a. $\begin{bmatrix} 7 & 4 \\ -5 & -2 \end{bmatrix}$
b. $\begin{bmatrix} -3 & 4 \\ -6 & -7 \end{bmatrix}$
c. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Answer

a. $\begin{bmatrix} 5(3^n)-2^{n+2} & 4(3^n)-2^{n+2} \\ 5(2^n)-5(3^n) & 5(2^n)-4(3^n) \end{bmatrix}$
b. $\begin{bmatrix} 3-2(3^n) & 2(3^n)-2 \\ 3-3^{n+1} & 3^{n+1}-2 \end{bmatrix}$
c. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if n is even, and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ if n is odd.

5.E.9: 3.9: Nonhomogeneous Systems

Exercise $5.E.3.9.1$

Find a particular solution to $x' = x + 2y + 2t, \ y' = 3x + 2y - 4$,

a. using integrating factor method,
b. using eigenvector decomposition,
c. using undetermined coefficients.

Exercise $5.E.3.9.2$

Find the general solution to $x' = 4x + y - 1, \ y' = x + 4y - e^t$,

a. using integrating factor method,
b. using eigenvector decomposition,
c. using undetermined coefficients.

Exercise $5.E.3.9.3$

Find the general solution to $x_1'' = -6x_1 + 3x_2 + \cos(t), \ x_2'' = 2x_1 - 7x_2 + 3\cos(t)$,

a. using eigenvector decomposition,
b. using undetermined coefficients.

Exercise $5.E.3.9.4$

Find the general solution to $x_1'' = -6x_1 + 3x_2 + \cos(2t), \ x_2'' = 2x_1 - 7x_2 + \cos(2t)$,

a. using eigenvector decomposition,
b. using undetermined coefficients.

? Exercise $5.E.3.9.5$

Take the equation

$$\vec{x}' = \begin{bmatrix} \frac{1}{t} & -1 \\ 1 & \frac{1}{t} \end{bmatrix} \vec{x} + \begin{bmatrix} t^2 \\ -t \end{bmatrix}. \tag{5.E.3}$$

a. Check that

$$\vec{x}_c = c_1 \begin{bmatrix} t \sin t \\ -t \cos t \end{bmatrix} + c_2 \begin{bmatrix} t \cos t \\ t \sin t \end{bmatrix} \tag{5.E.4}$$

is the complementary solution.

b. Use variation of parameters to find a particular solution.

? Exercise $5.E.3.9.6$

Find a particular solution to $x' = 5x + 4y + t,\ y' = x + 8y - t$,

a. using integrating factor method,
b. using eigenvector decomposition,
c. using undetermined coefficients.

Answer

The general solution is (particular solutions should agree with one of these): $x(t) = C_1 e^{9t} + 4C_2 e^{4t} - \frac{t}{3} - \frac{5}{54}$, $y(t) = C_1 e^{9t} - C_2 e^{4t} + \frac{t}{6} + \frac{7}{216}$

? Exercise $5.E.3.9.7$

Find a particular solution to $x' = y + e^t,\ y' = x + e^t$,

a. using integrating factor method,
b. using eigenvector decomposition,
c. using undetermined coefficients.

Answer

The general solution is (particular solutions should agree with one of these): $x(t) = C_1 e^t + C_2 e^{-t} + te^t$, $y(t) = C_1 e^t - C_2 e^{-t} + te^t$

? Exercise $5.E.3.9.8$

Solve $x_1' = x_2 + t, x_2' = x_1 + t$ with initial conditions $x_1(0) = 1, x_2(0) = 2$, using eigenvector decomposition.

Answer

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{5}{2} e^t - t - 1 \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{-1}{2} e^{-t}$$

? Exercise $5.E.3.9.9$

Solve $x_1'' = -3x_1 + x_2 + t, x_2'' = 9x_1 + 5x_2 + \cos(t)$ with initial conditions $x_1(0) = 0, x_2(0) = 0, x_1'(0) = 0, x_2'(0) = 0$, using eigenvector decomposition.

Answer

$$\vec{x} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \left(\left(\frac{1}{140} + \frac{1}{120\sqrt{6}} \right) e^{\sqrt{6}t} + \left(\frac{1}{140} + \frac{1}{120\sqrt{6}} \right) e^{-\sqrt{6}t} - \frac{t}{60} - \frac{\cos(t)}{70} \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(\frac{-9}{80} \sin(2t) + \frac{1}{30} \cos(2t) + \frac{9t}{40} - \frac{\cos(t)}{30} \right)$$

CHAPTER OVERVIEW

6: Fourier series and PDEs

Contributors and Attributions

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6.1: Boundary value problems

Before we tackle the Fourier series, we need to study the so-called **boundary value problems** (or endpoint problems). For example, suppose we have

$$x'' + \lambda x = 0, \quad x(a) = 0, \quad x(b) = 0,$$

for some constant λ, where $x(t)$ is defined for t in the interval $[a, b]$. Unlike before, when we specified the value of the solution and its derivative at a single point, we now specify the value of the solution at two different points. Note that $x = 0$ is a solution to this equation, so existence of solutions is not an issue here. Uniqueness of solutions is another issue. The general solution to $x'' + \lambda x = 0$ has two arbitrary constants present.[1] It is, therefore, natural (but wrong) to believe that requiring two conditions guarantees a unique solution.

Example 6.1.1

Take $\lambda = 1, a = 0, b = \pi$. That is,

$$x'' + x = 0, \quad x(0) = 0, \quad x(\pi) = 0.$$

Then $x = \sin t$ is another solution (besides $x = 0$) satisfying both boundary conditions. There are more. Write down the general solution of the differential equation, which is $x = A\cos t + B\sin t$. The condition $x(0) = 0$ forces $A = 0$. Letting $x(\pi) = 0$ does not give us any more information as $x = B\sin t$ already satisfies both boundary conditions. Hence, there are infinitely many solutions of the form $x = B\sin t$, where B is an arbitrary constant.

Example 6.1.2

On the other hand, change to $\lambda = 2$.

$$x'' + 2x = 0, \quad x(0) = 0, \quad x(\pi) = 0.$$

Then the general solution is $x = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$. Letting $x(0) = 0$ still forces $A = 0$. We apply the second condition to find $0 = x(\pi) = B\sin(\sqrt{2}t)$. As $\sin(\sqrt{2}t) \neq 0$ we obtain $B = 0$. Therefore $x = 0$ is the unique solution to this problem.

What is going on? We will be interested in finding which constants λ allow a nonzero solution, and we will be interested in finding those solutions. This problem is an analogue of finding eigenvalues and eigenvectors of matrices.

6.1.1: Eigenvalue Problems

For basic Fourier series theory we will need the following three eigenvalue problems. We will consider more general equations, but we will postpone this until Chapter 5.

$$x'' + \lambda x = 0, \qquad x(a) = 0, \quad x(b) = 0, \tag{6.1.1}$$

$$x'' + \lambda x = 0, \qquad x'(a) = 0, \quad x'(b) = 0, \tag{6.1.2}$$

and

$$x'' + \lambda x = 0, \quad x(a) = x(b), \quad x'(a) = x'(b), \tag{6.1.3}$$

A number λ is called an eigenvalue of (6.1.1) (resp. (6.1.2) or (6.1.3)) if and only if there exists a nonzero (not identically zero) solution to (6.1.1) (resp. (6.1.2) or (6.1.3)) given that specific λ. The nonzero solution we found is called the corresponding *eigenfunction*.

Note the similarity to eigenvalues and eigenvectors of matrices. The similarity is not just coincidental. If we think of the equations as differential operators, then we are doing the same exact thing. Think of a function $x(t)$ as a vector with infinitely many components (one for each t). Let $L = -\frac{d^2}{dt^2}$ be the linear operator. Then the eigenvalue/eigenfunction pair should be λ and nonzero x such that $Lx = \lambda x$. In other words, we are looking for nonzero functions x satisfying certain endpoint conditions that solve $(L - \lambda)x = 0$. A lot of the formalism from linear algebra still applies here, though we will not pursue this line of reasoning too far.

Example 6.1.3

Let us find the eigenvalues and eigenfunctions of

$$x'' + \lambda x = 0, \quad x(0) = 0, \quad x(\pi) = 0.$$

For reasons that will be clear from the computations, we will have to handle the cases $\lambda > 0, \lambda = 0, \lambda < 0$ separately. First suppose that $\lambda > 0$, then the general solution to $x'' + \lambda x = 0$ is

$$x = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t).$$

The condition $x(0) = 0$ implies immediately A. Next

$$0 = x(\pi) = B\sin(\sqrt{\lambda}\pi).$$

If B is zero, then x is not a nonzero solution. So to get a nonzero solution we must have that $\sin(\sqrt{\lambda}\pi) = 0$. Hence, $\sqrt{\lambda}\pi$ must be an integer multiple of π. In other words, $\sqrt{\lambda} = k$ for a positive integer k. Hence the positive eigenvalues are k^2 for all integers $k \geq 1$. The corresponding eigenfunctions can be taken as $x = \sin(kt)$. Just like for eigenvectors, we get all the multiples of an eigenfunction, so we only need to pick one.

Now suppose that $\lambda = 0$. In this case the equation is $x'' = 0$ and the general solution is $x = At + B$. The condition $x(0) = 0$ implies that $B = 0$, and $x(\pi) = 0$ implies that $A = 0$. This means that $\lambda = 0$ is not an eigenvalue.

Finally, suppose that $\lambda < 0$. In this case we have the general solution[2]

$$x = A\cosh(\sqrt{-\lambda}t) + B\sinh(\sqrt{-\lambda}t).$$

Letting $x(0) = 0$ implies that $A = 0$ (recall $\cosh 0 = 1$ and $\sinh 0 = 0$). So our solution must be $x = B\sinh(\sqrt{-\lambda}t)$ and satisfy $x(\pi) = 0$. This is only possible if B is zero. Why? Because $\sinh\xi$ is only zero when $\xi = 0$. You should plot sinh to see this fact. We can also see this from the definition of sinh. We get $0 = \sinh t = \frac{e^t - e^{-t}}{2}$. Hence $e^t = e^{-t}$, which implies $t = -t$ and that is only true if $t = 0$. So there are no negative eigenvalues.

In summary, the eigenvalues and corresponding eigenfunctions are

$$\lambda_k = k^2 \quad \text{with an eigenfucntion} \quad x_k = \sin(kt) \quad \text{for all integers } k \geq 1.$$

Example 6.1.4

Let us compute the eigenvalues and eigenfunctions of

$$x'' + \lambda x = 0, \quad x'(0) = 0, \quad x'(\pi) = 0.$$

Again we will have to handle the cases $\lambda > 0, \lambda = 0, \lambda < 0$ separately. First suppose that $\lambda > 0$. The general solution to $x'' + \lambda x = 0$ is $x = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)$. So

$$x' = -A\sqrt{\lambda}\sin(\sqrt{\lambda}t) + B\sqrt{\lambda}\cos(\sqrt{\lambda}t).$$

The condition $x'(0) = 0$ implies immediately $B = 0$. Next

$$0 = x'(\pi) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\pi).$$

Again A cannot be zero if λ is to be an eigenvalue, and $\sin(\sqrt{\lambda}\pi)$ is only zero if $\sqrt{\lambda} = k$ for a positive integer k. Hence the positive eigenvalues are again k^2 for all integers $k \geq 1$. And the corresponding eigenfunctions can be taken as $x = \cos(kt)$.

Now suppose that $\lambda = 0$. In this case the equation is $x'' = 0$ and the general solution is $x = At + B$ so $x' = A$. The condition $x'(0) = 0$ implies that $A = 0$. Now $x'(\pi) = 0$ also simply implies $A = 0$. This means that B could be anything (let us take it to be 1). So $\lambda = 0$ is an eigenvalue and $x = 1$ is a corresponding eigenfunction.

Finally, let $\lambda < 0$. In this case we have the general solution $x = A\cosh(\sqrt{-\lambda}t) + B\sinh(\sqrt{-\lambda}t)$ and hence

$$x' = A\sqrt{-\lambda}\sinh(\sqrt{-\lambda}t) + B\sqrt{-\lambda}\cosh(\sqrt{-\lambda}t).$$

We have already seen (with roles of A and B switched) that for this to be zero at $t=0$ and $t=\pi$ it implies that $A=B=0$. Hence there are no negative eigenvalues.

In summary, the eigenvalues and corresponding eigenfunctions are

$$\lambda_k = k^2 \quad \text{with an eigenfunction} \quad x_k = \cos(kt) \quad \text{for all integers } k \geq 1,$$

and there is another eigenvalue

$$\lambda_0 = 0 \quad \text{with an eigenfunction} \quad x_0 = 1.$$

The following problem is the one that leads to the general Fourier series.

Example 6.1.5

Let us compute the eigenvalues and eigenfunctions of

$$x'' + \lambda x = 0, \quad x(-\pi) = x(\pi), \quad x'(-\pi) = x'(\pi).$$

Notice that we have not specified the values or the derivatives at the endpoints, but rather that they are the same at the beginning and at the end of the interval.

Let us skip $\lambda < 0$. The computations are the same as before, and again we find that there are no negative eigenvalues.

For $\lambda = 0$, the general solution is $x = At + B$. The condition $x(-\pi) = x(\pi)$ implies that $A=0$ ($A\pi + B = -A\pi + B$ implies $A=0$). The second condition $x'(-\pi) = x'(\pi)$ says nothing about B and hence $\lambda = 0$ is an eigenvalue with a corresponding eigenfunction $x = 1$.

For $\lambda > 0$ we get that $x = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)$. Now

$$\underbrace{A\cos(-\sqrt{\lambda}\,\pi) + B\sin(-\sqrt{\lambda}\,\pi)}_{x(-\pi)} = \underbrace{A\cos(\sqrt{\lambda}\,\pi) + B\sin(\sqrt{\lambda}\,\pi)}_{x(\pi)}.$$

We remember that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. Therefore,

$$A\cos(\sqrt{\lambda}\pi) - B\sin(\sqrt{\lambda}\pi) = A\cos(\sqrt{\lambda}\pi) + B\sin(\sqrt{\lambda}\pi).$$

Hence either $B=0$ or $\sin(\sqrt{\lambda}\pi) = 0$. Similarly (exercise) if we differentiate x and plug in the second condition we find that $A=0$ or $\sin(\sqrt{\lambda}\pi)=0$. Therefore, unless we want A and B to both be zero (which we do not) we must have $\sin(\sqrt{\lambda}\pi)=0$. Hence, $\sqrt{\lambda}$ is an integer and the eigenvalues are yet again $\lambda = k^2$ for an integer $k \geq 1$. In this case, however, $x = A\cos(kt) + B\sin(kt)$ is an eigenfunction for any A and any B. So we have two linearly independent eigenfunctions $\sin(kt)$ and $\cos(kt)$. Remember that for a matrix we could also have had two eigenvectors corresponding to a single eigenvalue if the eigenvalue was repeated.

In summary, the eigenvalues and corresponding eigenfunctions are

$$\begin{array}{llllll} \lambda_k = k^2 & \text{with eigenfunctions} & \cos(kt) & \text{and} & \sin(kt) & \text{for all integers } k \geq 1, \\ \lambda_0 = 0 & \text{with an eigenfunction} & x_0 = 1. & & & \end{array} \tag{6.1.4}$$

6.1.2: Orthogonality of Eigenfunctions

Something that will be very useful in the next section is the *orthogonality* property of the eigenfunctions. This is an analogue of the following fact about eigenvectors of a matrix. A matrix is called *symmetric* if $A = A^T$. *Eigenvectors for two distinct eigenvalues of a symmetric matrix are orthogonal*. That symmetry is required. We will not prove this fact here. The differential operators we are dealing with act much like a symmetric matrix. We, therefore, get the following theorem.

Theorem 6.1.1: Orthogonal

Suppose that $x_1(t)$ and $x_2(t)$ are two eigenfunctions of the problem (6.1.1), (6.1.2) or (6.1.3) for two different eigenvalues λ_1 and λ_2. Then they are **orthogonal** in the sense that

$$\int_a^b x_1(t)x_2(t)dt = 0.$$

Note that the terminology comes from the fact that the integral is a type of inner product. We will expand on this in the next section. The theorem has a very short, elegant, and illuminating proof so let us give it here. First note that we have the following two equations.

$$x_1'' + \lambda_1 x_1 = 0 \quad \text{and} \quad x_2'' + \lambda_2 x_2 = 0.$$

Multiply the first by x_2 and the second by x_1 and subtract to get

$$(\lambda_1 - \lambda_2)x_1x_2 = x_2''x_1 - x_2x_1''.$$

Now integrate both sides of the equation.

$$\begin{aligned}(\lambda_1-\lambda_2)\int_a^b x_1x_2dt &= \int_a^b x_2''x_1 - x_2x_1''dt \\ &= \int_a^b \frac{d}{dt}(x_2'x_1 - x_2x_1')dt \\ &= [x_2'x_1 - x_2x_1']_{t=a}^b = 0.\end{aligned} \tag{6.1.5}$$

The last equality holds because of the boundary conditions. For example, if we consider (6.1.1) we have $x_1(a) = x_1(b) = x_2(a) = x_2(b) = 0$ and so $x_2'x_1 - x_2x_1'$ is zero at both a and b. As $\lambda_1 \neq \lambda_2$, the theorem follows.

Exercise 6.1.1: (easy)

Finish the theorem (check the last equality in the proof) for the cases (6.1.2) and (6.1.3).

We have seen previously that $\sin(nt)$ was an eigenfunction for the problem $x'' + \lambda x = 0,\ x(0) = 0,\ x(\pi) = 0$. Hence we have the integral

$$\int_0^\pi \sin(mt)\sin(nt)\,dt = 0, \quad \text{when } m \neq n.$$

Similarly,

$$\int_0^\pi \cos(mt)\cos(nt)\,dt = 0, \quad \text{when } m \neq n, \qquad \text{and} \qquad \int_0^\pi \cos(nt)\,dt = 0.$$

And finally we also get

$$\int_{-\pi}^\pi \sin(mt)\sin(nt)\,dt = 0, \quad \text{when } m \neq n, \qquad \text{and} \qquad \int_{-\pi}^\pi \sin(nt)\,dt = 0,$$
$$\int_{-\pi}^\pi \cos(mt)\cos(nt)\,dt = 0, \quad \text{when } m \neq n, \qquad \text{and} \qquad \int_{-\pi}^\pi \cos(nt)\,dt = 0,$$

and

$$\int_{-\pi}^\pi \cos(mt)\sin(nt)\,dt = 0 \qquad (\text{even if } m = n).$$

6.1.3: Fredholm Alternative

We now touch on a very useful theorem in the theory of differential equations. The theorem holds in a more general setting than we are going to state it, but for our purposes the following statement is sufficient. We will give a slightly more general version in Chapter 5.

Theorem 6.1.2

Fredholm alternative[3]

Exactly one of the following statements holds. Either

$$x'' + \lambda x = 0, \quad x(a) = 0, \quad x(b) = 0$$

has a nonzero solution, or

$$x'' + \lambda x = f(t), \quad x(a) = 0, \quad x(b) = 0 \tag{6.1.6}$$

has a unique solution for every function f continuous on $[a, b]$.

The theorem is also true for the other types of boundary conditions we considered. The theorem means that if λ is not an eigenvalue, the nonhomogeneous equation (6.1.6) has a unique solution for every right hand side. On the other hand if λ is an eigenvalue, then (6.1.6) need not have a solution for every f, and furthermore, even if it happens to have a solution, the solution is not unique.

We also want to reinforce the idea here that linear differential operators have much in common with matrices. So it is no surprise that there is a finite dimensional version of Fredholm alternative for matrices as well. Let A be an $n \times n$ matrix. The Fredholm alternative then states that either $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution, or $(A - \lambda I)\vec{x} = \vec{b}$ has a solution for every $\vec{b}$.

A lot of intuition from linear algebra can be applied to linear differential operators, but one must be careful of course. For example, one difference we have already seen is that in general a differential operator will have infinitely many eigenvalues, while a matrix has only finitely many.

6.1.3.1: Application

Let us consider a physical application of an endpoint problem. Suppose we have a tightly stretched quickly spinning elastic string or rope of uniform linear density ρ. Let us put this problem into the $xy-$plane. The x axis represents the position on the string. The string rotates at angular velocity ω, so we will assume that the whole $xy-$plane rotates at angular velocity ω. We will assume that the string stays in this $xy-$plane and y will measure its deflection from the equilibrium position, $y = 0$, on the x axis. Hence, we will find a graph giving the shape of the string. We will idealize the string to have no volume to just be a mathematical curve. If we take a small segment and we look at the tension at the endpoints, we see that this force is tangential and we will assume that the magnitude is the same at both end points. Hence the magnitude is constant everywhere and we will call its magnitude T. If we assume that the deflection is small, then we can use Newton's second law to get an equation

$$Ty'' + \rho\omega^2 y = 0.$$

Let L be the length of the string and the string is fixed at the beginning and end points. Hence, $y(0) = 0$ and $y(L) = 0$. See Figure 6.1.1.

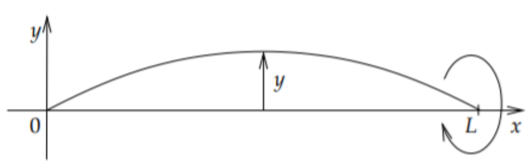

Figure 6.1.1: Whirling string.

We rewrite the equation as $y'' + \frac{\rho\omega^2}{T} y = 0$. The setup is similar to Example 6.1.3, except for the interval length being L instead of π. We are looking for eigenvalues of $y'' + \lambda y = 0, y(0) = 0, y(L) = 0$ where $\lambda = \frac{\rho\omega^2}{T}$. As before there are no nonpositive eigenvalues. With $\lambda > 0$, the general solution to the equation is $y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. The condition $y(0) = 0$ implies that $A = 0$ as before. The condition $y(L) = 0$ implies that $\sin(\sqrt{\lambda}L) = 0$ and hence $\sqrt{\lambda}L = k\pi$ for some integer $k > 0$, so

$$\frac{\rho\omega^2}{T} = \lambda = \frac{k^2\pi^2}{L^2}.$$

What does this say about the shape of the string? It says that for all parameters ρ, ω, T not satisfying the equation above, the string is in the equilibrium position, $y = 0$. When $\frac{\rho\omega^2}{T} = \frac{k^2\pi^2}{L^2}$, then the string will some distance B. We cannot compute B with the information we have.

Let us assume that ρ and T are fixed and we are changing ω. For most values of ω the string is in the equilibrium state. When the angular velocity ω hits a value $\omega = \frac{k\pi\sqrt{T}}{L\sqrt{\rho}}$, then the string pops out and has the shape of a sin wave crossing the x-axis $k-1$ times between the end points. For example, at $k=1$, the string does not cross the x-axis and the shape looks like in Figure $6.1.1$. On the other hand, when $k=3$ the string crosses the x-axis 2 times, see Figure $6.1.2$. When ω changes again, the string returns to the equilibrium position. The higher the angular velocity, the more times it crosses the x-axis when it is popped out.

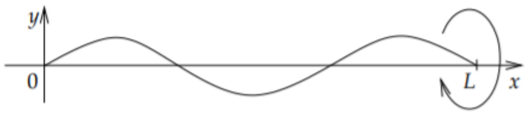

Figure $6.1.2$: Whirling string at the third eigenvalue $(k=3)$.

For another example, if you have a spinning jump rope (then $k=1$ as it is completely) and you pull on the ends to increase the tension, then the velocity also increases for the rope to stay "popped out".

6.1.4: Footnotes

[1] See Example 2.2.1 and Example 2.2.3.

[2] Recall that $\cosh s = \frac{1}{2}(e^s + e^{-s})$ and $\sinh s = \frac{1}{2}(e^s - e^{-s})$. As an exercise try the computation with the general solution written as $x = Ae^{\sqrt{-\lambda}t} + Be^{-\sqrt{-\lambda}t}$ (for different A and B of course).

[3] Named after the Swedish mathematician Erik Ivar Fredholm (1866–1927).

6.2: The Trigonometric Series

6.2.1: Periodic Functions and Motivation

As motivation for studying Fourier series, suppose we have the problem

$$x'' + \omega_0^2 x = f(t), \tag{6.2.1}$$

for some periodic function $f(t)$. We have already solved

$$x'' + \omega_0^2 x = F_0 \cos(\omega t). \tag{6.2.2}$$

One way to solve (6.2.1) is to decompose $f(t)$ as a sum of cosines (and sines) and then solve many problems of the form (6.2.2). We then use the principle of superposition, to sum up all the solutions we got to get a solution to (6.2.1).

Before we proceed, let us talk a little bit more in detail about periodic functions. A function is said to be *periodic* with period P if $f(t)$ for all t. For brevity we will say $f(t)$ is $P-$periodic. Note that a $P-$periodic function is also $2P-$periodic, $3P-$periodic and so on. For example, $\cos(t)$ and $\sin(t)$ are $2\pi-$periodic. So are $\cos(kt)$ and $\sin(kt)$ for all integers k. The constant functions are an extreme example. They are periodic for any period (exercise).

Normally we will start with a function $f(t)$ defined on some interval $[-L, L]$ and we will want to *extend* $f(t)$ *periodically* to make it a $2L-$periodic function. We do this extension by defining a new function $F(t)$ such that for t in$[-L, L]$, $F(t) = f(t)$. For t in $[L, 3L]$, we define $F(t) = f(t-2L)$, for t in $[-3L, -L]$, $F(t) = f(t+2L)$, and so on. We assumed that $f(-L) = f(L)$. We could have also started with f defined only on the half-open interval $(-L, L]$ and then define $f(-L) = f(L)$.

Example 6.2.1

Define $f(t) = 1 - t^2$ on $[-1, 1]$. Now extend $f(t)$ periodically to a 2-periodic function. See Figure $6.2.1$ on the facing page.

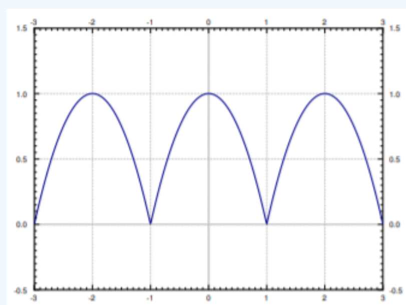

Figure $6.2.1$: Periodic extension of the function $1 - t^2$.

You should be careful to distinguish between $f(t)$ and its extension. A common mistake is to assume that a formula for $f(t)$ holds for its extension. It can be confusing when the formula for $f(t)$ is periodic, but with perhaps a different period.

Exercise 6.2.1

Define $f(t) = \cos t$ on $[\frac{-\pi}{2}, \frac{\pi}{2}]$. Take the $\pi-$periodic extension and sketch its graph. How does it compare to the graph of $\cos t$?

6.2.2: Inner Product and Eigenvector Decomposition

Suppose we have a *symmetric matrix*, that is $A^T = A$. We have said before that the eigenvectors of A are then orthogonal. Here the word *orthogonal* means that if $\vec{v}$ and $\vec{w}$ are two distinct (and not multiples of each other) eigenvectors of A, then $\langle \vec{v}, \vec{w} \rangle = 0$. In this case the inner product $\langle \vec{v}, \vec{w} \rangle$ is the dot product, which can be computed as $\vec{v}^T \vec{w}$.

To decompose a vector $\vec{v}$ in terms of mutually orthogonal vectors $\vec{w}_1$ and $\vec{w}_2$ we write

$$\vec{v} = a_1 \vec{w}_1 + a_2 \vec{w}_2.$$

Let us find the formula for a_1 and a_2. First let us compute

$$\langle \vec{v}, \overrightarrow{w_1} \rangle = \langle a_1 \vec{w}_1 + a_2 \vec{w}_2, \overrightarrow{w_1} \rangle = a_1 \langle \vec{w}_1, \overrightarrow{w_1} \rangle + a_2 \underbrace{\langle \vec{w}_2, \overrightarrow{w_1} \rangle}_{=0} = a_1 \langle \vec{w}_1, \overrightarrow{w_1} \rangle.$$

Therefore,

$$a_1 = \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}.$$

Similarly

$$a_2 = \frac{\langle \vec{v}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle}.$$

You probably remember this formula from vector calculus.

Example 6.2.2

Write $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ as a linear combination of $\vec{w}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

First note that $\vec{w}_1$ and $\vec{w}_2$ are orthogonal as $\langle \vec{w}_1, \vec{w}_2 \rangle = 1(1) + (-1)1 = 0$. Then

$$\begin{aligned} a_1 &= \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} = \frac{2(1) + 3(-1)}{1(1) + (-1)(-1)} = \frac{-1}{2}, \\ a_2 &= \frac{\langle \vec{v}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle} = \frac{2+3}{1+1} = \frac{5}{2}. \end{aligned} \tag{6.2.3}$$

Hence

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

6.2.3: Trigonometric Series

Instead of decomposing a vector in terms of eigenvectors of a matrix, we will decompose a function in terms of eigenfunctions of a certain eigenvalue problem. The eigenvalue problem we will use for the Fourier series is

$$x'' + \lambda x = 0, \quad x(-\pi) = x(\pi) \quad x'(-\pi) = x'(\pi).$$

We have previously computed that the eigenfunctions are $1, \cos(kt), \sin(kt)$. That is, we will want to find a representation of a $2\pi-$periodic function $f(t)$ as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

This series is called the *Fourier series*[1] or the *trigonometric series* for $f(t)$. We write the coefficient of the eigenfunction 1 as $\frac{a_0}{2}$ for convenience. We could also think of $1 = \cos(0t)$, so that we only need to look at $\cos(kt)$ and $\sin(kt)$.

As for matrices we want to find a *projection* of $f(t)$ onto the subspace generated by the eigenfunctions. So we will want to define an *inner product of functions*. For example, to find a_n we want to compute $\langle f(t), \cos(nt) \rangle$. We define the inner product as

$$\langle f(t), g(t) \rangle = \int_{-\pi}^{\pi} f(t)g(t)dt.$$

With this definition of the inner product, we have seen in the previous section that the eigenfunctions $\cos(kt)$(including the constant eigenfunction), and $\sin(kt)$ are *orthogonal* in the sense that

$$\begin{aligned} \langle \cos(mt), \cos(nt) \rangle &= 0 && \text{for } m \neq n, \\ \langle \sin(mt), \sin(nt) \rangle &= 0 && \text{for } m \neq n, \\ \langle \sin(mt), \cos(nt) \rangle &= 0 && \text{for all } m \text{ and } n. \end{aligned} \tag{6.2.4}$$

By elementary calculus for $n = 1, 2, 3, \ldots$. we have $\langle \cos(nt), \cos(nt) \rangle = \pi$ and $\langle \sin(nt), \sin(nt) \rangle = \pi$. For the constant we get

$$\langle 1, 1 \rangle = \int_{\pi}^{\pi} 1 \cdot 1 \, dt = 2\pi.$$

The coefficients are given by

$$\begin{aligned} a_n &= \frac{\langle f(t), \cos(nt) \rangle}{\langle \cos(nt), \cos(nt) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, \\ b_n &= \frac{\langle f(t), \sin(nt) \rangle}{\langle \sin(nt), \sin(nt) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt. \end{aligned} \tag{6.2.5}$$

Compare these expressions with the finite-dimensional example. For a_0 we get a similar formula

$$a_0 = 2 \frac{\langle f(t), 1 \rangle}{\langle 1, 1 \rangle} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt.$$

Let us check the formulas using the orthogonality properties. Suppose for a moment that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

Then for $m \geq 1$ we have

$$\begin{aligned} \langle f(t), \cos(mt) \rangle &= \left\langle \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt), \cos(mt) \right\rangle \\ &= \frac{a_0}{2} \langle 1, \cos(mt) \rangle + \sum_{n=1}^{\infty} a_n \langle \cos(nt), \cos(mt) \rangle + b_n \langle \sin(nt), \sin(mt) \rangle \\ &= a_m \langle \cos(mt), \cos(mt) \rangle . \end{aligned} \tag{6.2.6}$$

And hence $a_m = \frac{\langle f(t), \cos(mt) \rangle}{\langle \cos(mt), \cos(mt) \rangle}$.

Exercise 6.2.2

Carry out the calculation for a_0 and b_m.

Example 6.2.3

Take the function

$$f(t) = t$$

for t in $(-\pi, \pi]$. Extend $f(t)$ periodically and write it as a Fourier series. This function is called the *sawtooth*.

The plot of the extended periodic function is given in Figure $6.2.2$. Let us compute the coefficients.

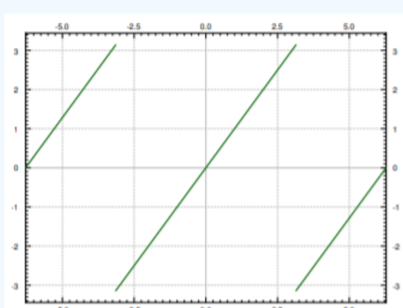

Figure $6.2.2$: The graph of the sawtooth function.

Solution

We start with a_0,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t dt = 0.$$

We will often use the result from calculus that says that the integral of an odd function over a symmetric interval is zero. Recall that an odd function is a function $\varphi(t)$ such that $\varphi(-t) = -\varphi(t)$. For example the functions $t, \sin t$, or (importantly for us) $t\cos(nt)$ are all odd functions. Thus

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos(nt) dt = 0.$$

Let us move to b_n. Another useful fact from calculus is that the integral of an even function over a symmetric interval is twice the integral of the same function over half the interval. Recall an *even function* is a function $\varphi(t)$ such that $\varphi(-t) = \varphi(t)$. For example $t\sin(nt)$ is even.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt) dt \\ &= \frac{2}{\pi}\int_{0}^{\pi} t\sin(nt) dt \\ &= \frac{2}{\pi}\left(\left[\frac{-t\cos(nt)}{n}\right]_{t=0}^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nt) dt\right) \\ &= \frac{2}{\pi}\left(\frac{-\pi\cos(n\pi)}{n} + 0\right) \\ &= \frac{-2\cos(n\pi)}{n} = \frac{2(-1)^{n+1}}{n}. \end{aligned} \tag{6.2.7}$$

We have used the fact that

$$\cos(n\pi) = (-1)^n = \begin{cases} 1 & \text{if } n \text{ even}, \\ -1 & \text{if } n \text{ odd}. \end{cases}$$

The series, therefore, is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nt).$$

Let us write out the first 3 harmonics of the series for $f(t)$.

$$2\sin(t) - \sin(2t) + \frac{2}{3}\sin(3t) + \cdots$$

The plot of these first three terms of the series, along with a plot of the first 20 terms is given in Figure $6.2.3$.

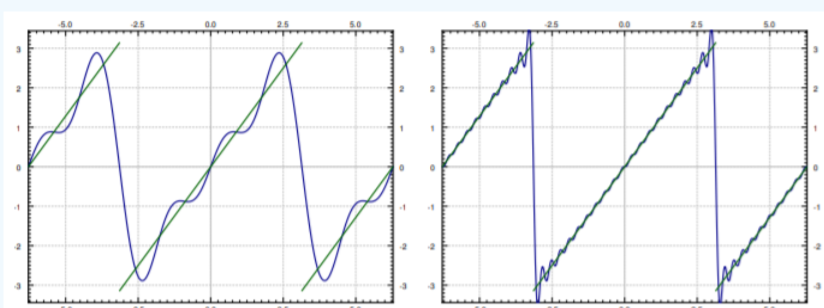

Figure $6.2.3$: First 3 (left graph) and 20 (right graph) harmonics of the sawtooth function.

Example 6.2.4

Take the function

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t \leq 0, \\ \pi & \text{if } 0 < t \leq \pi. \end{cases}$$

Extend $f(t)$ periodically and write it as a Fourier series. This function or its variants appear often in applications and the function is called the square wave.

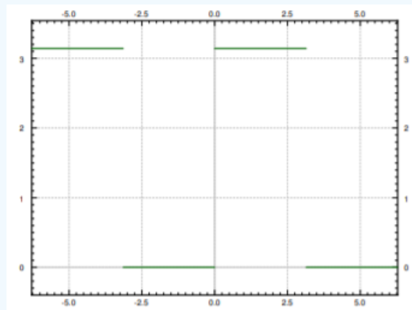

Figure 6.2.4: The graph of the square wave function.

The plot of the extended periodic function is given in Figure $6.2.4$. Now we compute the coefficients. Let us start with a_0

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt = \frac{1}{\pi} \int_{0}^{\pi} \pi dt = \pi.$$

Next,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt = \frac{1}{\pi} \int_{0}^{\pi} \pi \cos(nt) dt = 0.$$

And finally

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \\ &= \frac{1}{\pi} \int_{0}^{\pi} \pi \sin(nt) dt \\ &= \left[\frac{-\cos(nt)}{n} \right]_{t=0}^{\pi} \\ &= \frac{1-\cos(\pi n)}{n} = \frac{1-(-1)^n}{n} = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned} \tag{6.2.8}$$

The Fourier series is

$$\frac{\pi}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n} \sin(nt) + \sum_{k=1}^{\infty} \frac{2}{2k-1} \sin((2k-1)t).$$

Let us write out the first 3 harmonics of the series for $f(t)$.

$$\frac{\pi}{2} + 2\sin(t) + \frac{2}{3}\sin(3t) + \cdots$$

The plot of these first three and also of the first 20 terms of the series is given in Figure $6.2.5$.

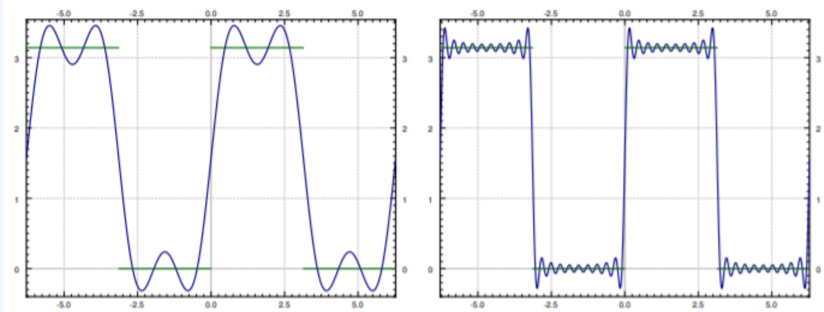

Figure $6.2.5$: First 3 (left graph) and 20 (right graph) harmonics of the square wave function.

We have so far skirted the issue of convergence. For example, if $f(t)$ is the square wave function, the equation

$$f(t) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{2}{2k-1} \sin((2k-1)t).$$

is only an equality for such t where $f(t)$ is continuous. That is, we do not get an equality for $t = -\pi, 0, \pi$ and all the other discontinuities of $f(t)$. It is not hard to see that when t is an integer multiple of π (which includes all the discontinuities), then

$$\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{2}{2k-1} \sin((2k-1)t) = \frac{\pi}{2}.$$

We redefine $f(t)$ on $[-\pi, \pi]$ as

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t < 0, \\ \pi & \text{if } 0 < t < \pi, \\ \pi/2 & \text{if } t = -\pi, t = 0, \text{ or } t = \pi, \end{cases}$$

and extend periodically. The series equals this extended $f(t)$ everywhere, including the discontinuities. We will generally not worry about changing the function values at several (finitely many) points.

We will say more about convergence in the next section. Let us however mention briefly an effect of the discontinuity. Let us zoom in near the discontinuity in the square wave. Further, let us plot the first 100 harmonics, see Figure $6.2.6$. You will notice that while the series is a very good approximation away from the discontinuities, the error (the overshoot) near the discontinuity at $t = \pi$ does not seem to be getting any smaller. This behavior is known as the *Gibbs phenomenon*. The region where the error is large does get smaller, however, the more terms in the series we take.

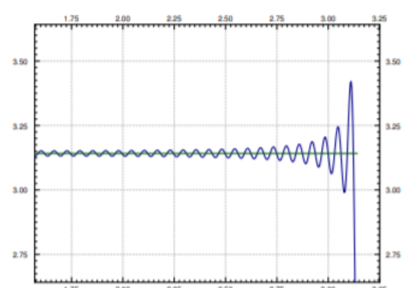

Figure $6.2.6$: Gibbs phenomenon in action.

We can think of a periodic function as a "signal" being a superposition of many signals of pure frequency. For example, we could think of the square wave as a tone of certain base frequency. This base frequency is called the *fundamental frequency*. The square wave will be a superposition of many different pure tones of frequencies that are multiples of the fundamental frequency. In music, the higher frequencies are called the *overtones*. All the frequencies that appear are called the *spectrum* of the signal. On the other hand a simple sine wave is only the pure tone (no overtones). The simplest way to make sound using a computer is the square wave, and the sound is very different from a pure tone. If you ever played video games from the 1980s or so, then you heard what square waves sound like.

6.2.4: Footnotes

[1] Named after the French mathematician Jean Baptiste Joseph Fourier (1768 – 1830).

6.3: More on the Fourier Series

Before reading the lecture, it may be good to first try Project IV (Fourier series) from the IODE website: https://conf.math.illinois.edu/iode/fsgui.html. After reading the lecture it may be good to continue with Project V (Fourier series again).

6.3.1: 2L-Periodic Functions

We have computed the Fourier series for a 2π-periodic function, but what about functions of different periods. Well, fear not, the computation is a simple case of change of variables. We can just rescale the independent axis. Suppose that we have a $2L$-periodic function $f(t)$ (L is called the *half period*). Let $S = \frac{\pi}{L}t$. Then the function

$$g(s) = f\left(\frac{L}{\pi}s\right)$$

is 2π-periodic. We want to also rescale all our sines and cosines. We want to write

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right).$$

If we change variables to s we see that

$$g(s) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(ns) + b_n \sin(ns).$$

We compute a_n and b_n as before. After we write down the integrals we change variables from s back to t.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)ds = \frac{1}{L}\int_{-L}^{L} f(t)dt, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\cos(ns)ds = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\sin(ns)ds = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt.
\end{aligned}$$

The two most common half periods that show up in examples are π and 1 because of the simplicity. We should stress that we have done no new mathematics, we have only changed variables. If you understand the Fourier series for 2π-periodic functions, you understand it for $2L$-periodic functions. All that we are doing is moving some constants around, but all the mathematics is the same.

Example 6.3.1

Let

$$f(t) = |t| \quad \text{for } -1 < t \leq 1,$$

extended periodically. The plot of the periodic extension is given in Figure 6.3.1. Compute the Fourier series of $f(t)$.

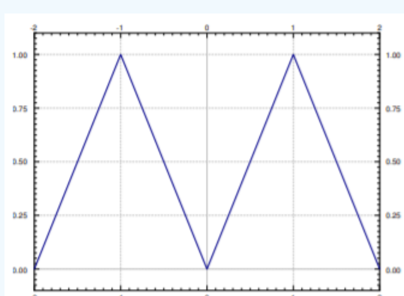

Figure 6.3.1: Periodic extension of the function $f(t)$.

Solution

We want to write $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$. For $n \geq 1$ we note that $|t| \cos(n\pi t)$ is even and hence

$$\begin{aligned} a_n &= \int_{-1}^{1} f(t)\cos(n\pi t)dt \\ &= 2\int_0^1 t\cos(n\pi t)dt \\ &= 2\left[\frac{t}{n\pi}\sin(n\pi t)\right]_{t=0}^{1} - 2\int_0^1 \frac{1}{n\pi}\sin(n\pi t)dt \\ &= 0 + \frac{1}{n^2\pi^2}[\cos(n\pi t)]_{t=0}^{1} = \frac{2((-1)^n - 1)}{n^2\pi^2} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{-4}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{6.3.1}$$

Next we find a_0

$$a_0 = \int_{-1}^{1} |t| dt = 1.$$

You should be able to find this integral by thinking about the integral as the area under the graph without doing any computation at all. Finally we can find b_n. Here, we notice that $|t| \sin(n\pi t)$ is odd and, therefore,

$$b_n = \int_{-1}^{1} f(t)\sin(n\pi t)dt = 0.$$

Hence, the series is

$$\frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^2\pi^2}\cos(n\pi t)$$

Let us explicitly write down the first few terms of the series up to the 3^{rd} harmonic.

$$\frac{1}{2} - \frac{4}{\pi^2}\cos(\pi t) - \frac{4}{9\pi^2}\cos(3\pi t) - \cdots$$

The plot of these few terms and also a plot up to the 20^{th} harmonic is given in Figure 6.3.2. You should notice how close the graph is to the real function. You should also notice that there is no "Gibbs phenomenon" present as there are no discontinuities.

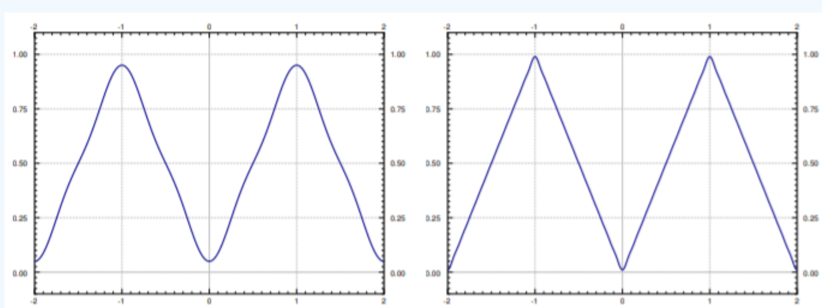

Figure 6.3.2: Fourier series of $f(t)$ up to the 3^{rd} harmonic (left graph) and up to the 20^{th} harmonic (right graph).

6.3.2: Convergence

We will need the one sided limits of functions. We will use the following notation

$$f(c-) = \lim_{t \uparrow c} f(t), \qquad \text{and} \qquad f(c+) = \lim_{t \downarrow c} f(t).$$

If you are unfamiliar with this notation, $\lim_{t \uparrow c} f(t)$ means we are taking a limit of $f(t)$ as t approaches c from below (i.e. $t < c$) and $\lim_{t \downarrow c} f(t)$ means we are taking a limit of $f(t)$ as t approaches c from above (i.e. $t > c$). For example, for the square wave function

$$f(t) = \begin{cases} 0 & \text{if} \quad -\pi < t \leq 0, \\ \pi & \text{if} \quad 0 < t \leq \pi, \end{cases} \tag{6.3.2}$$

we have $f(0-) = 0$ and $f(0+) = \pi$.

Let $f(t)$ be a function defined on an interval $[a, b]$. Suppose that we find finitely many points $a = t_0, t_1, t_2, \ldots, t_k = b$ in the interval, such that $f(t)$ is continuous on the intervals $(t_0, t_1), (t_1, t_2), \ldots, (t_{k-1}, t_k)$. Also suppose that all the one sided limits exist, that is, all of $f(t_0+), f(t_1-), f(t_1+), f(t_2-), f(t_2+), \ldots, f(t_k-)$ exist and are finite. Then we say $f(t)$ is *piecewise continuous*.

If moreover, $f(t)$ is differentiable at all but finitely many points, and $f'(t)$ is piecewise continuous, then $f(t)$ is said to be *piecewise smooth*.

Example 6.3.2

The square wave function (6.3.2) is piecewise smooth on $[-\pi, \pi]$ or any other interval. In such a case we simply say that the function is *piecewise smooth*.

Example 6.3.3

The function $f(t) = |t|$ is piecewise smooth.

Example 6.3.4

The function $f(t) = \dfrac{1}{t}$ is not piecewise smooth on $[-1, 1]$ (or any other interval containing zero). In fact, it is not even piecewise continuous.

Example 6.3.5

The function $f(t) = \sqrt[3]{t}$ is not piecewise smooth on $[-1, 1]$ (or any other interval containing zero). $f(t)$ is continuous, but the derivative of $f(t)$ is unbounded near zero and hence not piecewise continuous.

Piecewise smooth functions have an easy answer on the convergence of the Fourier series.

Theorem 6.3.1

Suppose $f(t)$ is a $2L$-periodic piecewise smooth function. Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right)$$

be the Fourier series for $f(t)$. Then the series converges for all t. If $f(t)$ is continuous near t, then

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right).$$

Otherwise

$$\frac{f(t-) + f(t+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right).$$

If we happen to have that $f(t) = \frac{f(t-)+f(t+)}{2}$ at all the discontinuities, the Fourier series converges to $f(t)$ everywhere. We can always just redefine $f(t)$ by changing the value at each discontinuity appropriately. Then we can write an equals sign between $f(t)$ and the series without any worry. We mentioned this fact briefly at the end last section.

Note that the theorem does not say how fast the series converges. Think back to the discussion of the Gibbs phenomenon in the last section. The closer you get to the discontinuity, the more terms you need to take to get an accurate approximation to the function.

6.3.3: Differentiation and Integration of Fourier Series

Not only does Fourier series converge nicely, but it is easy to differentiate and integrate the series. We can do this just by differentiating or integrating term by term.

Theorem 6.3.2

Suppose

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$$

is a piecewise smooth continuous function and the derivative $f'(t)$ is piecewise smooth. Then the derivative can be obtained by differentiating term by term,

$$f'(t) = \sum_{n=1}^{\infty} \frac{-a_n n\pi}{L} \sin\left(\frac{n\pi}{L}t\right) + \frac{b_n n\pi}{L} \cos\left(\frac{n\pi}{L}t\right).$$

It is important that the function is continuous. It can have corners, but no jumps. Otherwise the differentiated series will fail to converge. For an exercise, take the series obtained for the square wave and try to differentiate the series. Similarly, we can also integrate a Fourier series.

Theorem 6.3.3

Suppose

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$$

is a piecewise smooth function. Then the antiderivative is obtained by antidifferentiating term by term and so

$$F(t) = \frac{a_0 t}{2} + C + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\left(\frac{n\pi}{L}t\right) + \frac{-b_n L}{n\pi} \cos\left(\frac{n\pi}{L}t\right),$$

where $F'(t) = f(t)$ and C is an arbitrary constant.

Note that the series for $F(t)$ is no longer a Fourier series as it contains the $\frac{a_0 t}{2}$ term. The antiderivative of a periodic function need no longer be periodic and so we should not expect a Fourier series.

6.3.4: Rates of Convergence and Smoothness

Let us do an example of a periodic function with one derivative everywhere.

Example 6.3.6

Take the function

$$f(t) = \begin{cases} (t+1)t & \text{if} \quad -1 < t \leq 0, \\ (1-t)t & \text{if} \quad 0 < t \leq 1, \end{cases}$$

and extend to a 2-periodic function. The plot is given in Figure $6.3.3$.

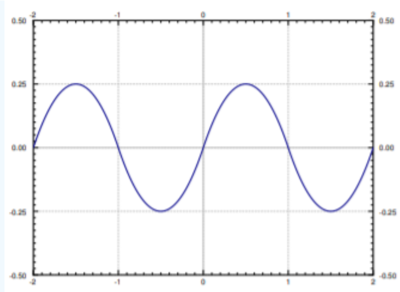

Figure 6.3.3: Smooth 2-periodic function.

Note that this function has one derivative everywhere, but it does not have a second derivative whenever t is an integer.

? Exercise 6.3.1

Compute $f''(0+)$ and $f''(0-)$.

Let us compute the Fourier series coefficients. The actual computation involves several integration by parts and is left to student.

$$\begin{aligned} a_0 &= \int_{-1}^{1} f(t)dt = \int_{-1}^{0}(t+1)tdt + \int_{0}^{1}(1-t)tdt = 0, \\ a_n &= \int_{-1}^{1} f(t)\cos(n\pi t)dt = \int_{-1}^{0}(t+1)t\cos(n\pi t)dt + \int_{0}^{1}(1-t)t\cos(n\pi t)dt = 0 \\ b_n &= \int_{-1}^{1} f(t)\sin(n\pi t)dt = \int_{-1}^{0}(t+1)t\sin(n\pi t)dt + \int_{0}^{1}(1-t)t\sin(n\pi t)dt \\ &= \frac{4(1-(-1)^n)}{\pi^3 n^3} = \begin{cases} \dfrac{8}{\pi^3 n^3} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases} \end{aligned} \tag{6.3.3}$$

That is, the series is

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{\pi^3 n^3}\sin(n\pi t).$$

This series converges very fast. If you plot up to the third harmonic, that is the function

$$\frac{8}{\pi^3}\sin(\pi t) + \frac{8}{27\pi^3}\sin(3\pi t),$$

it is almost indistinguishable from the plot of $f(t)$ in Figure $6.3.3$. In fact, the coefficient $\frac{8}{27\pi^3}$ is already just 0.0096 (approximately). The reason for this behavior is the n^3 term in the denominator. The coefficients b_n in this case go to zero as fast as $\frac{1}{n^3}$ goes to zero.

For functions constructed piecewise from polynomials as above, it is generally true that if you have one derivative, the Fourier coefficients will go to zero approximately like $\frac{1}{n^3}$. If you have only a continuous function, then the Fourier coefficients will go to zero as $\frac{1}{n^2}$. If you have discontinuities, then the Fourier coefficients will go to zero approximately as $\frac{1}{n}$. For more general functions the story is somewhat more complicated but the same idea holds, the more derivatives you have, the faster the coefficients go to zero. Similar reasoning works in reverse. If the coefficients go to zero like $\frac{1}{n^2}$ you always obtain a continuous function. If they go to zero like $\frac{1}{n^3}$ you obtain an everywhere differentiable function.

To justify this behavior, take for example the function defined by the Fourier series

$$f(t)=\sum_{n=1}^{\infty}\frac{1}{n^3}\sin(nt).$$

When we differentiate term by term we notice

$$f'(t)=\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(nt).$$

Therefore, the coefficients now go down like $\frac{1}{n^2}$, which means that we have a continuous function. The derivative of $f'(t)$ is defined at most points, but there are points where $f'(t)$ is not differentiable. It has corners, but no jumps. If we differentiate again (where we can) we find that the function $f''(t)$, now fails to be continuous (has jumps)

$$f''(t)=\sum_{n=1}^{\infty}\frac{-1}{n}\sin(nt).$$

This function is similar to the sawtooth. If we tried to differentiate the series again we would obtain

$$\sum_{n=1}^{\infty}-\cos(nt),$$

which does not converge!

? Exercise 6.3.2

Use a computer to plot the series we obtained for $f(t)$, $f'(t)$ and $f''(t)$. That is, plot say the first 5 harmonics of the functions. At what points does $f''(t)$ have the discontinuities?

6.4: Sine and Cosine Series

6.4.1: 4.4.1:Even Periodic Functions

You may have noticed by now that an odd function has no cosine terms in the Fourier series and an even function has no sine terms in the Fourier series. This observation is not a coincidence. Let us look at even and odd periodic function in more detail.

Recall that a function $f(t)$ is odd if $f(-t) = -f(t)$. A function $f(t)$ is even if $f(-t) = f(t)$. For example, $\cos(nt)$ is even and $\sin(nt)$ is odd. Similarly the function t^k is even if k is even and odd when k is odd.

Exercise 6.4.1

Take two functions $f(t)$ and $g(t)$ and define their product $h(t) = f(t)\,g(t)$.

a. Suppose both $f(t)$ and $g(t)$ are odd, is $h(t)$ odd or even?
b. Suppose one is even and one is odd, is $h(t)$ odd or even?
c. Suppose both are even, is $h(t)$ odd or even?

If $f(t)$ and $g(t)$ are both odd, then $f(t) + g(t)$ is odd. Similarly for even functions. On the other hand, if $f(t)$ is odd and $g(t)$ even, then we cannot say anything about the sum $f(t) + g(t)$. In fact, the Fourier series of any function is a sum of an odd (the sine terms) and an even (the cosine terms) function.

In this section we consider odd and even periodic functions. We have previously defined the $2L$-periodic extension of a function defined on the interval $[-L, L]$. Sometimes we are only interested in the function on the range $[0, L]$ and it would be convenient to have an odd (resp. even) function. If the function is odd (resp. even), all the cosine (resp. sine) terms will disappear. What we will do is take the odd (resp. even) extension of the function to $[-L, L]$ and then extend periodically to a $2L$-periodic function.

Take a function $f(t)$ defined on $[0, L]$. On $(-L, L]$ define the functions

$$\begin{aligned} F_{\text{odd}}(t) &\stackrel{\text{def}}{=} \begin{cases} f(t) & \text{if } \quad 0 \le t \le L, \\ -f(-t) & \text{if } \quad -L < t < 0, \end{cases} \\ F_{\text{even}}(t) &\stackrel{\text{def}}{=} \begin{cases} f(t) & \text{if } \quad 0 \le t \le L, \\ f(-t) & \text{if } \quad -L < t < 0. \end{cases} \end{aligned} \tag{6.4.1}$$

Extend $F_{odd}(t)$ and $F_{even}(t)$ to be $2L$-periodic. Then $F_{odd}(t)$ is called the *odd periodic extension* of $f(t)$, and $F_{even}(t)$ is called the *even periodic extension* of $f(t)$. For the odd extension we generally assume that $f(0) = f(L) = 0$.

Exercise 6.4.2

Check that $F_{odd}(t)$ is odd and that $F_{even}(t)$ is even. For F_{odd}, assume $f(0) = f(L) = 0$.

Example 6.4.1

Take the function $f(t) = t(1-t)$ defined on $[0, 1]$. Figure $6.4.1$ shows the plots of the odd and even extensions of $f(t)$.

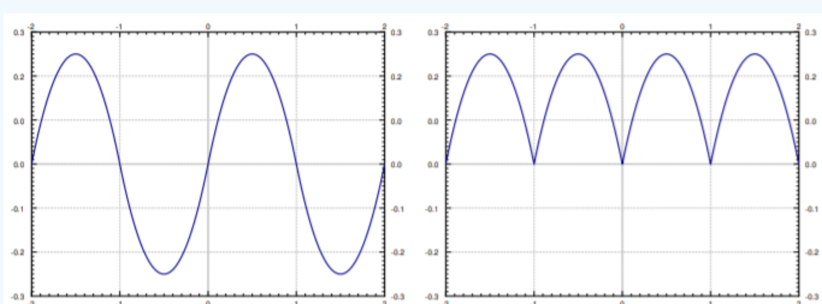

Figure 6.4.1: Odd and even 2-periodic extension of $f(t) = t(1-t)$, $0 \le t \le 1$.

6.4.2: Sine and Cosine Series

Let $f(t)$ be an odd $2L$-periodic function. We write the Fourier series for $f(t)$. First, we compute the coefficients a_n (including $n=0$) and get

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt = 0.$$

That is, there are no cosine terms in the Fourier series of an odd function. The integral is zero because $f(t)\cos\left(\frac{n\pi}{L}t\right)$ is an odd function (product of an odd and an even function is odd) and the integral of an odd function over a symmetric interval is always zero. The integral of an even function over a symmetric interval $[-L, L]$ is twice the integral of the function over the interval $[0, L]$. The function $f(t)\sin\left(\frac{n\pi}{L}t\right)$ is the product of two odd functions and hence is even.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt = \frac{2}{L}\int_{0}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt.$$

We now write the Fourier series of $f(t)$ as

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}t\right).$$

Similarly, if $f(t)$ is an even $2L$-periodic function. For the same exact reasons as above, we find that $b_n = 0$ and

$$a_n = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt.$$

The formula still works for $n=0$, in which case it becomes

$$a_0 = \frac{2}{L}\int_{0}^{L} f(t)\,dt.$$

The Fourier series is then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right).$$

An interesting consequence is that the coefficients of the Fourier series of an odd (or even) function can be computed by just integrating over the half interval $[0, L]$. Therefore, we can compute the Fourier series of the odd (or even) extension of a function by computing certain integrals over the interval where the original function is defined.

Theorem 6.4.1

Let $f(t)$ be a piecewise smooth function defined on $[0, L]$. Then the odd periodic extension of $f(t)$ has the Fourier series

$$F_{odd}(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}t\right),$$

where

$$b_n = \frac{2}{L}\int_{0}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt.$$

The even periodic extension of $f(t)$ has the Fourier series

$$F_{even}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right),$$

where

$$a_n = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt.$$

The series $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}t\right)$ is called the *sine series* of $f(t)$ and the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right)$ is called the *cosine series* of $f(t)$. We often do not actually care what happens outside of $[0, L]$. In this case, we pick whichever series fits our problem better.

It is not necessary to start with the full Fourier series to obtain the sine and cosine series. The sine series is really the eigenfunction expansion of $f(t)$ using eigenfunctions of the eigenvalue problem $x'' + \lambda x = 0 \;\; x(0) = 0,\, x(L) = L$. The cosine series is the eigenfunction expansion of $f(t)$ using eigenfunctions of the eigenvalue problem $x'' + \lambda x = 0$, $x'(0) = 0$, $x'(L) = L$. We could have, therefore, gotten the same formulas by defining the inner produ

$$\langle f(t), g(y)\rangle = \int_0^L f(t)g(t)\,dt,$$

and following the procedure of Section 4.2. This point of view is useful, as we commonly use a specific series that arose because our underlying question led to a certain eigenvalue problem. If the eigenvalue problem is not one of the three we covered so far, you can still do an eigenfunction expansion, generalizing the results of this chapter. We will deal with such a generalization in Chapter 5.

Example 6.4.2

Find the Fourier series of the even periodic extension of the function $f(t) = t^2$ for $0 \leq t \leq \pi$.

Solution

We want to write

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt),$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t^2 dt = \frac{2\pi^2}{3},$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} t^2 \cos(nt)dt = \frac{2}{\pi}\left[t^2 \frac{1}{n}\sin(nt)\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} t\sin(nt)dt \\ &= \frac{4}{n^2\pi}[t\cos(nt)]_0^{\pi} + \frac{4}{n^2\pi}\int_0^{\pi}\cos(nt)dt = \frac{4(-1)^n}{n^2}. \end{aligned} \tag{6.4.2}$$

Note that we have "detected" the continuity of the extension since the coefficients decay as $\frac{1}{n^2}$. That is, the even extension of t^2 has no jump discontinuities. It does have corners, since the derivative, which is an odd function and a sine series, has jumps; it has a Fourier series whose coefficients decay only as $\frac{1}{n}$.

Explicitly, the first few terms of the series are

$$\frac{\pi^2}{3} - 4\cos(t) + \cos(2t) - \frac{4}{9}\cos(3t) + \cdots$$

Exercise 6.4.3

a. Compute the derivative of the even extension of $f(t)$ above and verify it has jump discontinuities. Use the actual definition of $f(t)$, not its cosine series!
b. Why is it that the derivative of the even extension of $f(t)$ is the odd extension of $f'(t)$?

6.4.3: Application

Fourier series ties in to the boundary value problems we studied earlier. Let us see this connection in more detail.

Suppose we have the boundary value problem for $0 < t < L$.

$$x''(t) + \lambda x(t) = f(t),$$

for the *Dirichlet boundary conditions* $x(0) = 0, x(L) = 0$. By using the Fredholm alternative (Theorem 4.1.2) we note that as long as λ is *not* an eigenvalue of the underlying homogeneous problem, there exists a unique solution. Note that the eigenfunctions of this eigenvalue problem are the functions $\sin\left(\frac{n\pi}{L}t\right)$. Therefore, to find the solution, we first find the Fourier sine series for $f(t)$. We write x also as a sine series, but with unknown coefficients. We substitute the series for x into the equation and solve for the unknown coefficients. If we have the *Neumann boundary conditions* $x'(0) = 0$ and $x'(L) = 0$, we do the same procedure using the cosine series.

Let us see how this method works on examples.

Example 6.4.3

Take the boundary value problem for $0 < t < 1$,

$$x''(t) + 2x(t) = f(t),$$

where $f(t) = t$ on $0 < t < 1$, and satisfying the **Dirichlet boundary** conditions $x(0) = 0$ and $x(1) = 0$. We write $f(t)$ as a sine series

$$f(t) = \sum_{n=1}^{\infty} c_n \sin(n\pi t).$$

Compute

$$c_n = 2\int_0^1 t\,\sin(n\pi t)\,dt = \frac{2(-1)^{n+1}}{n\pi}.$$

We write $x(t)$ as

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t).$$

We plug in to obtain

$$\begin{aligned} x''(t) + 2x(t) &= \underbrace{\sum_{n=1}^{\infty} -b_n n^2\pi^2 \sin(n\pi t)}_{x''} + \underbrace{2\sum_{n=1}^{\infty} b_n \sin(n\pi t)}_{x} \\ &= \sum_{n=1}^{\infty} b_n (2 - n^2\pi^2)\sin(n\pi t) \\ &= f(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi t). \end{aligned} \tag{6.4.3}$$

Therefore,

$$b_n(2 - n^2\pi^2) = \frac{2(-1)^{n+1}}{n\pi}.$$

or

$$b_n = \frac{2(-1)^{n+1}}{n\pi(2 - n^2\pi^2)}.$$

That $2 - n^2\pi^2$ is not zero for any n, and that we can solve for b_n, is precisely because 2 is not an eigenvale of the problem. We have thus obtained a Fourier series for the solution

$$x(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(2 - n^2\pi^2)}\sin(n\pi t).$$

See Figure 6.4.2 for a graph of the solution. Notice that because the eigenfunctions satisfy the boundary conditions, and x is written in terms of the boundary conditions, then x satisfies the boundary conditions.

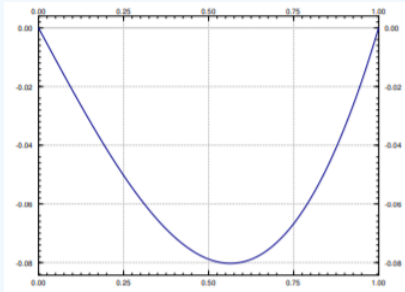

Figure 6.4.2: Plot of the solution of $x'' + 2x = t$, $x(0) = 0$, $x(1) = 0$.

Example 6.4.4

Similarly we handle the Neumann conditions. Take the boundary value problem for $0 < t < 1$,

$$x''(t) + 2x(t) = f(t),$$

where again $f(t) = t$ on $0 < t < 1$, but now satisfying the Neumann boundary conditions $x'(0) = 0$ and $x'(1) = 0$. We write $f(t)$ as a cosine series

$$f(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\pi t),$$

where

$$c_0 = 2\int_0^1 t\,dt = 1$$

and

$$c_n = 2\int_0^1 t\cos(n\pi t)dt = \frac{2((-1)^n - 1)}{\pi^2 n^2} = \begin{cases} \frac{-4}{\pi^2 n^2} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

We also write $x(t)$ as a cosine series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t).$$

We plug in to obtain

$$\begin{aligned} x''(t) + 2x(t) &= \sum_{n=1}^{\infty} \left[-a_n n^2 \pi^2 \cos(n\pi t)\right] + a_0 + 2\sum_{n=1}^{\infty} [a_n \cos(n\pi t)] \\ &= a_0 + \sum_{n=1}^{\infty} a_n (2 - n^2\pi^2) \cos(n\pi t) \\ &= f(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{\pi^2 n^2} \cos(n\pi t). \end{aligned} \tag{6.4.4}$$

Therefore, $a = \frac{1}{2}$ and $a_n = 0$ for n even ($n \geq 2$) and for n odd we have

$$a_n(2 - n^2\pi^2) = \frac{-4}{\pi^2 n^2},$$

or

$$a_n = \frac{-4}{n^2\pi^2(2 - n^2\pi^2)}.$$

The Fourier series for the solution $x(t)$ is

$$x(t) = \frac{1}{4} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^2\pi^2(2-n^2\pi^2)} \cos(n\pi t).$$

6.5: Applications of Fourier Series

6.5.1: Periodically Forced Oscillation

Let us return to the forced oscillations. Consider a mass-spring system as before, where we have a mass m on a spring with spring constant k, with damping c, and a force $F(t)$ applied to the mass. Suppose the forcing function $F(t)$ is $2L$-periodic for some $L>0$. We have already seen this problem in chapter 2 with a simple $F(t)$.

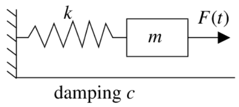

Figure 6.5.1

The equation that governs this particular setup is

$$mx''(t)+cx'(t)+kx(t)=F(t). \tag{6.5.1}$$

The general solution consists of (6.5.1) consists of the complementary solution x_c, which solves the associated homogeneous equation $mx''+cx'+kx=0$, and a particular solution of Equation (6.5.1) we call x_p. For $c>0$, the complementary solution x_c will decay as time goes by. Therefore, we are mostly interested in a particular solution x_p that does not decay and is periodic with the same period as $F(t)$. We call this particular solution the steady periodic solution and we write it as x_{sp} as before. What will be new in this section is that we consider an arbitrary forcing function $F(t)$ instead of a simple cosine.

For simplicity, let us suppose that $c=0$. The problem with $c>0$ is very similar. The equation

$$mx''+kx=0$$

has the general solution

$$x(t)=A\cos(\omega_0 t)+B\sin(\omega_0 t),$$

where $\omega_0=\sqrt{\frac{k}{m}}$. Any solution to $mx''(t)+kx(t)=F(t)$ is of the form $A\cos(\omega_0 t)+B\sin(\omega_0 t)+x_{sp}$. The steady periodic solution x_{sp} has the same period as $F(t)$.

In the spirit of the last section and the idea of undetermined coefficients we first write

$$F(t)=\frac{c_0}{2}+\sum_{n=1}^{\infty} c_n\cos\left(\frac{n\pi}{L}t\right)+d_n\sin\left(\frac{n\pi}{L}t\right).$$

Then we write a proposed steady periodic solution x as

$$x(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}t\right)+b_n\sin\left(\frac{n\pi}{L}t\right),$$

where a_n and b_n are unknowns. We plug x into the differential equation and solve for a_n and b_n in terms of c_n and d_n. This process is perhaps best understood by example.

Example 6.5.1

Suppose that $k=2$, and $m=1$. The units are again the mks units (meters-kilograms-seconds). There is a jetpack strapped to the mass, which fires with a force of 1 newton for 1 second and then is off for 1 second, and so on. We want to find the steady periodic solution.

Solution

The equation is, therefore,

$$x''+2x=F(t),$$

where $F(t)$ is the step function

$$F(t)=\begin{cases}0 & \text{if} \quad -1<t<0,\\ 1 & \text{if} \quad 0<t<1,\end{cases}$$

extended periodically. We write

$$F(t)=\frac{c_0}{2}+\sum_{n=1}^{\infty} c_n\cos(n\pi t)+d_n\sin(n\pi t).$$

We compute

$$\begin{aligned}
c_n &=\int_{-1}^{1} F(t)\cos(n\pi t)dt=\int_0^1 \cos(n\pi t)dt=0 \quad \text{for } n\geq 1,\\
c_0 &=\int_{-1}^{1} F(t)dt=\int_0^1 dt=1,\\
d_n &=\int_{-1}^{1} F(t)\sin(n\pi t)dt\\
&=\int_0^1 \sin(n\pi t)dt\\
&=\left[\frac{-\cos(n\pi t)}{n\pi}\right]_{t=0}^{1}\\
&=\frac{1-(-1)^n}{\pi n}=\begin{cases}\frac{2}{\pi n} & \text{if } n \text{ odd},\\ 0 & \text{if } n \text{ even}.\end{cases}
\end{aligned}\tag{6.5.2}$$

So

$$F(t)=\frac{1}{2}+\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{2}{\pi n}\sin(n\pi t).$$

We want to try

$$x(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(n\pi t)+b_n\sin(n\pi t).$$

Once we plug into the differential equation $x''+2x=F(t)$, it is clear that $a_n=0$ for $n\geq 1$ as there are no corresponding terms in the series for $F(t)$. Similarly $b_n=0$ for n even. Hence we try

$$x(t)=\frac{a_0}{2}+\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} b_n\sin(n\pi t).$$

We plug into the differential equation and obtain

$$\begin{aligned}
x''+2x &=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\left[-b_n n^2\pi^2\sin(n\pi t)\right]+a_0+2\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\left[b_n\sin(n\pi t)\right]\\
&=a_0+\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} b_n(2-n^2\pi^2)\sin(n\pi t)\\
&=F(t)=\frac{1}{2}+\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{2}{\pi n}\sin(n\pi t).
\end{aligned}\tag{6.5.3}$$

So $a_0=\frac{1}{2}$, $b_n=0$ for even n, and for odd n we get

$$b_n=\frac{2}{\pi n(2-n^2\pi^2)}.$$

The steady periodic solution has the Fourier series

$$x_{sp}(t) = \frac{1}{4} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{\pi n(2 - n^2\pi^2)} \sin(n\pi t).$$

We know this is the steady periodic solution as it contains no terms of the complementary solution and it is periodic with the same period as $F(t)$ itself. See Figure $6.5.1$ for the plot of this solution.

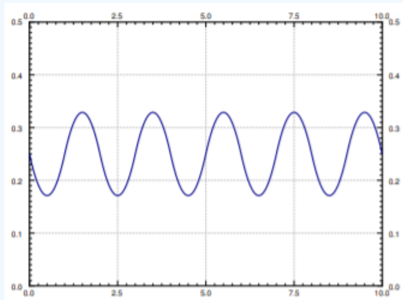

Figure 6.5.1: Plot of the steady periodic solution x_{sp} of Example $6.5.1$.

6.5.1.1: Resonance

Just like when the forcing function was a simple cosine, resonance could still happen. Let us assume $c = 0$ and we will discuss only pure resonance. Again, take the equation

$$mx''(t) + kx(t) = F(t).$$

When we expand $F(t)$ and find that some of its terms coincide with the complementary solution to $mx'' + kx = 0$, we cannot use those terms in the guess. Just like before, they will disappear when we plug into the left hand side and we will get a contradictory equation (such as $0 = 1$). That is, suppose

$$x_c = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

where $\omega_0 = \dfrac{N\pi}{L}$ for some positive integer N. In this case we have to modify our guess and try

$$x(t) = \frac{a_0}{2} + t\left(a_N \cos\left(\frac{N\pi}{L}t\right) + b_N \sin\left(\frac{N\pi}{L}t\right)\right) + \sum_{\substack{n=1 \\ n\neq N}}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right).$$

In other words, we multiply the offending term by t. From then on, we proceed as before.

Of course, the solution will not be a Fourier series (it will not even be periodic) since it contains these terms multiplied by t. Further, the terms $t\left(a_N \cos\left(\frac{N\pi}{L}t\right) + b_N \sin\left(\frac{N\pi}{L}t\right)\right)$ will eventually dominate and lead to wild oscillations. As before, this behavior is called *pure resonance* or just *resonance*.

Note that there now may be infinitely many resonance frequencies to hit. That is, as we change the frequency of F (we change L), different terms from the Fourier series of F may interfere with the complementary solution and will cause resonance. However, we should note that since everything is an approximation and in particular c is never actually zero but something very close to zero, only the first few resonance frequencies will matter.

Example $6.5.2$

Find the steady periodic solution to the equation

$$2x'' + 18\pi^2 x = F(t), \tag{6.5.4}$$

where

$$F(t) = \begin{cases} -1 & \text{if} \quad -1 < t < 0, \\ 1 & \text{if} \quad 0 < t < 1, \end{cases}$$

extended periodically. We note that

$$F(t)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(n\pi t).$$

Exercise 6.5.1

Compute the Fourier series of F to verify the above equation.

As $\sqrt{\frac{k}{m}}=\sqrt{\frac{18\pi^2}{2}}=3\pi$, the solution to (6.5.4) is

$$x(t)=c_1\cos(3\pi t)+c_2\sin(3\pi t)+x_p(t)$$

for some particular solution x_p.

If we just try an x_p given as a Fourier series with $\sin(n\pi t)$ as usual, the complementary equation, $2x''+18\pi^2x=0$, eats our 3^{rd} harmonic. That is, the term with $\sin(3\pi t)$ is already in in our complementary solution. Therefore, we pull that term out and multiply it by t. We also add a cosine term to get everything right. That is, we try

$$x_p(t)=a_3t\cos(3\pi t)+b_3t\sin(3\pi t)+\sum_{\substack{n=1\\ n\text{ odd}\\ n\neq 3}}^{\infty}b_n\sin(n\pi t).$$

Let us compute the second derivative.

$$x_p''(t)=-6a_3\pi\sin(3\pi t)-9\pi^2a_3t\cos(3\pi t)+6b_3\pi\cos(3\pi t)-9\pi^2b_3t\sin(3\pi t)+\sum_{\substack{n=1\\ n\text{ odd}\\ n\neq 3}}^{\infty}(-n^2\pi^2b_n)\sin(n\pi t).$$

We now plug into the left hand side of the differential equation.

$$\begin{aligned}2x_p''+18\pi^2x_p= &-12a_3\pi\sin(3\pi t)-18\pi^2a_3t\cos(3\pi t)+12b_3\pi\cos(3\pi t)-18\pi^2b_3t\sin(3\pi t)\\ &\qquad\qquad +18\pi^2a_3t\cos(3\pi t)\qquad\qquad\qquad +18\pi^2b_3t\sin(3\pi t)\\ &+\sum_{\substack{n=1\\ n\text{ odd}\\ n\neq 3}}^{\infty}(-2n^2\pi^2b_n+18\pi^2b_n)\,\sin(n\pi t).\end{aligned}\tag{6.5.5}$$

If we simplify we obtain

$$2x_p''+18\pi^2x=-12a_3\pi\sin(3\pi t)+12b_3\pi\cos(3\pi t)+\sum_{\substack{n=1\\ n\text{ odd}\\ n\neq 3}}^{\infty}(-2n^2\pi^2b_n+18\pi^2b_n)\sin(n\pi t.)$$

This series has to equal to the series for $F(t)$. We equate the coefficients and solve for a_3 and b_n.

$$\begin{aligned}a_3&=\frac{4/(3\pi)}{-12\pi}=\frac{-1}{9\pi^2},\\ b_3&=0,\\ b_n&=\frac{4}{n\pi(18\pi^2-2n^2\pi^2)}=\frac{2}{\pi^3n(9-n^2)}\qquad\text{for } n \text{ odd and } n\neq 3.\end{aligned}\tag{6.5.6}$$

That is,

$$x_p(t)=\frac{-1}{9\pi^2}t\cos(3\pi t)+\sum_{\substack{n=1\\ n\text{ odd}\\ n\neq 3}}^{\infty}\frac{2}{\pi^3n(9-n^2)}\sin(n\pi t.)$$

When $c > 0$, you will not have to worry about pure resonance. That is, there will never be any conflicts and you do not need to multiply any terms by t. There is a corresponding concept of practical resonance and it is very similar to the ideas we already explored in Chapter 2. Basically what happens in practical resonance is that one of the coefficients in the series for x_{sp} can get very big. We will not go into details here.

6.5.2: Contributors and Attributions

- ◦ Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

6.6: PDEs, Separation of Variables, and The Heat Equation

Let us recall that a *partial differential equation* or PDE is an equation containing the partial derivatives with respect to several independent variables. Solving PDEs will be our main application of Fourier series.

A PDE is said to be *linear* if the dependent variable and its derivatives appear at most to the first power and in no functions. We will only talk about linear PDEs. Together with a PDE, we usually have specified some *boundary conditions*, where the value of the solution or its derivatives is specified along the boundary of a region, and/or some *initial conditions* where the value of the solution or its derivatives is specified for some initial time. Sometimes such conditions are mixed together and we will refer to them simply as *side conditions*.

We will study three specific partial differential equations, each one representing a more general class of equations. First, we will study the *heat equation*, which is an example of a *parabolic* PDE. Next, we will study the *wave equation*, which is an example of a *hyperbolic PDE*. Finally, we will study the *Laplace equation*, which is an example of an *elliptic PDE*. Each of our examples will illustrate behavior that is typical for the whole class.

6.6.1: Heat on an Insulated Wire

Let us first study the heat equation. Suppose that we have a wire (or a thin metal rod) of length L that is insulated except at the endpoints. Let x denote the position along the wire and let t denote time. See Figure $6.6.1$.

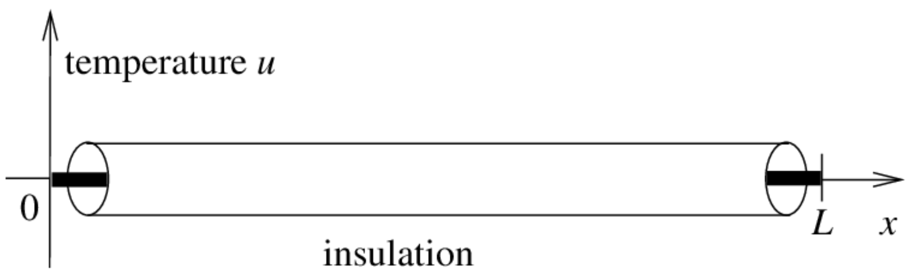

Figure $6.6.1$: Insulated wire.

Let $u(x,t)$ denote the temperature at point x at time t. The equation governing this setup is the so-called *one-dimensional heat equation*:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where $k > 0$ is a constant (the *thermal conductivity* of the material). That is, the change in heat at a specific point is proportional to the second derivative of the heat along the wire. This makes sense; if at a fixed t the graph of the heat distribution has a maximum (the graph is concave down), then heat flows away from the maximum. And vice-versa.

We will generally use a more convenient notation for partial derivatives. We will write u_t instead of $\frac{\partial u}{\partial t}$, and we will write u_{xx} instead of $\frac{\partial^2 u}{\partial x^2}$. With this notation the heat equation becomes

$$u_t = ku_{xx}.$$

For the heat equation, we must also have some boundary conditions. We assume that the ends of the wire are either exposed and touching some body of constant heat, or the ends are insulated. For example, if the ends of the wire are kept at temperature 0, then we must have the conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

If, on the other hand, the ends are also insulated we get the conditions

$$u_x(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0.$$

Let us see why that is so. If u_x is positive at some point x_0, then at a particular time, u is smaller to the left of x_0, and higher to the right of x_0. Heat is flowing from high heat to low heat, that is to the left. On the other hand if u_x is negative then heat is again flowing from high heat to low heat, that is to the right. So when u_x is zero, that is a point through which heat is not flowing. In other words, $u_x(0,t) = 0$ means no heat is flowing in or out of the wire at the point $x = 0$.

We have two conditions along the x-axis as there are two derivatives in the x direction. These side conditions are said to be *homogeneous* (i.e., u or a derivative of u is set to zero).

We also need an initial condition—the temperature distribution at time $t=0$. That is,

$$u(x,0)=f(x),$$

for some known function $f(x)$. This initial condition is not a homogeneous side condition.

6.6.2: Separation of Variables

The heat equation is linear as u and its derivatives do not appear to any powers or in any functions. Thus the principle of superposition still applies for the heat equation (without side conditions). If u_1 and u_2 are solutions and c_1, c_2 are constants, then $u=c_1u_1+c_2u_2$ is also a solution.

Exercise 6.6.1

Verify the principle of superposition for the heat equation.

Superposition also preserves some of the side conditions. In particular, if u_1 and u_2 are solutions that satisfy $u(0,t)=0$ and $u_(L,t)=0$, and c_1, c_2 are constants, then $u=c_1u_1+c_2u_2$ is still a solution that satisfies $u(0,t)=0$ and$u_(L,t)=0$. Similarly for the side conditions $u_x(0,t)=0$ and $u_x(L,t)=0$. In general, superposition preserves all homogeneous side conditions.

The method of *separation of variables* is to try to find solutions that are sums or products of functions of one variable. For example, for the heat equation, we try to find solutions of the form

$$u(x,t)=X(x)T(t).$$

That the desired solution we are looking for is of this form is too much to hope for. What is perfectly reasonable to ask, however, is to find enough "building-block" solutions of the form $u(x,t)=X(x)T(t)$ using this procedure so that the desired solution to the PDE is somehow constructed from these building blocks by the use of superposition.

Let us try to solve the heat equation

$$u_t=ku_{xx} \quad \text{with} \quad u(0,t)=0, \quad u(L,t)=0, \quad \text{and} \quad u(x,0)=f(x).$$

Let us guess $u(x,t)=X(x)T(t)$. We will try to make this guess satisfy the differential equation, $u_t=ku_{xx}$, and the homogeneous side conditions, $u(0,t)=0$ and $u(L,t)=0$. Then, as superposition preserves the differential equation and the homogeneous side conditions, we will try to build up a solution from these building blocks to solve the nonhomogeneous initial condition $u(x,0)=f(x)$.

First we plug $u(x,t)=X(x)T(t)$ into the heat equation to obtain

$$X(x)T'(t)=kX''(x)T(t).$$

We rewrite as

$$\frac{T'(t)}{kT(t)}=\frac{X''(x)}{X(x)}.$$

This equation must hold for all x and all t. But the left hand side does not depend on x and the right hand side does not depend on t. Hence, each side must be a constant. Let us call this constant $-\lambda$ (the minus sign is for convenience later). We obtain the two equations

$$\frac{T'(t)}{kT(t)}=-\lambda=\frac{X''(x)}{X(x)}.$$

In other words

$$\begin{aligned} X''(x)+\lambda X(x) &= 0, \\ T'(t)+\lambda kT(t) &= 0. \end{aligned} \tag{6.6.1}$$

The boundary condition $u(0,t)=0$ implies $X(0)T(t)=0$. We are looking for a nontrivial solution and so we can assume that $T(t)$ is not identically zero. Hence $X(0)=0$. Similarly, $u(L,t)=0$ implies $X(L)=0$. We are looking for nontrivial solutions X of the eigenvalue problem $X''+\lambda X=0, X(0)=0, X(L)=0$. We have previously found that the only eigenvalues are $\lambda_n=\frac{n^2\pi^2}{L^2}$, for integers $n\geq 1$, where eigenfunctions are $\sin\left(\frac{n\pi}{L}x\right)$. Hence, let us pick the solutions

$$X_n(x)=\sin\left(\frac{n\pi}{L}x\right).$$

The corresponding T_n must satisfy the equation

$$T_n'(t)+\frac{n^2\pi^2}{L^2}kT_n(t)=0.$$

By the method of integrating factor, the solution of this problem is

$$T_n(t)=e^{\frac{-n^2\pi^2}{L^2}kt}.$$

It will be useful to note that $T_n(0)=1$. Our building-block solutions are

$$u_n(x,t)=X_n(x)T_n(t)=\sin\left(\frac{n\pi}{L}x\right)e^{\frac{-n^2\pi^2}{L^2}kt}.$$

We note that $u_n(x,0)=\sin\left(\frac{n\pi}{L}x\right)$. Let us write $f(x)$ as the sine series

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{L}x\right).$$

That is, we find the Fourier series of the odd periodic extension of $f(x)$. We used the sine series as it corresponds to the eigenvalue problem for $X(x)$ above. Finally, we use superposition to write the solution as

$$u(x,t)=\sum_{n=1}^{\infty}b_nu_n(x,t)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{L}x\right)e^{\frac{-n^2\pi^2}{L^2}kt}.$$

Why does this solution work? First note that it is a solution to the heat equation by superposition. It satisfies $u(0,t)=0$ and $u(L,t)=0$, because $x=0$ or $x=L$ makes all the sines vanish. Finally, plugging in $t=0$, we notice that $T_n(0)=1$ and so

$$u(x,0)=\sum_{n=1}^{\infty}b_nu_n(x,0)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{L}x\right)=f(x).$$

Example 6.6.1

Suppose that we have an insulated wire of length 1, such that the ends of the wire are embedded in ice (temperature 0). Let $k=0.003$. Then suppose that initial heat distribution is $u(x,0)=50x(1-x)$. See Figure 6.6.2.

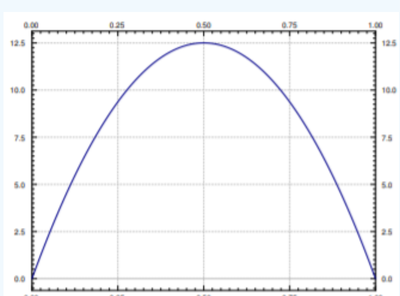

Figure 6.6.2: Initial distribution of temperature in the wire.

We want to find the temperature function $u(x,t)$. Let us suppose we also want to find when (at what t) does the maximum temperature in the wire drop to one half of the initial maximum of 12.5.

We are solving the following PDE problem:

$$\begin{aligned} u_t &= 0.003 u_{xx}, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= 50x(1-x) \quad \text{for } 0 < x < 1. \end{aligned} \tag{6.6.2}$$

We write $f(x) = 50x(1-x)$ for $0 < x < 1$ as a sine series. That is, $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$, where

$$b_n = 2\int_0^1 50x(1-x)\sin(n\pi x)dx = \frac{200}{\pi^3 n^3} - \frac{200(-1)^n}{\pi^3 n^3} = \begin{cases} 0 & \text{if } n \text{ even,} \\ \frac{400}{\pi^3 n^3} & \text{if } n \text{ odd.} \end{cases}$$

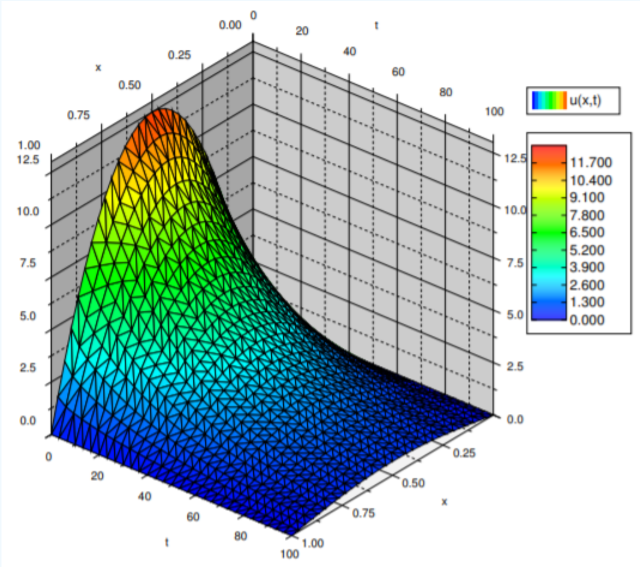

Figure 6.6.3: Plot of the temperature of the wire at position x at time t.

The solution $u(x,t)$, plotted in Figure 6.6.3 for \(0 \leq t \leq 100\), is given by the series:

$$u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{400}{\pi^3 n^3} \sin(n\pi x) e^{-n^2\pi^2 0.003t}.$$

Finally, let us answer the question about the maximum temperature. It is relatively easy to see that the maximum temperature will always be at $x = 0.5$, in the middle of the wire. The plot of $u(x,t)$ confirms this intuition.

If we plug in $x = 0.5$ we get

$$u(0.5,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{400}{\pi^3 n^3} \sin(n\pi 0.5) e^{-n^2\pi^2 0.003t}.$$

For $n = 3$ and higher (remember n is only odd), the terms of the series are insignificant compared to the first term. The first term in the series is already a very good approximation of the function. Hence

$$u(0.5,t) \approx \frac{400}{\pi^3} e^{-\pi^2 0.003t}.$$

The approximation gets better and better as t gets larger as the other terms decay much faster. Let us plot the function $0.5, t$, the temperature at the midpoint of the wire at time t, in Figure 6.6.4. The figure also plots the approximation by the first term.

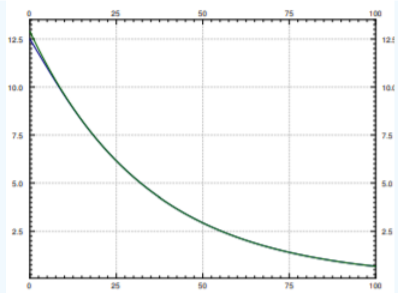

Figure 6.6.4: Temperature at the midpoint of the wire (the bottom curve), and the approximation of this temperature by using only the first term in the series (top curve).

After $t = 5$ or so it would be hard to tell the difference between the first term of the series for $u(x,t)$ and the real solution $u(x,t)$. This behavior is a general feature of solving the heat equation. If you are interested in behavior for large enough t, only the first one or two terms may be necessary.

Let us get back to the question of when is the maximum temperature one half of the initial maximum temperature. That is, when is the temperature at the midpoint $12.5/2 = 6.25$. We notice on the graph that if we use the approximation by the first term we will be close enough. We solve

$$6.25 = \frac{400}{\pi^3} e^{-\pi^2 0.003 t}.$$

That is,

$$t = \frac{\ln \frac{6.25\pi^3}{400}}{-\pi^2 0.003} \approx 24.5.$$

So the maximum temperature drops to half at about $t = 24.5$.

We mention an interesting behavior of the solution to the heat equation. The heat equation "smoothes" out the function $f(x)$ as t grows. For a fixed t, the solution is a Fourier series with coefficients $b_n e^{\frac{-n^2\pi^2}{L^2}kt}$. If $t > 0$, then these coefficients go to zero faster than any $\frac{1}{n^P}$ for any power p. In other words, the Fourier series has infinitely many derivatives everywhere. Thus even if the function $f(x)$ has jumps and corners, then for a fixed $t > 0$, the solution $u(x,t)$ as a function of x is as smooth as we want it to be.

Example 6.6.2

When the initial condition is already a sine series, then there is no need to compute anything, you just need to plug in. Consider

$$u_t = 0.3\, u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = 0.1 \sin(\pi t) + \sin(2\pi t).$$

The solution is then

$$u(x,t) = 0.1 \sin(\pi t) e^{-0.3\pi^2 t} + \sin(2\pi t) e^{-1.2\pi^2 t}.$$

6.6.3: Insulated Ends

Now suppose the ends of the wire are insulated. In this case, we are solving the equation

$$u_t = k u_{xx} \quad \text{with} \quad u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

Yet again we try a solution of the form $u(x,t) = X(x)T(t)$. By the same procedure as before we plug into the heat equation and arrive at the following two equations

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ T'(t) + \lambda k T(t) &= 0. \end{aligned} \tag{6.6.3}$$

At this point the story changes slightly. The boundary condition $u_x(0,t)=0$ implies $X'(0)T(t)=0$. Hence $X'(0)=0$. Similarly, $u_x(L,t)=0$ implies $X'(L)=0$. We are looking for nontrivial solutions X of the eigenvalue problem $X''+\lambda X=0$, $X'(0)=0,\ X'(L)=0,$. We have previously found that the only eigenvalues are $\lambda_n=\frac{n^2\pi^2}{L^2}$, for integers $n\geq 0$, where eigenfunctions are $\cos(\frac{n\pi}{L})X$ (we include the constant eigenfunction). Hence, let us pick solutions

$$X_n(x)=\cos(\frac{n\pi}{L}x) \quad \text{and} \quad X_0(x)=1.$$

The corresponding T_n must satisfy the equation

$$T_n'(t)+\frac{n^2\pi^2}{L^2}kT_n(t)=0.$$

For $n\geq 1$, as before,

$$T_n(t)=e^{\frac{-n^2\pi^2}{L^2}kt}.$$

For $n=0$, we have $T_0'(t)=0$ and hence $T_0(t)=1$. Our building-block solutions will be

$$u_n(x,t)=X_n(x)T_n(t)=\cos\left(\frac{n\pi}{L}x\right)e^{\frac{-n^2\pi^2}{L^2}kt},$$

and

$$u_0(x,t)=1.$$

We note that $u_n(x,0)=\cos\left(\frac{n\pi}{L}x\right)$. Let us write f using the cosine series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi}{L}x\right).$$

That is, we find the Fourier series of the even periodic extension of $f(x)$.

We use superposition to write the solution as

$$u(x,t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nu_n(x,t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi}{L}x\right)e^{\frac{-n^2\pi^2}{L^2}kt}.$$

Example 6.6.3

Let us try the same equation as before, but for insulated ends. We are solving the following PDE problem

$$\begin{aligned} u_t &= 0.003u_{xx}, \\ u_x(0,t) &= u_x(1,t)=0, \\ u(x,0) &= 50x(1-x) \quad \text{for } 0<x<1. \end{aligned} \tag{6.6.4}$$

For this problem, we must find the cosine series of $u(x,0)$. For $0<x<1$ we have

$$50x(1-x)=\frac{25}{3}+\sum_{\substack{n=2\\ n \text{ even}}}^{\infty}\left(\frac{-200}{\pi^2n^2}\right)\cos(n\pi x).$$

The calculation is left to the reader. Hence, the solution to the PDE problem, plotted in Figure $6.6.5$, is given by the series

$$u(x,t)=\frac{25}{3}+\sum_{\substack{n=2\\ n \text{ even}}}^{\infty}\left(\frac{-200}{\pi^2n^2}\right)\cos(n\pi x)e^{-n^2\pi^2 0.003t}.$$

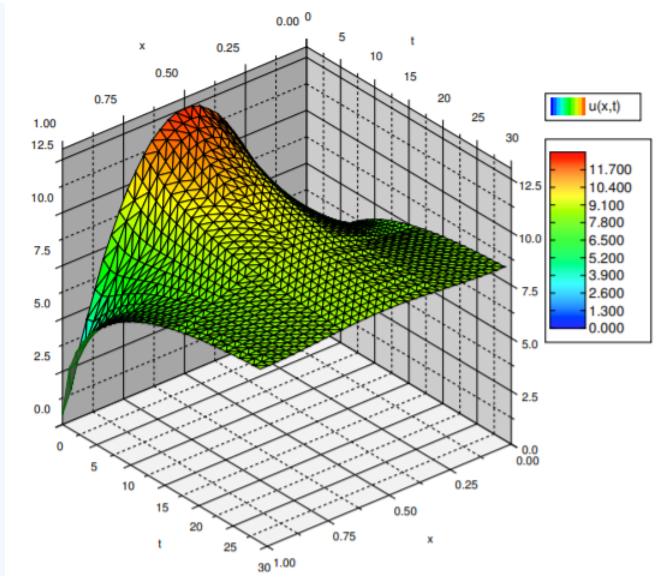

Figure 6.6.5: Plot of the temperature of the insulated wire at position x at time t.

Note in the graph that the temperature evens out across the wire. Eventually, all the terms except the constant die out, and you will be left with a uniform temperature of \(\frac{25}{3} \approx{8.33}\) along the entire length of the wire.

Let us expand on the last point. The constant term in the series is

$$\frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx.$$

In other words, $\frac{a_0}{2}$ is the average value of $f(x)$, that is, the average of the initial temperature. As the wire is insulated everywhere, no heat can get out, no heat can get in. So the temperature tries to distribute evenly over time, and the average temperature must always be the same, in particular it is always $\frac{a_0}{2}$. As time goes to infinity, the temperature goes to the constant $\frac{a_0}{2}$ everywhere.

6.7: One Dimensional Wave Equation

Imagine we have a tensioned guitar string of length L. Suppose we only consider vibrations in one direction. That is, let x denote the position along the string, let t denote time, and let y denote the displacement of the string from the rest position. See Figure 6.7.1.

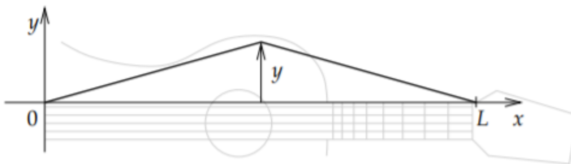

Figure 6.7.1: Vibrating string of length L, x is position, y is displacement.

The equation that governs this setup is the so-called *one-dimensional wave equation*:

$$y_{tt} = a^2 y_{xx},$$

for some constant $a > 0$. The intuition is similar to the heat equation, replacing velocity with acceleration: the acceleration at a specific point is proportional to the second derivative of the shape of the string. In other words when the string is concave down then u_{xx} is negative and the string wants to accelerate downwards, so u_{tt} should be negative. And vice versa. The wave equation is an example of a hyperbolic PDE.

Assume that the ends of the string are fixed in place:

$$y(0,t) = 0 \quad \text{and} \quad y(L,t) = 0.$$

Note that we have two conditions along the x axis as there are two derivatives in the x direction.

There are also two derivatives along the t direction and hence we need two further conditions here. We need to know the initial position and the initial velocity of the string. That is, for some known functions $f(x)$ and $g(x)$, we impose

$$y(x,0) = f(x) \quad \text{and} \quad y_t(x,0) = g(x).$$

As the equation is again linear, superposition works just as it did for the heat equation. And again we will use separation of variables to find enough building-block solutions to get the overall solution. There is one change however. It will be easier to solve two separate problems and add their solutions.

The two problems we will solve are

$$\begin{aligned} & w_{tt} = a^2 w_{xx}, & \\ & w(0,t) = w(L,t) = 0, & \\ & w(x,0) = 0 & \text{for } 0 < x < L, \\ & w_t(x,0) = g(x) & \text{for } 0 < x < L, \end{aligned} \tag{6.7.1}$$

and

$$\begin{aligned} & z_{tt} = a^2 z_{xx}, & \\ & z(0,t) = z(L,t) = 0, & \\ & z(x,0) = f(x) & \text{for } 0 < x < L, \\ & z_t(x,0) = 0 & \text{for } 0 < x < L. \end{aligned} \tag{6.7.2}$$

The principle of superposition implies that $y = w + z$ solves the wave equation and furthermore $y(x,0) = w(x,0) + z(x,0) = f(x)$ and $y_t(x,0) = w_t(x,0) + z_t(x,0) = g(x)$. Hence, y is a solution to

$$\begin{aligned} & y_{tt} = a^2 y_{xx}, & \\ & y(0,t) = y(L,t) = 0, & \\ & y(x,0) = f(x) & \text{for } 0 < x < L, \\ & y_t(x,0) = g(x) & \text{for } 0 < x < L. \end{aligned} \tag{6.7.3}$$

The reason for all this complexity is that superposition only works for homogeneous conditions such as $y(0,t)=y(L,t)=0$, $y(x,0)=0$, or $y_t(x,0)=0$. Therefore, we will be able to use the idea of separation of variables to find many building-block solutions solving all the homogeneous conditions. We can then use them to construct a solution solving the remaining nonhomogeneous condition.

Let us start with (6.7.1). We try a solution of the form $w(x,t)=X(x)T(t)$ again. We plug into the wave equation to obtain

$$X(x)T''(t)=a^2X''(x)T(t).$$

Rewriting we get

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}.$$

Again, left hand side depends only on t and the right hand side depends only on x. Therefore, both equal a constant, which we will denote by $-\lambda$.

$$\frac{T''(t)}{a^2T(t)}=-\lambda=\frac{X''(x)}{X(x)}.$$

We solve to get two ordinary differential equations

$$\begin{aligned}X''(x)+\lambda X(x)&=0,\\ T''(t)+\lambda a^2T(t)&=0.\end{aligned}\tag{6.7.4}$$

The conditions $0=w(0,t)=X(0)T(t)$ implies $X(0)=0$ and $w(L,t)=0$ implies that $X(L)=0$. Therefore, the only nontrivial solutions for the first equation are when $\lambda=\lambda_n=\frac{n^2\pi^2}{L^2}$ and they are

$$X_n(x)=\sin\left(\frac{n\pi}{L}x\right).$$

The general solution for T for this particular λ_n is

$$T_n(t)=A\cos\left(\frac{n\pi a}{L}t\right)+B\sin\left(\frac{n\pi a}{L}t\right).$$

We also have the condition that $w(x,0)=0$ or $X(x)T(0)=0$. This implies that $T(0)=0$, which in turn forces $A=0$. It is convenient to pick $B=\frac{L}{n\pi a}$ (you will see why in a moment) and hence

$$T_n(t)=\frac{L}{n\pi a}\sin\left(\frac{n\pi a}{L}t\right).$$

Our building-block solutions are

$$w_n(x,t)=\frac{L}{n\pi a}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right).$$

We differentiate in t, that is

$$\frac{\partial w_n}{\partial t}(x,t)=\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right).$$

Hence,

$$\frac{\partial w_n}{\partial t}(x,0)=\sin\left(\frac{n\pi}{L}x\right).$$

We expand $g(x)$ in terms of these sines as

$$g(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{L}x\right).$$

Using superposition we can just write down the solution to (6.7.1) as a series

$$w(x,t) = \sum_{n=1}^{\infty} b_n w_n(x,t) = \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi a}{L}t\right).$$

Exercise 6.7.1

Check that $w(x,0) = 0$ and $w_t(x,0) = g(x)$.

Similarly we proceed to solve (6.7.2). We again try $z(x,y) = X(x)T(t)$. The procedure works exactly the same at first. We obtain

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ T''(t) + \lambda a^2 T(t) &= 0. \end{aligned} \tag{6.7.5}$$

and the conditions $X(0) = 0$, $X(L) = 0$. So again $\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}$ and

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

This time the condition on T is $T'(0) = 0$. Thus we get that $B = 0$ and we take

$$T_n(t) = \cos\left(\frac{n\pi a}{L}t\right).$$

Our building-block solution will be

$$z_n(x,t) = \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right).$$

As $z_n(x,0) = \sin\left(\frac{n\pi}{L}x\right)$, we expand $f(x)$ in terms of these sines as

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right).$$

And we write down the solution to (6.7.2) as a series

$$z(x,t) = \sum_{n=1}^{\infty} c_n z_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right).$$

Exercise 6.7.2

Fill in the details in the derivation of the solution of (6.7.2). Check that the solution satisfies all the side conditions.

Putting these two solutions together, let us state the result as a theorem.

Theorem 6.7.1

Take the equation

$$\begin{aligned} &y_{tt} = a^2 y_{xx}, \\ &y(0,t) = y(L,t) = 0, \\ &y(x,0) = f(x) && \text{for } 0 < x < L, \\ &y_t(x,0) = g(x) && \text{for } 0 < x < L, \end{aligned} \tag{6.7.6}$$

where

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right),$$

and

$$g(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

Then the solution $y(x,t)$ can be written as a sum of the solutions of (4.7.4) and (4.7.5). In other words,

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right) + c_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right) \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[b_n \frac{L}{n\pi a}\sin\left(\frac{n\pi a}{L}t\right) + c_n \cos\left(\frac{n\pi a}{L}t\right)\right]. \end{aligned} \tag{6.7.7}$$

Example 6.7.1

Let us try a simple example of a plucked string. Suppose that a string of length 2 is plucked in the middle such that it has the initial shape given in Figure 6.7.2. That is

$$f(x)=\begin{cases} 0.1x & \text{if } 0 \le x \le 1, \\ 0.1(2-x) & \text{if } 1 < x \le 2 \end{cases}.$$

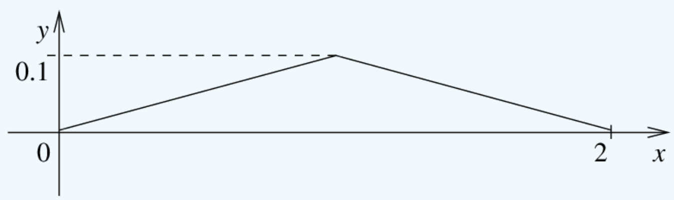

Figure 6.7.2: Initial shape of a plucked string from Example 6.7.1.

The string starts at rest ($g(x)=0$). Suppose that $a=1$ in the wave equation for simplicity. In other words, we wish to solve the problem:

$$\begin{aligned} &y_{tt}=y_{xx}, \\ &y(0,t)=y(2,t)=0, \\ &y(x,0)=f(x) \quad \text{and} \quad y_t(x,0)=0. \end{aligned} \tag{6.7.8}$$

We leave it to the reader to compute the sine series of $f(x)$. The series will be

$$f(x)=\sum_{n=1}^{\infty} \frac{0.8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}x\right).$$

Note that $\sin\left(\frac{n\pi}{2}\right)$ is the sequence $1,0,-1,0,1,0,-1,\ldots$ for $n=1,2,3,4,\ldots$. Therefore,

$$f(x)=\frac{0.8}{\pi^2}\sin\left(\frac{\pi}{2}x\right)-\frac{0.8}{9\pi^2}\sin\left(\frac{3\pi}{2}x\right)+\frac{0.8}{25\pi^2}\sin\left(\frac{5\pi}{2}x\right)-\cdots$$

The solution $y(x,t)$ is given by

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} \frac{0.8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}x\right)\cos\left(\frac{n\pi}{2}t\right) \\ &= \sum_{m=1}^{\infty} \frac{0.8(-1)^{m+1}}{(2m-1)^2\pi^2}\sin\left(\frac{(2m-1)\pi}{2}x\right)\cos\left(\frac{(2m-1)\pi}{2}t\right) \\ &= \frac{0.8}{\pi^2}\sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}t\right)-\frac{0.8}{9\pi^2}\sin\left(\frac{3\pi}{2}x\right)\cos\left(\frac{3\pi}{2}t\right) \\ &\qquad +\frac{0.8}{25\pi^2}\sin\left(\frac{5\pi}{2}x\right)\cos\left(\frac{5\pi}{2}t\right)-\cdots \end{aligned} \tag{6.7.9}$$

See Figure 6.7.3 for a plot for $0<t<3$. Notice that unlike the heat equation, the solution does not become "smoother," the "sharp edges" remain. We will see the reason for this behavior in the next section where we derive the solution to the wave equation in a different way.

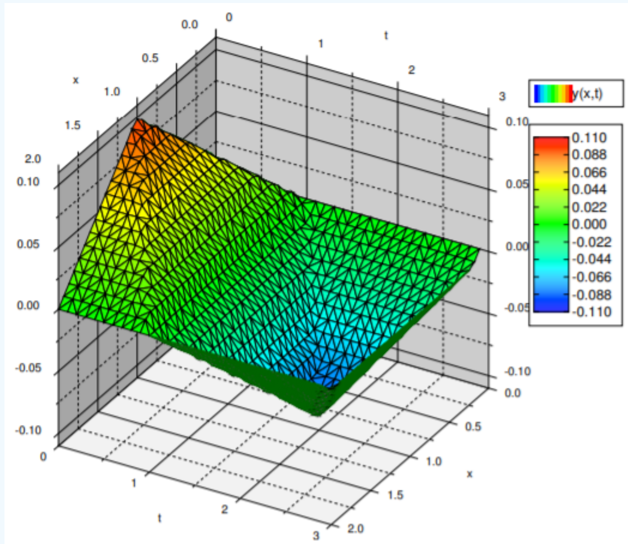

Figure 6.7.3: Shape of the plucked string for $0 < t < 3$.

Make sure you understand what the plot, such as the one in the figure, is telling you. For each fixed t, you can think of the function $y(x,t)$ as just a function of x. This function gives you the shape of the string at time t. See Figure 6.7.4 for plots of at y as a function of x at several different values of t. On this plot you can see the sharp edges remaining much better.

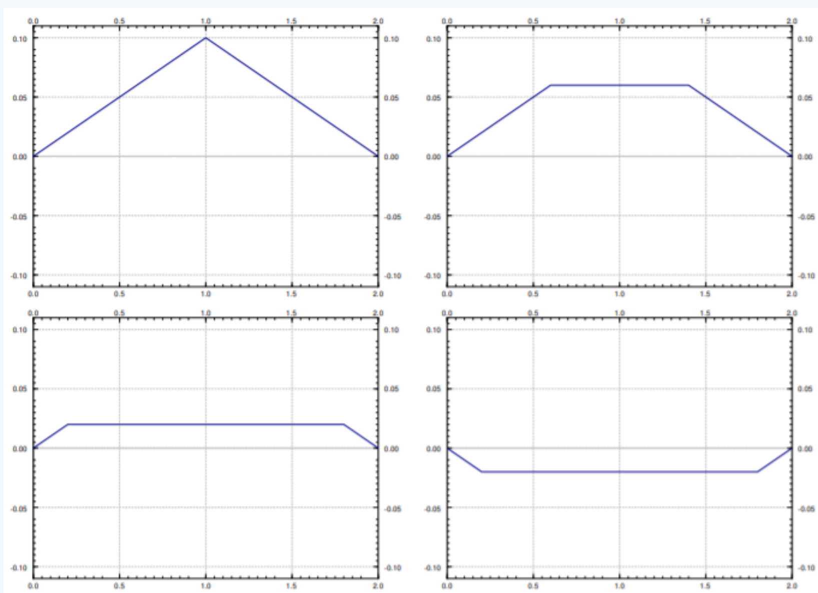

Figure 6.7.4: Plucked string for $t = 0$, $t = 0.4$, $t = 0.8$, and $t = 1.2$.

One thing to take away from all this is how a guitar sounds. Notice that the (angular) frequencies that come up in the solution are $n\frac{\pi a}{L}$. That is, there is a certain base *fundamental frequency* $\frac{\pi a}{L}$, and then we also get all the multiples of this frequency, which in music are called the *overtones*. Which overtones appear and with what amplitude is what musicians call the *timbre* of the note. Mathematicians usually call this the *spectrum*. Because all the frequencies are multiples of one frequency (the fundamental) we get a nice pleasing sound.

The fundamental frequency $\frac{\pi a}{L}$ increases as we decrease length L. That is, if we place a finger on the fingerboard and then pluck a string we get a higher note. The constant a is given by

$$a = \sqrt{\frac{T}{\rho}},$$

where T is tension and ρ is the linear density of the string. Tightening the string (turning the tuning peg on a guitar) increases a and hence produces a higher fundamental frequency (a higher note). On the other hand using a heavier string reduces a and produces a lower fundamental frequency (a lower note). A bass guitar has longer thicker strings, while a ukulele has short strings made of lighter material.

Something rather interesting is the almost symmetry between space and time. In its simplest form we see this symmetry in the solutions

$$\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right).$$

Except for the a, time and space are just the same.

In general, the solution for a fixed x is a Fourier series in t, for a fixed t it is a Fourier series in x, and the coefficients are related. If the shape $f(x)$ or the initial velocity have lots of corners, then the sound wave will have lots of corners. That is because the Fourier coefficients of the initial shape decay to zero (as $n \to \infty$) at the same rate as the Fourier coefficients of the wave in time (for some fixed x). So if you use a sharp object to pick the string, you get a sharper sound with lots of high frequency components, while if you use your thumb, you get a softer sound without so many high overtones. Similarly if you pluck close to the bridge, you are getting a pluck that looks more like the sawtooth, and you get an even sharper sound.

In fact, if you look at the formula for the solution, you see that for any fixed x we get an almost arbitrary Fourier series in t, everything except the constant term. You can essentially obtain any sound you want by plucking the string in just the right way. Of course we are considering an ideal string of no stiffness and no air resistance. Those variables clearly impact the sound as well.

6.8: D’Alembert Solution of The Wave Equation

We have solved the wave equation by using Fourier series. But it is often more convenient to use the so-called *d’Alembert solution to the wave equation*.[1] While this solution can be derived using Fourier series as well, it is really an awkward use of those concepts. It is easier and more instructive to derive this solution by making a correct change of variables to get an equation that can be solved by simple integration.

Suppose we have the wave equation

$$y_{tt} = a^2 y_{xx}. \tag{6.8.1}$$

We wish to solve the equation (6.8.1) given the conditions

$$\begin{aligned} y(0,t) = y(L,t) &= 0 && \text{for all } t, \\ y(x,0) &= f(x) && 0 < x < L, && (6.8.2) \\ y_t(x,0) &= g(x) && 0 < x < L. && (6.8.3) \end{aligned}$$

6.8.1: Change of Variables

We will transform the equation into a simpler form where it can be solved by simple integration. We change variables to $\xi = x - at$, $\eta = x + at$. The chain rule says:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -a\frac{\partial}{\partial \xi} + a\frac{\partial}{\partial \eta}. \end{aligned} \tag{6.8.4}$$

We compute

$$\begin{aligned} y_{xx} &= \frac{\partial^2 y}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}\right) = \frac{\partial^2 y}{\partial \xi^2} + 2\frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial \eta^2}, \\ y_{tt} &= \frac{\partial^2 y}{\partial^2} = \left(-a\frac{\partial}{\partial \xi} + a\frac{\partial}{\partial \eta}\right)\left(-a\frac{\partial y}{\partial \xi} + a\frac{\partial y}{\partial \eta}\right) = a^2\frac{\partial^2 y}{\partial \xi^2} - 2a^2\frac{\partial^2 y}{\partial \xi \partial \eta} + a^2\frac{\partial^2 y}{\partial \eta^2}. \end{aligned} \tag{6.8.5}$$

In the above computations, we used the fact from calculus that $\frac{\partial^2 y}{\partial \xi \partial \eta} = \frac{\partial^2 y}{\partial \eta \partial \xi}$. We plug what we got into the wave equation,

$$0 = a^2 y_{xx} - y_{tt} = 4a^2 \frac{\partial^2 y}{\partial \xi \partial \eta} = 4a^2 y_{\xi\eta}.$$

Therefore, the wave equation (6.8.1) transforms into $y_{\xi\eta} = 0$. It is easy to find the general solution to this equation by integrating twice. Keeping ξ constant, we integrate with respect to η first[2] and notice that the constant of integration depends on ξ; for each ξ we might get a different constant of integration. We get $y_\xi = C(\xi)$. Next, we integrate with respect to ξ and notice that the constant of integration must depend on η. Thus, $y = \int C(\xi)d\xi + B(\eta)$. The solution must, therefore, be of the following form for some functions $A(\xi)$ and $B(\eta)$:

$$y = A(\xi) + B(\eta) = A(x - at) + B(x + at).$$

The solution is a superposition of two functions (waves) traveling at speed a in opposite directions. The coordinates ξ and η are called the *characteristic coordinates*, and a similar technique can be applied to more complicated hyperbolic PDE. And in fact, in Section 1.9 it is used to solve first order linear PDE. Basically, to solve the wave equation (or more general hyperbolic equations) we find certain characteristic curves along which the equation is really just an ODE, or a pair of ODEs. In this case these are the curves where ξ and η are constant.

6.8.2: D’Alembert’s Formula

We know what any solution must look like, but we need to solve for the given side conditions. We will just give the formula and see that it works. First let $F(x)$ denote the odd extension of $f(x)$, and let $G(x)$ denote the odd extension of $g(x)$. Define

$$A(x) = \frac{1}{2}F(x) - \frac{1}{2a}\int_0^x G(s)ds, \quad B(x) = \frac{1}{2}F(x) + \frac{1}{2a}\int_0^x G(s)ds.$$

We claim this $A(x)$ and $B(x)$ give the solution. Explicitly, the solution is $y(x,t) = A(x-at) + B(x+at)$ or in other words:

$$\begin{aligned} y(x,t) &= \frac{1}{2}F(x-at) - \frac{1}{2a}\int_0^{x-at} G(s)ds + \frac{1}{2}F(x+at) + \frac{1}{2a}\int_0^{x+at} G(s)ds \qquad (6.8.6) \\ &= \frac{F(x-at)+F(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} G(s)ds. \qquad (6.8.7) \end{aligned}$$

Let us check that the d'Alembert formula really works.

$$y(x,0) = \frac{1}{2}F(x) - \frac{1}{2a}\int_0^{x} G(s)ds + \frac{1}{2}F(x) + \frac{1}{2a}\int_0^{x} G(s)ds = F(x).$$

So far so good. Assume for simplicity F is differentiable. And we use the first form of (6.8.6) as it is easier to differentiate. By the fundamental theorem of calculus we have

$$y_t(x,t) = \frac{-a}{2}F'(x-at) + \frac{1}{2}G(x-at) + \frac{a}{2}F'(x+at) + \frac{1}{2}G(x+at).$$

So

$$y_t(x,0) = \frac{-a}{2}F'(x) + \frac{1}{2}G(x) + \frac{a}{2}F'(x) + \frac{1}{2}G(x) = G(x).$$

Yay! We're smoking now. OK, now the boundary conditions. Note that $F(x)$ and $G(x)$ are odd. Also $\int_0^x G(s)ds$ is an even function of x because $G(x)$ is odd (to see this fact, do the substitution $s = -v$). So

$$\begin{aligned} y(0,t) &= \frac{1}{2}F(-at) - \frac{1}{2a}\int_0^{-at} G(s)ds + \frac{1}{2}F(at) + \frac{1}{2a}\int_0^{at} G(s)ds \\ &= \frac{-1}{2}F(at) - \frac{1}{2a}\int_0^{at} G(s)ds + \frac{1}{2}F(at) + \frac{1}{2a}\int_0^{at} G(s)ds = 0. \end{aligned} \qquad (6.8.8)$$

Note that $F(x)$ and $G(x)$ are $2L$ periodic. We compute

$$\begin{aligned} y(L,t) &= \frac{1}{2}F(L-at) - \frac{1}{2a}\int_0^{L-at} G(s)ds + \frac{1}{2}F(L+at) + \frac{1}{2a}\int_0^{L+at} G(s)ds \\ &= \frac{1}{2}F(-L-at) - \frac{1}{2a}\int_0^{L} G(s)ds - \frac{1}{2a}\int_0^{-at} G(s)ds + \\ &= \frac{1}{2}F(L+at) + \frac{1}{2a}\int_0^{L} G(s)ds + \frac{1}{2a}\int_0^{at} G(s)ds \\ &= \frac{-1}{2}F(L+at) - \frac{1}{2a}\int_0^{at} G(s)ds + \frac{1}{2}F(L+at) + \frac{1}{2a}\int_0^{at} G(s)ds = 0. \end{aligned} \qquad (6.8.9)$$

And voilà, it works.

Example 6.8.1

D'Alembert says that the solution is a superposition of two functions (waves) moving in the opposite direction at "speed" a. To get an idea of how it works, let us work out an example. Consider the simpler setup

$$\begin{aligned} y_{tt} &= y_{xx}, \\ y(0,t) &= y(1,t) = 0, \\ y(x,0) &= f(x), \\ y_t(x,0) &= 0. \end{aligned} \qquad (6.8.10)$$

Here $f(x)$ is an impulse of height 1 centered at $x = 0.5$:

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 0.45, \\ 20(x-0.45) & \text{if} \quad 0.45 \le x < 0.5, \\ 20(0.55-x) & \text{if} \quad 0.5 \le x < 0.55 \\ 0 & \text{if} \quad 0.55 \le x \le 1. \end{cases}$$

The graph of this impulse is the top left plot in Figure $6.8.1$.

Let $F(x)$ be the odd periodic extension of $f(x)$. Then from (6.8.6) we know that the solution is given as

$$y(x,t) = \frac{F(x-t)+F(x+t)}{2}.$$

It is not hard to compute specific values of $y(x,t)$. For example, to compute $y(0.1,0.6)$ we notice $x-t=-0.5$ and $x+t=0.7$. Now $F(-0.5)=-f(0.5)=-20(0.55-0.5)=-1$ and $F(0.7)=f(0.7)=0$. Hence $y(0.1,0.6)=\frac{-1+0}{2}=-0.5$. As you can see the d'Alembert solution is much easier to actually compute and to plot than the Fourier series solution. See Figure $6.8.1$ for plots of the solution y for several different t.

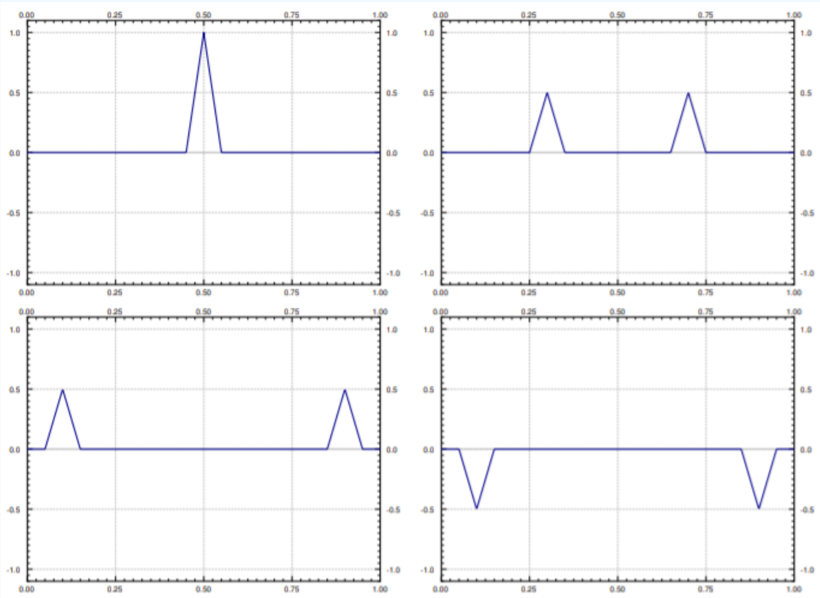

Figure 6.8.1: Plot of the d'Alembert solution for $t=0$, $t=0.2$, $t=0.4$, and $t=0.6$.

6.8.3: Another Way to Solve for the Side Conditions

It is perhaps easier and more useful to memorize the procedure rather than the formula itself. The important thing to remember is that a solution to the wave equation is a superposition of two waves traveling in opposite directions. That is,

$$y(x,t) = A(x-at)+B(x+at).$$

If you think about it, the exact formulas for A and B are not hard to guess once you realize what kind of side conditions $y(x,t)$ is supposed to satisfy. Let us give the formula again, but slightly differently. Best approach is to do this in stages. When $g(x)=0$ (and hence $G(x)=0$) we have the solution

$$\frac{F(x-at)+F(x+at)}{2}.$$

On the other hand, when $f(x)=0$ (and hence $F(x)=0$), we let

$$H(x) = \int_0^x G(s)ds.$$

The solution in this case is

$$\frac{1}{2a}\int_{x-at}^{x+at} G(s)ds = \frac{-H(x-at)+H(x+at)}{2a}.$$

By superposition we get a solution for the general side conditions (6.8.2) (when neither $f(x)$ nor $g(x)$ are identically zero).

$$y(x,t) = \frac{F(x-at)+F(x+at)}{2} + \frac{-H(x-at)+H(x+at)}{2a}. \tag{6.8.11}$$

Do note the minus sign before the H, and the a in the second denominator.

? Exercise 6.8.1

Check that the new formula (6.8.11) satisfies the side conditions (6.8.2).

Warning: Make sure you use the odd extensions $F(x)$ and $G(x)$, when you have formulas for $f(x)$ and $g(x)$. The thing is, those formulas in general hold only for $0 < x < L$, and are not usually equal to $F(x)$ and $G(x)$ for other x.

6.8.4: Remarks

Let us remark that the formula $y(x,t) = A(x-at) + B(x+at)$ is the reason why the solution of the wave equation doesn't get as time goes on, that is, why in the examples where the initial conditions had corners, the solution also has corners at every time t.

The corners bring us to another interesting remark. Nobody ever notices at first that our example solutions are not even differentiable (they have corners): In Example $6.8.1$ above, the solution is not differentiable whenever $x = t + 0.5$ or $x = -t + 0.5$ for example. Really to be able to compute u_{xx} or u_{tt}, you need not one, but two derivatives. Fear not, we could think of a shape that is very nearly $F(x)$ but does have two derivatives by rounding the corners a little bit, and then the solution would be very nearly $\frac{F(x-t)+F(x+t)}{2}$ and nobody would notice the switch.

One final remark is what the d'Alembert solution tells us about what part of the initial conditions influence the solution at a certain point. We can figure this out by Let us suppose that the string is very long (perhaps infinite) for simplicity. Since the solution at time t is

$$y(x,t) = \frac{F(x-at)+F(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} G(s)\,ds,$$

we notice that we have only used the initial conditions in the interval $[x-at, x+at]$. These two endpoints are called the *wavefronts*, as that is where the wave front is given an initial ($t = 0$) disturbance at x. So if $a = 1$, an observer sitting at $x = 0$ at time $t = 1$ has only seen the initial conditions for x in the range $[-1, 1]$ and is blissfully unaware of anything else. This is why for example we do not know that a supernova has occurred in the universe until we see its light, millions of years from the time when it did in fact happen.

6.8.5: Footnotes

[1] Named after the French mathematician Jean le Rond d'Alembert (1717 – 1783).

[2] There is nothing special about η, you can integrate with ξ first, if you wish.

6.8.6: Contributors and Attributions

- ◦ Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

6.9: Steady State Temperature and the Laplacian

Suppose we have an insulated wire, a plate, or a 3-dimensional object. We apply certain fixed temperatures on the ends of the wire, the edges of the plate, or on all sides of the 3-dimensional object. We wish to find out what is the steady state temperature distribution. That is, we wish to know what will be the temperature after long enough period of time.

We are really looking for a solution to the heat equation that is not dependent on time. Let us first solve the problem in one space variable. We are looking for a function u that satisfies

$$u_t = k u_{xx},$$

but such that $u_t = 0$ for all x and t. Hence, we are looking for a function of x alone that satisfies $u_{xx} = 0$. It is easy to solve this equation by integration and we see that $u = Ax + B$ for some constants A and B.

Suppose we have an insulated wire, and we apply constant temperature T_1 at one end (say where $x = 0$) and T_2 on the other end (at $x = L$ where L is the length of the wire). Then our steady state solution is

$$u(x) = \frac{T_2 - T_1}{L} x + T_1.$$

This solution agrees with our common sense intuition with how the heat should be distributed in the wire. So in one dimension, the steady state solutions are basically just straight lines.

Things are more complicated in two or more space dimensions. Let us restrict to two space dimensions for simplicity. The heat equation in two space variables is

$$u_t = k(u_{xx} + u_{yy}), \tag{6.9.1}$$

or more commonly written as $u_t = k\Delta u$ or $u_t = k\nabla^2 u$. Here the Δ and ∇^2 symbols mean $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. We will use Δ from now on. The reason for using such a notation is that you can define Δ to be the right thing for any number of space dimensions and then the heat equation is always $u_t = k\Delta u$. The operator Δ is called the **Laplacian**.

OK, now that we have notation out of the way, let us see what does an equation for the steady state solution look like. We are looking for a solution to Equation (6.9.1) that does not depend on t, or in other words $u_t = 0$. Hence we are looking for a function $u(x, y)$ such that

$$\Delta u = u_{xx} + u_{yy} = 0.$$

This equation is called the *Laplace equation*[1]. Solutions to the Laplace equation are called **harmonic functions** and have many nice properties and applications far beyond the steady state heat problem.

Harmonic functions in two variables are no longer just linear (plane graphs). For example, you can check that the functions $x^2 - y^2$ and xy are harmonic. However, if you remember your multi-variable calculus we note that if u_{xx} is positive, u is concave up in the x direction, then u_{yy} must be negative and u must be concave down in the y direction. Therefore, a harmonic function can never have any “hilltop” or “valley” on the graph. This observation is consistent with our intuitive idea of steady state heat distribution; the hottest or coldest spot will not be inside.

Commonly the Laplace equation is part of a so-called *Dirichlet problem*[2]. That is, we have a region in the xy-plane and we specify certain values along the boundaries of the region. We then try to find a solution u defined on this region such that u agrees with the values we specified on the boundary.

For simplicity, we consider a rectangular region. Also for simplicity we specify boundary values to be zero at 3 of the four edges and only specify an arbitrary function at one edge. As we still have the principle of superposition, we can use this simpler solution to derive the general solution for arbitrary boundary values by solving 4 different problems, one for each edge, and adding those solutions together. This setup is left as an exercise.

We wish to solve the following problem. Let h and w be the height and width of our rectangle, with one corner at the origin and lying in the first quadrant.

$$\Delta u = 0, \tag{6.9.2}$$
$$u(0,y) = 0 \qquad \text{for } 0 < y < h, \tag{6.9.3}$$
$$u(x,h) = 0 \qquad \text{for } 0 < x < w, \tag{6.9.4}$$
$$u(w,y) = 0 \qquad \text{for } 0 < y < h, \tag{6.9.5}$$
$$u(x,0) = f(x) \qquad \text{for } 0 < x < w. \tag{6.9.6}$$

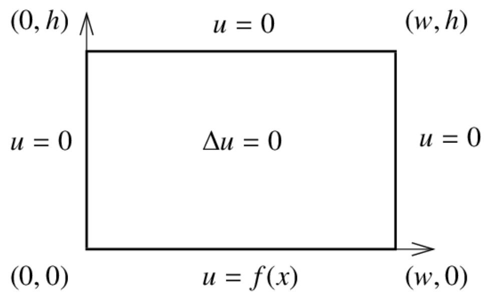

Figure 6.9.1

The method we apply is separation of variables. Again, we will come up with enough building-block solutions satisfying all the homogeneous boundary conditions (all conditions except (6.9.6)). We notice that superposition still works for the equation and all the homogeneous conditions. Therefore, we can use the Fourier series for $f(x)$ to solve the problem as before.

We try $u(x,y) = X(x)Y(y)$. We plug u into the equation to get

$$X''Y + XY'' = 0.$$

We put the Xs on one side and the Ys on the other to get

$$-\frac{X''}{X} = \frac{Y''}{Y}.$$

The left hand side only depends on x and the right hand side only depends on y. Therefore, there is some constant λ such that $\lambda = \frac{-X''}{X} = \frac{Y''}{Y}$. And we get two equations

$$X'' + \lambda X = 0,$$
$$Y'' - \lambda Y = 0.$$

Furthermore, the homogeneous boundary conditions imply that $X(0) = X(w) = 0$ and $Y(h) = 0$. Taking the equation for X we have already seen that we have a nontrivial solution if and only if $\lambda = \lambda_n = \frac{n^2\pi^2}{w^2}$ and the solution is a multiple of

$$X_n(x) = \sin\left(\frac{n\pi}{w}x\right).$$

For these given λ_n, the general solution for Y (one for each n) is

$$Y_n(y) = A_n \cosh\left(\frac{n\pi}{w}y\right) + B_n \sinh\left(\frac{n\pi}{w}y\right). \tag{6.9.7}$$

We only have one condition on Y_n and hence we can pick one of A_n or B_n to be something convenient. It will be useful to have $Y_n(0) = 1$, so we let $A_n = 1$. Setting $Y_n(h) = 0$ and solving for B_n we get that

$$B_n = \frac{-\cosh\left(\frac{n\pi h}{w}\right)}{\sinh\left(\frac{n\pi h}{w}\right)}.$$

After we plug the A_n and B_n we into (6.9.7) and simplify, we find

$$Y_n(y) = \frac{\sinh\left(\frac{n\pi(h-y)}{w}\right)}{\sinh\left(\frac{n\pi h}{w}\right)}.$$

We define $u_n(x,y) = X_n(x)Y_n(y)$. And note that u_n satisfies (6.9.2) - (6.9.5).

Observe that

$$u_n(x,0) = X_n(x)Y_n(0) = \sin\left(\frac{n\pi}{w}x\right).$$

Suppose

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{w}\right).$$

Then we get a solution of (6.9.2) - (6.9.6) of the following form.

$$u(x,y) = \sum_{n=1}^{\infty} b_n u_n(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{w}x\right)\left(\frac{\sinh\left(\frac{n\pi(h-y)}{w}\right)}{\sinh\left(\frac{n\pi h}{w}\right)}\right).$$

As u_n satisfies Equation (6.9.2) - (6.9.5) and any linear combination (finite or infinite) of u_n must also satisfy (6.9.2) - (6.9.5), we see that u must satisfy Equations (6.9.2) - (6.9.5). By plugging in $y = 0$ it is easy to see that u satisfies (6.9.6) as well.

Example 6.9.1

Suppose that we take $w = h = \pi$ and we let $f(x) = \pi$. We compute the sine series for the function π(we will get the square wave). We find that for $0 < x < \pi$ we have

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n}\sin(nx).$$

Therefore the solution $u(x,y)$, see Figure 6.9.2, to the corresponding Dirichlet problem is given as

$$u(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n}\sin(nx)\left(\frac{\sinh(n(\pi-y))}{\sinh(n\pi)}\right).$$

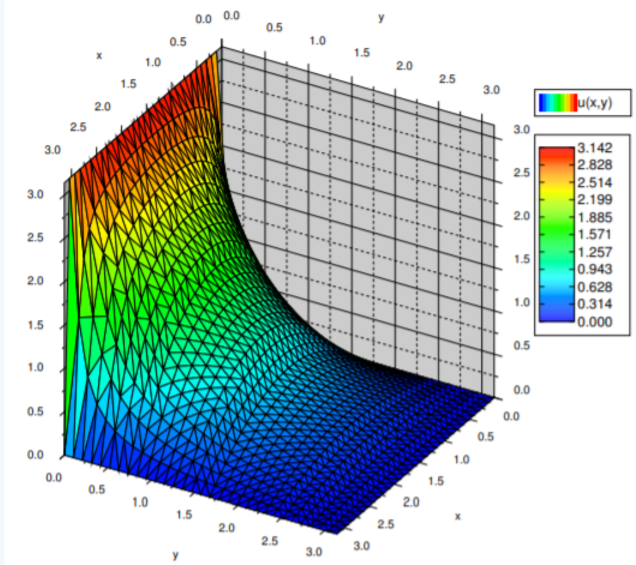

Figure 6.9.2: Steady state temperature of a square plate, three sides held at zero and one side held at π.

This scenario corresponds to the steady state temperature on a square plate of width π with 3 sides held at 0 degrees and one side held at π degrees. If we have arbitrary initial data on all sides, then we solve four problems, each using one piece of nonhomogeneous data. Then we use the principle of superposition to add up all four solutions to have a solution to the original problem.

A different way to visualize solutions of the Laplace equation is to take a wire and bend it so that it corresponds to the graph of the temperature above the boundary of your region. Cut a rubber sheet in the shape of your region—a square in our case—and stretch it fixing the edges of the sheet to the wire. The rubber sheet is a good approximation of the graph of the solution to the Laplace equation with the given boundary data.

6.9.1: Footnotes

[1] Named after the French mathematician Pierre-Simon, marquis de Laplace (1749–1827).

[2] Named after the German mathematician Johann Peter Gustav Lejeune Dirichlet (1805–1859).

6.10: Dirichlet Problem in the Circle and the Poisson Kernel

6.10.1: Laplace in Polar Coordinates

A more natural setting for the Laplace equation $\Delta u = 0$ is the circle rather than the square. On the other hand, what makes the problem somewhat more difficult is that we need polar coordinates.

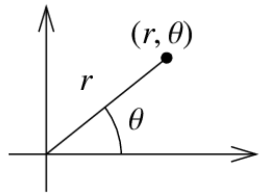

Figure 6.10.1

Recall that the polar coordinates for the (x, y)-plane are (r, θ):

$$x = r\cos\theta, \quad y = r\sin\theta,$$

where $r \geq 0$ and $-\pi < \theta < \pi$. So (x, y) is distance r from the origin at angle θ from the positive x-axis.

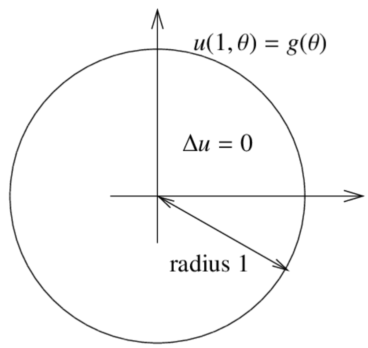

Figure 6.10.2

Now that we know our coordinates, let us give the problem we wish to solve. We have a circular region of radius 1, and we are interested in the Dirichlet problem for the Laplace equation for this region. Let $u(r, \theta)$ denote the temperature at the point (r, θ) in polar coordinates. We have the problem:

$$\Delta u = 0, \qquad \text{for } r < 1, \tag{6.10.1}$$

$$u(1, \theta) = g(\theta), \qquad \text{for } \pi < \theta \leq \pi. \tag{6.10.2}$$

The first issue we face is that we do not know what the Laplacian is in polar coordinates. Normally we would find u_{xx} and u_{yy} in terms of the derivatives in r and θ. We would need to solve for r and θ in terms of x and y. While this is certainly possible, it happens to be more convenient to work in reverse. Let us instead compute derivatives in r and θ in terms of derivatives in x and y and then solve. The computations are easier this way. First

$$\begin{aligned} x_r &= \cos\theta, & x_\theta &= -r\sin\theta, \\ y_r &= \sin\theta, & y_\theta &= r\cos\theta. \end{aligned} \tag{6.10.3}$$

Next by chain rule we obtain

$$\begin{aligned} u_r &= u_x x_r + u_y y_r = \cos(\theta) u_x + \sin(\theta) u_y, \\ u_{rr} &= \cos(\theta)(u_{xx} x_r + u_{xy} y_r) + \sin(\theta)(u_{yx} x_r + u_{yy} y_r) \\ &= \cos^2(\theta) u_{xx} + 2\cos(\theta)\sin(\theta) u_{xy} + \sin^2(\theta) u_{yy}. \end{aligned} \tag{6.10.4}$$

Similarly for the θ derivative. Note that we have to use product rule for the second derivative.

$$\begin{aligned} u_\theta &= u_x x_\theta + u_y y_\theta = -r\sin(\theta) u_x + r\cos(\theta) u_y, \\ u_{\theta\theta} &= -r\cos(\theta)(u_x) - r\sin(\theta)(u_{xx} x_\theta + u_{xy} y_\theta) - r\sin(\theta)(u_y) + r\cos(\theta)(u_{yx} x_\theta + u_{yy} y_\theta) \\ &= -r\cos(\theta) u_x - r\sin(\theta) u_y + r^2\sin^2(\theta) u_{xx} - r^2 2\sin(\theta)\cos(\theta) u_{xy} + r^2\cos^2(\theta) u_{yy}. \end{aligned} \tag{6.10.5}$$

Let us now try to solve for $u_{xx}+u_{yy}$. We start with $\frac{1}{r^2}u_{\theta\theta}$ to get rid of those pesky r^2. If we add u_{rr} and use the fact that $\cos^2(\theta)+\sin^2(\theta)=1$, we get

$$\frac{1}{r^2}u_{\theta\theta}+u_{rr}=u_{xx}+u_{yy}-\frac{1}{r}\cos(\theta)u_x-\frac{1}{r}\sin(\theta)u_y.$$

We're not quite there yet, but all we are lacking is $\frac{1}{r}u_r$. Adding it we obtain the *Laplacian in polar coordinates*:

$$\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r+u_{rr}=u_{xx}+u_{yy}=\Delta u.$$

Notice that the Laplacian in polar coordinates no longer has constant coefficients.

6.10.2: Series Solution

Let us separate variables as usual. That is let us try $u(r,\theta)=R(r)\Theta(\theta)$. Then

$$0=\Delta u=\frac{1}{r^2}R\Theta''+\frac{1}{r}R'\Theta+R''\Theta.$$

Let us put R on one side and Θ on the other and conclude that both sides must be constant.

$$\begin{aligned}\frac{1}{r^2}R\Theta'' &= -\left(\frac{1}{r}R'+R''\right)\Theta.\\ \frac{\Theta''}{\Theta} &= -\frac{rR'+r^2R''}{R}+-\lambda.\end{aligned}\tag{6.10.6}$$

We get two equations:

$$\begin{aligned}\Theta''+\lambda\Theta &=0,\\ r^2R''+rR'-\lambda R &=0.\end{aligned}\tag{6.10.7}$$

Let us first focus on Θ. We know that $u(r,\theta)$ ought to be 2π-periodic in θ, that is, $u(r,\theta)=u(r,\theta+2\pi)$. Therefore, the solution to $\Theta''+\lambda\Theta=0$ must be 2π-periodic. We have seen such a problem in Example 4.1.5. We conclude that $\lambda=n^2$ for a nonnegative integer $n=0,1,2,3,\ldots$. The equation becomes $\Theta''+n^2\Theta=0$. When $n=0$ the equation is just $\Theta''=0$, so we have the general solution $A\theta+B$. As Θ is periodic, $A=0$. For convenience let us write this solution as

$$\Theta_0=\frac{a_0}{2}$$

for some constant a_0. For positive n, the solution to $\Theta''+n^2\Theta=0$ is

$$\Theta_n=a_n\cos(n\theta)+b_n\sin(n\theta),$$

for some constants a_n and b_n.

Next, we consider the equation for R,

$$r^2R''+rR'-n^2R=0.$$

This equation has appeared in exercises before—we solved it in Exercise 2.E.2.1.6 and Exercise 2.E.1.7. The idea is to try a solution r^s and if that does not work out try a solution of the form $r^s\ln r$. When $n=0$ we obtain

$$R_0=Ar^0+Br^0\ln r=A+B\ln r,$$

and if $n>0$, we get

$$R_n=Ar^n+Br^{-n}.$$

The function $u(r,\theta)$ must be finite at the origin, that is, when $r=0$. Therefore, $B=0$ in both cases. Let us set $A=1$ in both cases as well, the constants in Θ_n will pick up the slack so we do not lose anything. Therefore let

$$R_0=1,\quad \text{and}\quad R_n=r^n.$$

Hence our building block solutions are

$$u_0(r,\theta)=\frac{a_0}{2}, \quad u_n(r,\theta)=a_n r^n\cos(n\theta)+b_n r^n\sin(n\theta).$$

Putting everything together our solution is:

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n r^n\cos(n\theta)+b_n r^n\sin(n\theta).$$

We look at the boundary condition in (6.10.1),

$$g(\theta)=u(1,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(n\theta)+b_n\sin(n\theta).$$

Therefore, the solution (6.10.1) is to expand $g(\theta)$, which is a 2π-periodic function as a Fourier series, and then the n^{th} coordinate is multiplied by r^n. In other words, to compute a^n and b^n from the formula we can, as usual, compute

$$a_n=\frac{1}{\pi}\int_{\pi}^{-\pi} g(\theta)\cos(n\theta)d\theta, \quad \text{and} \quad b_n=\frac{1}{\pi}\int_{\pi}^{-\pi} g(\theta)\sin(n\theta)d\theta.$$

Example 6.10.1

Suppose we wish to solve

$$\begin{aligned} &\Delta u=0, \qquad 0\le r<1, \quad -\pi<\theta\le\pi, \\ &u(1,\theta)=\cos(10\,\theta), \qquad -\pi<\theta\le\pi. \end{aligned} \tag{6.10.8}$$

The solution is

$$u(r,\theta)=r^{10}\cos(10\theta).$$

See the plot in Figure 6.10.3. The thing to notice in this example is that the effect of a high frequency is mostly felt at the boundary. In the middle of the disc, the solution is very close to zero. That is because r^{10} rather small when r is close to 0.

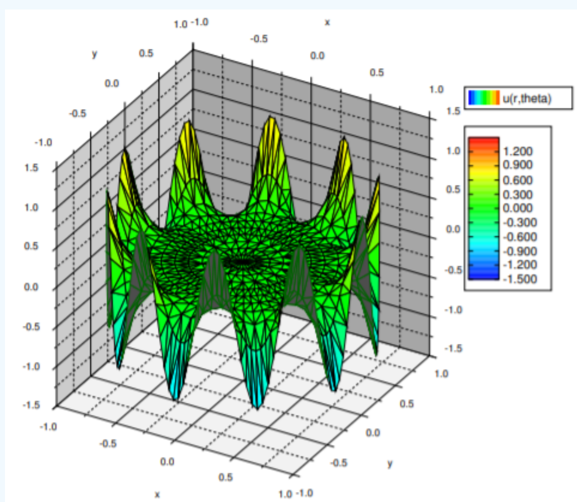

Figure 6.10.3: The solution of the Dirichlet problem in the disc with $\cos(10\theta)$ as boundary data.

Example 6.10.2

Let us solve a more difficult problem. Suppose we have a long rod with circular cross section of radius 1 and we wish to solve the steady state heat problem. If the rod is long enough we simply need to solve the Laplace equation in two dimensions. Let us put the center of the rod at the origin and we have exactly the region we are currently studying—a circle of radius 1. For the boundary conditions, suppose in Cartesian coordinates x and y, the temperature is fixed at 0 when $y<0$ and at $2y$ when $y>0$.

We set the problem up. As $y=r\sin(\theta)$, then on the circle of radius 1 we have $2y=2\sin(\theta)$. So

$$\begin{aligned} \Delta u &= 0, \qquad 0 \le r < 1, \quad -\pi < \theta \le \pi, \\ u(1,\theta) &= \begin{cases} 2\sin(\theta) & \text{if} \quad 0 \le \theta \le \pi, \\ 0 & \text{if} \ -\pi < \theta < 0. \end{cases} \end{aligned} \tag{6.10.9}$$

We must now compute the Fourier series for the boundary condition. By now the reader has plentiful experience in computing Fourier series and so we simply state that

$$u(1,\theta) = \frac{2}{\pi} + \sin(\theta) + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)}\cos(2n\theta).$$

? Exercise 6.10.1

Compute the series for $u(1,\theta)$ and verify that it really is what we have just claimed. Hint: Be careful, make sure not to divide by zero.

We now simply write the solution (see Figure 6.10.4) by multiplying by r^n in the right places.

$$u(r,\theta) = \frac{2}{\pi} + r\sin(\theta) + \sum_{n=1}^{\infty} \frac{-4r^{2n}}{\pi(4n^2-1)}\cos(2n\theta).$$

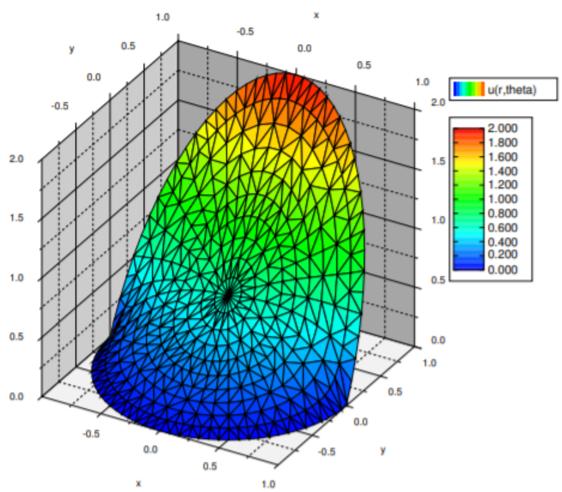

Figure 6.10.4: The solution of the Dirichlet problem with boundary data 0 for $y < 0$ and $2y$ for $y > 0$.

6.10.3: Poisson Kernel

There is another way to solve the Dirichlet problem with the help of an integral kernel. That is, we will find a function $P(r,\theta,\alpha)$ called the *Poisson kernel*[1] such that

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta,\alpha)g(\alpha)d\alpha.$$

While the integral will generally not be solvable analytically, it can be evaluated numerically. In fact, unless the boundary data is given as a Fourier series already, it will be much easier to numerically evaluate this formula as there is only one integral to evaluate.

The formula also has theoretical applications. For instance, as $P(r,\theta,\alpha)$ will have infinitely many derivatives, then via differentiating under the integral we find that the solution $u(r,\theta)$ has infinitely many derivatives, at least when inside the circle, $r < 1$. By infinitely many derivatives what you should think of is that $u(r,\theta)$ has “no corners” and all of its partial derivatives exist too and also have “no corners”.

We will compute the formula for $P(r,\theta,\alpha)$ from the series solution, and this idea can be applied anytime you have a convenient series solution where the coefficients are obtained via integration. Hence you can apply this reasoning to obtain such integral kernels for other equations, such as the heat equation. The computation is long and tedious, but not overly difficult. Since the ideas are often applied in similar contexts, it is good to understand how this computation works.

What we do is start with the series solution and replace the coefficients with the integrals that compute them. Then we try to write everything as a single integral. We must use a different dummy variable for the integration and hence we use α instead of θ.

$$\begin{aligned} u(r,\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \\ &= \underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\alpha)d\alpha\right)}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} g(\alpha)\cos(n\alpha)d\alpha\right)}_{a_n} r^n \cos(n\theta) \\ &+ \underbrace{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} g(\alpha)\sin(n\alpha)d\alpha\right)}_{b_n} r^n \sin(n\theta) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(g(\alpha) + 2\sum_{n=1}^{\infty} g(\alpha)\cos(n\alpha)r^n\cos(n\theta) + g(\alpha)\sin(n\alpha)r^n\sin(n\theta)\right)d\alpha \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\left(1 + 2\sum_{n=1}^{\infty} r^n\left(\cos(n\alpha)\cos(n\theta) + \sin(n\alpha)\sin(n\theta)\right)\right)}_{P(r,\theta,\alpha)} g(\alpha)d\alpha \end{aligned} \tag{6.10.10}$$

OK, so we have what we wanted, the expression in the parentheses is the Poisson kernel, $P(r,\theta,\alpha)$. However, we can do a lot better. It is still given as a series, and we would really like to have a nice simple expression for it. We must work a little harder. The trick is to rewrite everything in terms of complex exponentials. Let us work just on the kernel.

$$\begin{aligned} P(r,\theta,\alpha) &= 1 + 2\sum_{n=1}^{\infty} r^n(\cos(n\alpha)\cos(n\theta) + \sin(n\alpha)\sin(n\theta)) \\ &= 1 + 2\sum_{n=1}^{\infty} r^n\cos(n(\theta-\alpha)) \\ &= 1 + 2\sum_{n=1}^{\infty} r^n(e^{in(\theta-\alpha)} + e^{-in(\theta-\alpha)}) \\ &= 1 + \sum_{n=1}^{\infty}(re^{i(\theta-\alpha)})^n + \sum_{n=1}^{\infty}(re^{-i(\theta-\alpha)})^n. \end{aligned} \tag{6.10.11}$$

In the above expression we recognize the *geometric series*. That is, recall from calculus that as long as $|z| < 1$, then

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Note that n starts at 1 and that is why we have the z in the numerator. It is the standard geometric series multiplied by z. Let us continue with the computation.

$$\begin{aligned} P(r,\theta,\alpha) &= 1 + \sum_{n=1}^{\infty}(re^{i(\theta-\alpha)})^n + \sum_{n=1}^{\infty}(re^{-i(\theta-\alpha)})^n \\ &= 1 + \frac{re^{i(\theta-\alpha)}}{1-re^{i(\theta-\alpha)}} + \frac{re^{-i(\theta-\alpha)}}{1-re^{-i(\theta-\alpha)}} \\ &= \frac{(1-re^{i(\theta-\alpha)})(1-re^{-i(\theta-\alpha)}) + (1-re^{-i(\theta-\alpha)})re^{i(\theta-\alpha)} + (1-re^{i(\theta-\alpha)})re^{-i(\theta-\alpha)}}{(1-re^{i(\theta-\alpha)})(1-re^{-i(\theta-\alpha)})} \\ &= \frac{1-r^2}{1-re^{i(\theta-\alpha)} - re^{-i(\theta-\alpha)} + r^2} \\ &= \frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}. \end{aligned} \tag{6.10.12}$$

Now that's a formula we can live with. The solution to the Dirichlet problem using the Poisson kernel is

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}g(\alpha)d\alpha.$$

Sometimes the formula for the Poisson kernel is given together with the constant $\frac{1}{2\pi}$, in which case we should of course not leave it in front of the integral. Also, often the limits of the integral are given as 0 to 2π; everything inside is 2π-periodic in α, so this does not change the integral.

Let us not leave the Poisson kernel without explaining its geometric meaning. Let s be the distance from (r,θ) to $(1,\alpha)$. You may recall from calculus that this distance s in polar coordinates is given precisely by the square root of $1-2r\cos(\theta-\alpha)+r^2$. That is, the Poisson kernel is really the formula

$$\frac{1-r^2}{s^2}.$$

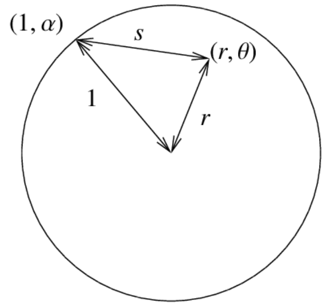

Figure 6.10.5

One final note we make about the formula is to note that it is really a weighted average of the boundary values. First let us look at what happens at the origin, that is when $r=0$.

$$\begin{aligned} u(0,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-0^2}{1-2(0)\cos(\theta-\alpha)+0^2}g(\alpha)d\alpha \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}g(\alpha)d\alpha. \end{aligned} \tag{6.10.13}$$

So $u(0,0)$ is precisely the average value of $g(\theta)$ and therefore the average value of u on the boundary. This is a general feature of harmonic functions, the value at some point p is equal to the average of the values on a circle centered at p.

What the formula says is that the value of the solution at any point in the circle is a weighted average of the boundary data $g(\theta)$. The kernel is bigger when (r,θ) is closer to $(1,\alpha)$. Therefore when computing $u(r,\theta)$ we give more weight to the values $g(\alpha)$ when $(1,\alpha)$ is closer to (r,θ) and less weight to the values $g(\theta)$ when $(1,\alpha)$ far from (r,θ).

6.10.4: Footnotes

[1] Named for the French mathematician Siméon Denis Poisson (1781 – 1840).

6.E: Fourier Series and PDEs (Exercises)

These are homework exercises to accompany Libl's "Differential Equations for Engineering" Textmap. This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Prerequisite for the course is the basic calculus sequence.

6.E.1: 4.1: Boundary value problems

Hint for the following exercises: Note that when $\lambda > 0$, then $\cos(\sqrt{\lambda}(t-a))$ and $\sin(\sqrt{\lambda}(t-a))$ are also solutions of the homogeneous equation.

Exercise $6.E.4.1.1$

Compute all eigenvalues and eigenfunctions of $x'' + \lambda x = 0, x(a) = 0, x(b) = 0$ (assume $a < b$).

Exercise $6.E.4.1.2$

Compute all eigenvalues and eigenfunctions of $x'' + \lambda x = 0, x'(a) = 0, x'(b) = 0$ (assume $a < b$).

Exercise $6.E.4.1.3$

Compute all eigenvalues and eigenfunctions of $x'' + \lambda x = 0, x'(a) = 0, x(b) = 0$ (assume $a < b$).

Exercise $6.E.4.1.4$

Compute all eigenvalues and eigenfunctions of $x'' + \lambda x = 0, x(a) = x(b), x'(a) = x'(b)$ (assume $a < b$).

Exercise $6.E.4.1.5$

We have skipped the case of $\lambda < 0$ for the boundary value problem $x'' + \lambda x = 0, x(-\pi) = x(\pi), x'(-\pi) = x'(\pi)$. Finish the calculation and show that there are no negative eigenvalues.

Exercise $6.E.4.1.6$

Consider a spinning string of length 2 and linear density 0.1 and tension 3. Find smallest angular velocity when the string pops out.

Answer

$$\omega = \pi\sqrt{\frac{15}{2}}$$

Exercise $6.E.4.1.7$

Suppose $x'' + \lambda x = 0$ and $x(0) = 1, x(1) = 1$. Find all λ for which there is more than one solution. Also find the corresponding solutions (only for the eigenvalues).

Answer

$\lambda_k = 4k^2\pi^2$ for $k = 1,\ 2,\ 3, \ldots$ $x_k = \cos(2k\pi t) + B\sin(2k\pi t)$ (for any B)

Exercise $6.E.4.1.8$

Suppose $x'' + x = 0$ and $x(0) = 0, x'(\pi) = 1$. Find all the solution(s) if any exist.

Answer

$x(t) = -\sin(t)$

Exercise $6.E.4.1.9$

Consider $x' + \lambda x = 0$ and $x(0) = 0, x(1) = 0$. Why does it not have any eigenvalues? Why does any first order equation with two endpoint conditions such as above have no eigenvalues?

Answer

General solution is $x = Ce^{-\lambda t}$. Since $x(0) = 0$ then $C = 0$, and so $x(t) = 0$. Therefore, the solution is always identically zero. One condition is always enough to guarantee a unique solution for a first order equation.

Exercise $6.E.4.1.10$: (challenging)

Suppose $x''' + \lambda x = 0$ and $x(0) = 0, x'(0) = 0, x(1) = 0$. Suppose that $\lambda > 0$. Find an equation that all such eigenvalues must satisfy. Hint: Note that $-\sqrt[3]{\lambda}$ is a root of $r^3 + \lambda = 0$.

Answer

$$\frac{\sqrt{3}}{3} e^{\frac{-3}{2}\sqrt[3]{\lambda}} - \frac{\sqrt{3}}{3}\cos\left(\frac{\sqrt{3}\sqrt[3]{\lambda}}{2}\right) + \sin\left(\frac{\sqrt{3}\sqrt[3]{\lambda}}{2}\right) = 0$$

6.E.2: 4.2: The Trigonometric Series

Exercise $6.E.4.2.1$

Suppose $f(t)$ is defined on $[-\pi, \pi]$ as $\sin(5t) + \cos(3t)$. Extend periodically and compute the Fourier series of $f(t)$.

Exercise $6.E.4.2.2$

Suppose $f(t)$ is defined on $[-\pi, \pi]$ as $|t|$. Extend periodically and compute the Fourier series of $f(t)$.

Exercise $6.E.4.2.3$

Suppose $f(t)$ is defined on $[-\pi, \pi]$ as $|t|^3$. Extend periodically and compute the Fourier series of $f(t)$.

Exercise $6.E.4.2.4$

Suppose $f(t)$ is defined on $(-\pi, \pi]$ as

$$f(t) = \begin{cases} -1 & if -\pi < t \leq 0, \\ 1 & if 0 < t \leq \pi. \end{cases} \tag{6.E.1}$$

Extend periodically and compute the Fourier series of $f(t)$.

Exercise $6.E.4.2.5$

Suppose $f(t)$ is defined on $(-\pi, \pi]$ as t^3. Extend periodically and compute the Fourier series of $f(t)$.

Exercise $6.E.4.2.6$

Suppose $f(t)$ is defined on $[-\pi, \pi]$ as t^2. Extend periodically and compute the Fourier series of $f(t)$.

There is another form of the Fourier series using complex exponentials e^{nt} for $n = \dots, -2, -1, 0, 1, 2, \dots$ instead of $\cos(nt)$ and $\sin(nt)$ for positive n. This form may be easier to work with sometimes. It is certainly more compact to write, and there is only one formula for the coefficients. On the downside, the coefficients are complex numbers.

Exercise $6.E.4.2.7$

Let

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos(nt)+b_n \sin(nt). \tag{6.E.2}$$

Use Euler's formula $e^{i\theta}=\cos(\theta)+i\sin(\theta)$ to show that there exist complex numbers c_m such that

$$f(t)=\sum_{m=-\infty}^{\infty} c_m e^{imt}. \tag{6.E.3}$$

Note that the sum now ranges over all the integers including negative ones. Do not worry about convergence in this calculation. Hint: It may be better to start from the complex exponential form and write the series as

$$c_0+\sum_{m=1}^{\infty} c_m e^{imt}+c_{-m}e^{-imt}. \tag{6.E.4}$$

Exercise $6.E.4.2.8$

Suppose $f(t)$ is defined on $[-\pi,\pi]$ as $f(t)=\sin(t)$. Extend periodically and compute the Fourier series.

Answer

$\sin(t)$

Exercise $6.E.4.2.9$

Suppose $f(t)$ is defined on $(-\pi,\pi]$ as $f(t)=\sin(\pi t)$. Extend periodically and compute the Fourier series.

Answer

$\sum_{n=1}^{\infty}\frac{(\pi-n)\sin(\pi n+\pi^2)+(\pi+n)\sin(\pi n-\pi^2)}{\pi n^2-\pi^3}\sin(nt)$

Exercise $6.E.4.2.10$

Suppose $f(t)$ is defined on $(-\pi,\pi]$ as $f(t)=\sin^2(t)$. Extend periodically and compute the Fourier series.

Answer

$\frac{1}{2}-\frac{1}{2}\cos(2t)$

Exercise $6.E.4.2.11$

Suppose $f(t)$ is defined on $(-\pi,\pi]$ as $f(t)=t^4$. Extend periodically and compute the Fourier series.

Answer

$\frac{\pi^4}{5}+\sum_{n=1}^{\infty}\frac{(-1)^n(8\pi^2n^2-48)}{n^4}\cos(nt)$

6.E.3: 4.3: More on the Fourier series

? Exercise $6.E.4.3.1$

Let

$$f(t)=\begin{cases}0 & \text{if} \quad -1<t\leq 0,\\ t & \text{if} \quad 0<t\leq 1,\end{cases} \tag{6.E.5}$$

extended periodically.

a. Compute the Fourier series for $f(t)$.
b. Write out the series explicitly up to the 3^{rd} harmonic.

? Exercise $6.E.4.3.2$

Let

$$f(t)=\begin{cases}-t & \text{if} \quad -1<t\leq 0,\\ t^2 & \text{if} \quad 0<t\leq 1,\end{cases} \tag{6.E.6}$$

extended periodically.

a. Compute the Fourier series for $f(t)$.
b. Write out the series explicitly up to the 3^{rd} harmonic.

? Exercise $6.E.4.3.3$

Let

$$f(t)=\begin{cases}\dfrac{-t}{10} & \text{if} \quad -10<t\leq 0,\\ \dfrac{t}{10} & \text{if} \quad 0<t\leq 10,\end{cases} \tag{6.E.7}$$

extended periodically (period is 20).

a. Compute the Fourier series for $f(t)$.
b. Write out the series explicitly up to the 3^{rd} harmonic.

? Exercise $6.E.4.3.4$

Let $f(t)=\sum_{n=1}^{\infty}\dfrac{1}{n^3}\cos(nt)$. Is $f(t)$ continuous and differentiable everywhere? Find the derivative (if it exists everywhere) or justify why $f(t)$ is not differentiable everywhere.

? Exercise $6.E.4.3.5$

Let $f(t)=\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin(nt)$. Is $f(t)$ differentiable everywhere? Find the derivative (if it exists everywhere) or justify why $f(t)$ is not differentiable everywhere.

? Exercise $6.E.4.3.6$

Let

$$f(t)=\begin{cases}0 & \textit{if} -2<t\leq 0,\\ t & \textit{if } 0<t\leq 1,\\ -t+2 & \textit{if } 1<t\leq 2,\end{cases} \tag{6.E.8}$$

extended periodically.

a. Compute the Fourier series for $f(t)$.
b. Write out the series explicitly up to the 3^{rd} harmonic.

? Exercise $6.E.4.3.7$

Let

$$f(t) = e^t \quad \mathit{for} -1 < t \leq 1 \tag{6.E.9}$$

extended periodically.

a. Compute the Fourier series for $f(t)$.
b. Write out the series explicitly up to the 3^{rd} harmonic.
c. What does the series converge to at $t = 1$.

? Exercise $6.E.4.3.8$

Let

$$f(t) = t^2 \quad \mathit{for} -1 < t \leq 1 \tag{6.E.10}$$

extended periodically.

a. Compute the Fourier series for $f(t)$.
b. By plugging in $t = 0$, evaluate $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} = 1 - \dfrac{1}{4} + \dfrac{1}{9} - \cdots$.
c. Now evaluate $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = 1 + \dfrac{1}{4} + \dfrac{1}{9} + \cdots$.

? Exercise $6.E.4.3.9$

Let

$$f(t) = \begin{cases} 0 & \text{if} \quad -3 < t \leq 0, \\ t & \text{if} \quad 0 < t \leq 3, \end{cases} \tag{6.E.11}$$

extended periodically. Suppose $F(t)$ is the function given by the Fourier series of f. Without computing the Fourier series evaluate.

a. $F(2)$
b. $F(-2)$
c. $F(4)$
d. $F(-4)$
e. $F(3)$
f. $F(-9)$

? Exercise $6.E.4.3.10$

Let

$$f(t) = t^2 \quad \mathit{for} -2 < t \leq 2 \tag{6.E.12}$$

extended periodically.

a. Compute the Fourier series for $f(t)$.
b. Write out the series explicitly up to the 3^{rd} harmonic.

Answer

a. $\frac{8}{6} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{\pi^2 n^2} \cos\left(\frac{n\pi}{2} t\right)$

b. $\frac{8}{6} - \frac{16}{\pi^2}\cos\left(\frac{\pi}{2}t\right) + \frac{4}{\pi^2}\cos(\pi t) - \frac{16}{9\pi^2}\cos\left(\frac{3\pi}{2}t\right) + \cdots$

Exercise $6.E.4.3.11$

Let

$$f(t) = t \quad \mathit{for}\ \lambda < t \leq \lambda\ \mathit{(for\ some}\ \lambda\mathit{)} \tag{6.E.13}$$

extended periodically.

a. Compute the Fourier series for $f(t)$.
b. Write out the series explicitly up to the 3^{rd} harmonic.

Answer

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2\lambda}{n\pi} \sin\left(\frac{n\pi}{\lambda}t\right)$
b. $\frac{2\lambda}{\pi}\sin\left(\frac{\pi}{\lambda}t\right) - \frac{\lambda}{\pi}\sin\left(\frac{2\pi}{\lambda}t\right) + \frac{2\lambda}{3\pi}\sin\left(\frac{3\pi}{\lambda}t\right) - \cdots$

Exercise $6.E.4.3.12$

Let

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n(n^2+1)} \sin(n\pi t). \tag{6.E.14}$$

Compute $f'(t)$.

Answer

$f'(t) = \sum_{n=1}^{\infty} \frac{\pi}{n^2+1} \cos(n\pi t)$

Exercise $6.E.4.3.13$

Let

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n^3)} \cos(nt). \tag{6.E.15}$$

a. Find the antiderivative.
b. Is the antiderivative periodic?

Answer

a. $F(t) = \frac{t}{2} + C + \sum_{n=1}^{\infty} \frac{1}{n^4} \sin(nt)$
b. no

Exercise $6.E.4.3.14$

Let

$$f(t) = \frac{t}{2} \quad \mathit{for}\ -\pi < t \leq \pi \tag{6.E.16}$$

extended periodically.

a. Compute the Fourier series for $f(t)$.
b. Plug in $t = \frac{\pi}{2}$ to find a series representation for $\frac{\pi}{4}$.

c. Using the first 4 terms of the result from part b) approximate $\frac{\pi}{4}$.

Answer

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nt)$
b. f is continuous at $t=\frac{\pi}{2}$ so the Fourier series converges to $f\left(\frac{\pi}{2}\right)=\frac{\pi}{4}$. Obtain
$\frac{\pi}{4}=\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$.
c. Using the first 4 terms get $76/105 \approx 0.72$(quite a bad approximation, you would have to take about 50 terms to start to get to within $0.01$of $\frac{\pi}{4}$).

Exercise $6.E.4.3.15$

Let

$$f(t)=\begin{cases}0 & \text{if} \quad -2<t\leq 0, \\ 2 & \text{if} \quad 0<t\leq 2,\end{cases} \tag{6.E.17}$$

extended periodically. Suppose $F(t)$ is the function given by the Fourier series of f. Without computing the Fourier series evaluate.

a. $F(0)$
b. $F(-1)$
c. $F(1)$
d. $F(-2)$
e. $F(4)$
f. $F(-8)$

Answer

a. $F(0)=1$
b. $F(-1)=0$
c. $F(1)=2$
d. $F(-2)=1$
e. $F(4)=1$
f. $F(-9)=0$

6.E.4: 4.4: Sine and Cosine Series

Exercise $6.E.4.4.1$

Take $f(t)=(t-1)^2$ defined on $0\leq t\leq 1$.

a. Sketch the plot of the even periodic extension of f.
b. Sketch the plot of the odd periodic extension of f.

Exercise $6.E.4.4.2$

Find the Fourier series of both the odd and even periodic extension of the function $f(t)=(t-1)^2$ for $0\leq t\leq 1$. Can you tell which extension is continuous from the Fourier series coefficients?

Exercise $6.E.4.4.3$

Find the Fourier series of both the odd and even periodic extension of the function $f(t)=t$ for $0\leq t\leq \pi$.

Exercise $6.E.4.4.4$

Find the Fourier series of the even periodic extension of the function $f(t) = \sin t$ for $0 \leq t \leq \pi$.

Exercise $6.E.4.4.5$

Consider

$$x''(t) + 4x(t) = f(t), \tag{6.E.18}$$

where $f(t) = 1$ on $0 < t < 1$.

a. Solve for the Dirichlet conditions $x(0) = 0, x(1) = 0$.
b. Solve for the Neumann conditions $x'(0) = 0$, $x'(1) = 0$.

Exercise $6.E.4.4.6$

Consider

$$x''(t) + 9x(t) = f(t), \tag{6.E.19}$$

for $f(t) = \sin(2\pi t)$ on $0 < t < 1$.

a. Solve for the Dirichlet conditions $x(0) = 0,\ x(1) = 0$.
b. b) Solve for the Neumann conditions $x'(0) = 0,\ x'(1) = 0$.

Exercise $6.E.4.4.7$

Consider

$$x''(t) + 3x(t) = f(t), \quad x(0) = 0, \quad x(1) = 0, \tag{6.E.20}$$

where $f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$. Write the solution $x(t)$ as a Fourier series, where the coefficients are given in terms of b_n.

Exercise $6.E.4.4.8$

Let $f(t) = t^2(2-t)$ for $0 \leq t \leq 2$. Let $F(t)$ be the odd periodic extension. Compute $F(1),\ F(2),\ F(3),\ F(-1),\ F(\frac{9}{2}),\ F(101),\ F(103)$Note: Do **not** compute using the sine series.

Exercise $6.E.4.4.9$

Let $f(t) = \frac{t}{3}$ on $0 \leq t < 3$.

a. Find the Fourier series of the even periodic extension.
b. Find the Fourier series of the odd periodic extension.

Answer

a. $\frac{1}{2} + \sum\limits_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{\pi^2 n^2} \cos\left(\frac{n\pi}{3} t\right)$
b. $\sum\limits_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \sin\left(\frac{n\pi}{3} t\right)$

Exercise $6.E.4.4.10$

Let $f(t) = \cos(2t)$ on $0 \leq t < \pi$.

a. Find the Fourier series of the even periodic extension.
b. Find the Fourier series of the odd periodic extension.

Answer

a. $\cos(2t)$

b. $\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4n}{\pi n^2-4\pi}\sin(nt)$

Exercise $6.E.4.4.11$

Let $f(t)$ be defined on $0 \le t < 1$. Now take the average of the two extensions $g(t) = \frac{F_{odd}(t)+F_{even}(t)}{2}$.

a. What is $g(t)$ if $0 \le t < 1$ (Justify!)
b. What is $g(t)$ if $-1 < t < 0$ (Justify!)

Answer

a. $f(t)$
b. 0

Exercise $6.E.4.4.12$

Let $f(t) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(nt)$. Solve $x'' - x = f(t)$ for the Dirichlet conditions $x(0) = 0$ and $x(\pi) = 0$.

Answer

$$\sum_{n=1}^{\infty} \frac{-1}{n^2(1+n^2)}\sin(nt)$$

Exercise $6.E.4.4.13$: (challenging)

Let $f(t) = t + \sum_{n=1}^{\infty} \frac{1}{2^n}\sin(nt)$. Solve $x'' + \pi x = f(t)$ for the Dirichlet conditions $x(0) = 0$ and $x(\pi) = 1$. Hint: Note that $\frac{t}{\pi}$ satisfies the given Dirichlet conditions.

Answer

$$\frac{t}{\pi} + \sum_{n=1}^{\infty} \frac{1}{2^n(\pi-n^2)}\sin(nt)$$

6.E.5: 4.5: Applications of Fourier series

Exercise $6.E.4.5.1$

Let $F(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos(n\pi t)$. Find the steady periodic solution to $x'' + 2x = F(t)$. Express your solution as a Fourier series.

Exercise $6.E.4.5.2$

Let $F(t) = \sum_{n=1}^{\infty} \frac{1}{n^3}\sin(n\pi t)$. Find the steady periodic solution to $x'' + x' + x = F(t)$. Express your solution as a Fourier series.

Exercise $6.E.4.5.3$

Let $F(t) = \sum_{n=1}^{\infty} \frac{1}{n^2}\cos(n\pi t)$. Find the steady periodic solution to $x'' + 4x = F(t)$. Express your solution as a Fourier series.

Exercise $6.E.4.5.4$

Let $F(t) = t$ for $-1 < t < 1$ and extended periodically. Find the steady periodic solution to $x'' + x = F(t)$. Express your solution as a series.

Exercise $6.E.4.5.5$

Let $F(t) = t$ for $-1 < t < 1$ and extended periodically. Find the steady periodic solution to $x'' + \pi^2 x = F(t)$. Express your solution as a series.

Exercise $6.E.4.5.6$

Let $F(t) = \sin(2\pi t) + 0.1\cos(10\pi t)$. Find the steady periodic solution to $x'' + \sqrt{2}x = F(t)$. Express your solution as a Fourier series.

Answer

$$x = \frac{1}{\sqrt{2} - 4\pi^2}\sin(2\pi t) + \frac{0.1}{\sqrt{2} - 100\pi^2}\cos(10\pi t)$$

Exercise $6.E.4.5.7$

Let $F(t) = \sum_{n=1}^{\infty} e^{-n}\cos(2nt)$. Find the steady periodic solution to $x'' + 3x = F(t)$. Express your solution as a Fourier series.

Answer

$$x = \sum_{n=1}^{\infty} \frac{e^{-n}}{3 - (2n)^2}\cos(2nt)$$

Exercise $6.E.4.5.8$

Let $F(t) = |t|$ for $-1 \le t \le 1$ extended periodically. Find the steady periodic solution to $x'' + \sqrt{3}x = F(t)$. Express your solution as a series.

Answer

$$x = \frac{1}{2\sqrt{3}} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^2\pi^2(\sqrt{3} - n^2\pi^2)}\cos(n\pi t)$$

Exercise $6.E.4.5.9$

Let $F(t) = |t|$ for $-1 \le t \le 1$ extended periodically. Find the steady periodic solution to $x'' + \pi^2 x = F(t)$. Express your solution as a series.

Answer

$$x = \frac{1}{2\sqrt{3}} - \frac{2}{\pi^3}t\sin(\pi t) + \sum_{\substack{n=3 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^2\pi^4(1 - n^2)}\cos(n\pi t)$$

6.E.6: 4.6: PDEs, Separation of Variables, and the Heat Equation

Exercise $6.E.4.6.1$

Imagine you have a wire of length 2, with $k = 0.001$ and an initial temperature distribution of $u(x,0) = 50x$. Suppose that both the ends are embedded in ice (temperature 0). Find the solution as a series.

Exercise $6.E.4.6.2$

Find a series solution of

$$\begin{aligned} u_t &= u_{xx}, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= 100 \quad \text{for } 0 < x < 1. \end{aligned} \tag{6.E.21}$$

Exercise $6.E.4.6.3$

Find a series solution of

$$\begin{aligned} u_t &= u_{xx}, \\ u_x(0,t) &= u_x(\pi,t) = 0, \\ u(x,0) &= 3\cos(x) + \cos(3x) \quad \text{for } 0 < x < \pi. \end{aligned} \tag{6.E.22}$$

Exercise $6.E.4.6.4$

Find a series solution of

$$\begin{aligned} u_t &= \frac{1}{3} u_{xx}, \\ u_x(0,t) &= u_x(\pi,t) = 0, \\ u(x,0) &= \frac{10x}{\pi} \quad \text{for } 0 < x < \pi. \end{aligned} \tag{6.E.23}$$

Exercise $6.E.4.6.5$

Find a series solution of

$$\begin{aligned} u_t &= u_{xx}, \\ u(0,t) &= 0, \quad u(1,t) = 100, \\ u(x,0) &= \sin(\pi x) \quad \text{for } 0 < x < 1. \end{aligned} \tag{6.E.24}$$

Hint: Use the fact that $u(x,t) = 100x$ is a solution satisfying $u_t = u_{xx},\ u(0,t) = 0,\ u(1,t) = 100$. Then usesuperposition.

Exercise $6.E.4.6.6$

Find the steady state temperature solution as a function of x alone, by letting $t \to \infty$ in the solution from exercises $6.E.4$ and $6.E.5$. Verify that it satisfies the equation $u_{xx} = 0$.

Exercise $6.E.4.6.7$

Use separation variables to find a nontrivial solution to $u_{xx} + u_{yy} = 0$, where $u(x,0) = 0$ and $u(0,y) = 0$. Hint: Try $u(x,y) = X(x)Y(y)$.

Exercise $6.E.4.6.8$: (challenging)

Suppose that one end of the wire is insulated (say at $x = 0$) and the other end is kept at zero temperature. That is, find a series solution of

$$\begin{aligned} u_t &= k u_{xx}, \\ u_x(0,t) &= u(L,t) = 0, \\ u(x,0) &= f(x) \quad \text{for } 0 < x < L. \end{aligned} \tag{6.E.25}$$

Express any coefficients in the series by integrals of $f(x)$.

? Exercise $6.E.4.6.9$: (challenging)

Suppose that the wire is circular and insulated, so there are no ends. You can think of this as simply connecting the two ends and making sure the solution matches up at the ends. That is, find a series solution of

$$\begin{aligned} u_t &= k u_{xx}, \\ u(0,t) &= u(L,t), \quad u_x(0,t) = u_x(L,t) \\ u(x,0) &= f(x) \quad \text{for } 0 < x < L. \end{aligned} \tag{6.E.26}$$

Express any coefficients in the series by integrals of $f(x)$.

? Exercise $6.E.4.6.10$

Consider a wire insulated on both ends, $L = 1$, $k = 1$, and $u(x,0) = \cos^2(\pi x)$.

a. Find the solution $u(x,t)$. Hint: a trig identity.
b. Find the average temperature.
c. Initially the temperature variation is 1 (maximum minus the minimum). Find the time when the variation is $\frac{1}{2}$.

? Exercise $6.E.4.6.11$

Find a series solution of

$$\begin{aligned} u_t &= 3u_{xx}, \\ u(0,t) &= u(\pi,t) = 0, \\ u(x,0) &= 5\sin(x) + 2\sin(5x) \quad \text{for } 0 < x < \pi. \end{aligned} \tag{6.E.27}$$

Answer

$u(x,t) = 5\sin(x)e^{3t} + 2\sin(5x)e^{-75t}$

? Exercise $6.E.4.6.12$

Find a series solution of

$$\begin{aligned} u_t &= 0.1u_{xx}, \\ u_x(0,t) &= u_x(\pi,t) = 0, \\ u(x,0) &= 1 + 2\cos(x) \quad \text{for } 0 < x < \pi. \end{aligned} \tag{6.E.28}$$

Answer

$u(x,t) = 1 + 2\cos(x)e^{-0.1t}$

? Exercise $6.E.4.6.13$

Use separation of variables to find a nontrivial solution to $u_{xt} = u_{xx}$.

Answer

$u(x,t) = e^{\lambda t}e^{\lambda x}$ for some λ

Exercise $6.E.4.6.14$

Use separation of variables to find a nontrivial solution to $u_x + u_t = u$. (Hint: try $u(x,t) = X(x) + T(t)$).

Answer

$u(x,t) = Ae^x + Be^t$

Exercise $6.E.4.6.15$

Suppose that the temperature on the wire is fixed at 0 at the ends, $L = 1$, $k = 1$, and $u(x,0) = 100\sin(2\pi x)$

a. What is the temperature at $x = \frac{1}{2}$ at any time.
b. What is the maximum and the minimum temperature on the wire at $t = 0$.
c. At what time is the maximum temperature on the wire exactly one half of the initial maximum at $t = 0$.

Answer

a. 0
b. minimum -100, maximum 100
c. $t = \frac{\ln 2}{4\pi^2}$

6.E.7: 4.7: One dimensional wave equation

Exercise $6.E.4.7.1$

Solve

$$\begin{aligned} y_{tt} &= 9y_{xx}, \\ y(0,t) &= y(1,t) = 0, \\ y(x,0) &= \sin(3\pi x) + \frac{1}{4}\sin(6\pi x) && \text{for } 0 < x < 1, \\ y_t(x,0) &= 0 && \text{for } 0 < x < 1. \end{aligned} \tag{6.E.29}$$

Exercise $6.E.4.7.2$

Solve

$$\begin{aligned} y_{tt} &= 4y_{xx}, \\ y(0,t) &= y(1,t) = 0, \\ y(x,0) &= \sin(3\pi x) + \frac{1}{4}\sin(6\pi x) && \text{for } 0 < x < 1, \\ y_t(x,0) &= \sin(9\pi x) && \text{for } 0 < x < 1. \end{aligned} \tag{6.E.30}$$

Exercise $6.E.4.7.3$

Derive the solution for a general plucked string of length L, where we raise the string some distance b at the midpoint and let go, and for any constant a (in the equation $y_{tt} = a^2 y_{xx}$).

Exercise $6.E.4.7.4$

Imagine that a stringed musical instrument falls on the floor. Suppose that the length of the string is 1 and $a = 1$. When the musical instrument hits the ground the string was in rest position and hence $y(x,0) = 0$. However, the string was moving at some velocity at impact ($t = 0$)), say $y_t(x,0) = -1$. Find the solution $y(x,t)$ for the shape of the string at time t.

Exercise $6.E.4.7.5$: (challenging)

Suppose that you have a vibrating string and that there is air resistance proportional to the velocity. That is, you have

$$\begin{aligned} y_{tt} &= a^2 y_{xx} - k y_t, & \\ y(0,t) &= y(1,t) = 0, & \\ y(x,0) &= f(x) & \text{for } 0 < x < 1, \\ y_t(x,0) &= 0 & \text{for } 0 < x < 1. \end{aligned} \tag{6.E.31}$$

Suppose that $0 < k < 2\pi a$. Derive a series solution to the problem. Any coefficients in the series should be expressed as integrals of $f(x)$.

Exercise $6.E.4.7.6$

Suppose you touch the guitar string exactly in the middle to ensure another condition $u\left(\frac{L}{2}, t\right) = 0$ for all time. Which multiples of the fundamental frequency $\frac{\pi a}{L}$ show up in the solution?

Exercise $6.E.4.7.7$

Solve

$$\begin{aligned} y_{tt} &= y_{xx}, & \\ y(0,t) &= y(\pi,t) = 0, & \\ y(x,0) &= \sin(x) & \text{for } 0 < x < \pi, \\ y_t(x,0) &= \sin(x) & \text{for } 0 < x < \pi. \end{aligned} \tag{6.E.32}$$

Answer

$y(x,t) = \sin(x)\,(\sin(t) + \cos(t))$

Exercise $6.E.4.7.8$

Solve

$$\begin{aligned} y_{tt} &= 25 y_{xx}, & \\ y(0,t) &= y(2,t) = 0, & \\ y(x,0) &= 0 & \text{for } 0 < x < 2, \\ y_t(x,0) &= \sin(\pi x) + 0.1 \sin(2\pi t) & \text{for } 0 < x < 2. \end{aligned} \tag{6.E.33}$$

Answer

$y(x,t) = \frac{1}{5\pi}\sin(\pi x)\sin(5\pi t) + \frac{1}{100\pi}\sin(2\pi x)\sin(10\pi t)$

Exercise $6.E.4.7.9$

Solve

$$\begin{aligned} y_{tt} &= 2 y_{xx}, & \\ y(0,t) &= y(\pi,t) = 0, & \\ y(x,0) &= x & \text{for } 0 < x < \pi, \\ y_t(x,0) &= 0 & \text{for } 0 < x < \pi. \end{aligned} \tag{6.E.34}$$

Answer

$y(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)\cos(n\sqrt{2}t)$

Exercise 6.E.4.7.10

Let's see what happens when $a = 0$. Find a solution to $y_{tt} = 0,\ y(0,t) = y(\pi,t) = 0,\ y(x,0) = \sin(2x),\ y_t(x,0) = \sin(x)$.

Answer

$$y(x,t) = \sin(2x) + t\sin(x)$$

6.E.8: 4.8: D'Alembert solution of the wave equation

Exercise 6.E.4.8.1

Using the d'Alembert solution solve $y_{tt} = 4y_{xx}$, $0 < x < \pi, t > 0, y(0,t) = y(\pi,t) = 0, y(x,0) = \sin x$, and $y_t(x,0) = \sin x$. Hint: Note that $\sin x$ is the odd extension of $y(x,0)$ and $y_t(x,0)$.

Exercise 6.E.4.8.2

Using the d'Alembert solution solve $y_{tt} = 2y_{xx}$, $0 < x < 1, t > 0, y(0,t) = y(1,t) = 0, y(x,0) = \sin^5(\pi x)$, and $y_t(x,0) = \sin^3(\pi x)$.

Exercise 6.E.4.8.3

Take $y_{tt} = 4y_{xx}$, $0 < x < \pi, t > 0, y(0,t) = y(\pi,t) = 0, y(x,0) = x(\pi - x)$, and $y_t(x,0) = 0$.

a. Solve using the d'Alembert formula. Hint: You can use the sine series for $y(x,0)$.
b. Find the solution as a function of x for a fixed $t = 0.5,\ t = 1,$ and $t = 2$. Do not use the sine series here.

Exercise 6.E.4.8.4

Derive the d'Alembert solution for $y_{tt} = a^2 y_{xx}$, $0 < x < \pi, t > 0, y(0,t) = y(\pi,t) = 0, y(x,0) = f(x)$, and $y_t(x,0) = 0$, using the Fourier series solution of the wave equation, by applying an appropriate trigonometric identity. Hint: Do it first for a single term of the Fourier series solution, in particular do it when y is $\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right)$.

Exercise 6.E.4.8.5

The d'Alembert solution still works if there are no boundary conditions and the initial condition is defined on the whole real line. Suppose that $y_{tt} = y_{xx}$ (for all x on the real line and $t \geq 0$), $y(x,0) = f(x)$, and $y_t(x,0)$, where

$$f(x) = \begin{cases} 0 & \text{if} \quad x < -1, \\ x+1 & \text{if} \quad -1 \leq x < 0, \\ -x+1 & \text{if} \quad 0 \leq x < 1 \\ 0 & \text{if} \quad x > 1. \end{cases}$$

Solve using the d'Alembert solution. That is, write down a piecewise definition for the solution. Then sketch the solution for $t = 0, t = 1/2, t = 1$, and $t = 2$.

Exercise 6.E.4.8.6

Using the d'Alembert solution solve $y_{tt} = 9y_{xx}$, $0 < x < 1, t > 0, y(0,t) = y(1,t) = 0, y(x,0) = \sin(2\pi x)$, and $y_t(x,0) = \sin(3\pi x)$.

Answer

$$y(x,t) = \frac{\sin(2\pi(x-3t)) + \sin(2\pi(3t+x))}{2} + \frac{\cos(3\pi(x-3t)) - \cos(3\pi(3t+x))}{18\pi}$$

Exercise $6.E.4.8.7$

Take $y_{tt} = 4y_{xx}$, $0 < x < 1, t > 0$, $y(0,t) = y(1,t) = 0$, $y(x,0) = x - x^2$, and $y_t(x,0) = 0$. Using the D'Alembert solution find the solution at

a. $t = 0.1$,
b. $t = 1/2$,
c. $t = 1$.

You may have to split your answer up by cases.

Answer

a. $y(x,t) = \begin{cases} x - x^2 - 0.04 & \text{if} \quad 0.2 \le x \le 0.8 \\ 0.6x & \text{if} \quad x \le 0.2 \\ 0.6 - 0.6x & \text{if} \quad x \ge 0.8 \end{cases}$

b. $y(x, \frac{1}{2}) = -x + x^2$

c. $y(x,1) = x - x^2$

Exercise $6.E.4.8.8$

Take $y_{tt} = 100y_{xx}$, $0 < x < 4, t > 0$, $y(0,t) = y(4,t) = 0$, $y(x,0) = F(x)$, and $y_t(x,0) = 0$. Suppose that $F(0) = 0$, $F(1) = 2$, $F(2) = 3$, $F(3) = 1$. Using the D'Alembert solution find

a. $y(1,1)$,
b. $y(4,3)$,
c. $y(3,9)$.

Answer

a. $y(1,1) = -\frac{1}{2}$

b. $y(4,3) = 0$

c. $y(3,9) = \frac{1}{2}$

6.E.9: 4.9: Steady state temperature and the Laplacian

Exercise $6.E.4.9.1$

Let R be the region described by $0 < x < \pi$ and $0 < y < \pi$. Solve the problem

$$\Delta u = 0, \quad u(x,0) = \sin x, \quad u(x,\pi) = 0, \quad u(0,y) = 0, \quad u(\pi,y) = 0.$$

Exercise $6.E.4.9.2$

Let R be the region described by $0 < x < 1$ and $0 < y < 1$. Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u(x,0) &= \sin(\pi x) - \sin(2\pi x), \quad u(x,1) = 0, \\ u(0,y) &= 0, \quad u(1,y) = 0. \end{aligned} \tag{6.E.35}$$

Exercise $6.E.4.9.3$

Let R be the region described by $0 < x < 1$ and $0 < y < 1$. Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u(x,0) &= u(x,1) = u(0,y) = u(1,y) = C. \end{aligned} \tag{6.E.36}$$

for some constant C. Hint: Guess, then check your intuition.

Exercise $6.E.4.9.4$

Let R be the region described by $0<x<\pi$ and $0<y<\pi$. Solve

$$\Delta u=0, \quad u(x, 0)=0, \quad u(x, \pi)=\pi, \quad u(0, y)=y, \quad u(\pi, y)=y.$$

Hint: Try a solution of the form $u(x, y)=X(x)+Y(y)$ (different separation of variables).

Exercise $6.E.4.9.5$

Use the solution of Exercise $6.E.4$ to solve

$$\Delta u=0, \quad u(x, 0)=\sin x, \quad u(x, \pi)=\pi, \quad u(0, y)=y, \quad u(\pi, y)=y.$$

Hint: Use superposition.

Exercise $6.E.4.9.6$

Let R be the region described by $0<x<w$ and $0<y<h$. Solve the problem

$$\begin{aligned} u_{xx}+u_{yy} &=0, & & \\ u(x, 0) &=0, & u(x, h) &=f(x), \\ u(0, y) &=0, & u(w, y) &=0. \end{aligned} \tag{6.E.37}$$

The solution should be in series form using the Fourier series coefficients of $f(x)$.

Exercise $6.E.4.9.7$

Let R be the region described by $0<x<w$ and $0<y<h$. Solve the problem

$$\begin{aligned} u_{xx}+u_{yy} &=0, & & \\ u(x, 0) &=0, & u(x, h) &=0, \\ u(0, y) &=f(y), & u(w, y) &=0. \end{aligned} \tag{6.E.38}$$

The solution should be in series form using the Fourier series coefficients of $f(y)$.

Exercise $6.E.4.9.8$

Let R be the region described by $0<x<w$ and $0<y<h$. Solve the problem

$$\begin{aligned} u_{xx}+u_{yy} &=0, & & \\ u(x, 0) &=0, & u(x, h) &=0, \\ u(0, y) &=0, & u(w, y) &=f(y). \end{aligned} \tag{6.E.39}$$

The solution should be in series form using the Fourier series coefficients of $f(y)$.

Exercise $6.E.4.9.9$

Let R be the region described by $0<x<1$ and $0<y<1$. Solve the problem

$$\begin{aligned} u_{xx}+u_{yy} &=0, & & \\ u(x, 0) &=\sin(9\pi x), & u(x, 1) &=\sin(2\pi x), \\ u(0, y) &=0, & u(1, y) &=0. \end{aligned} \tag{6.E.40}$$

Hint: Use superposition.

Exercise $6.E.4.9.10$

Let R be the region described by $0 < x < 1$ and $0 < y < 1$. Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u(x,0) = \sin(\pi x), &\qquad u(x,1) = \sin(\pi x), \\ u(0,y) = \sin(\pi y), &\qquad u(1,y) = \sin(\pi y). \end{aligned} \tag{6.E.41}$$

Hint: Use superposition.

Exercise $6.E.4.9.11$: (challenging)

Using only your intuition find $u(1/2,1/2)$, for the problem $\Delta u = 0$, where $u(0,y) = u(1,y) = 100$ for $0 < y < 1$, and $u(x,0) = u(x,1) = 0$ for $0 < x < 1$. Explain.

Exercise $6.E.4.9.12$

Let R be the region described by $0 < x < 1$ and $0 < y < 1$. Solve the problem

$$\Delta u = 0, \quad u(x,0) = \sum_{n=1}^{\infty} \sin(n\pi x), \quad u(x,1) = 0, \quad u(0,y) = 0, \quad u(1,y) = 0.$$

Answer

$u(x,y) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x) \left(\frac{\sinh(n\pi(1-y))}{\sinh(n\pi)} \right)$

Exercise $6.E.4.9.13$

Let R be the region described by $0 < x < 1$ and $0 < y < 2$. Solve the problem

$$\Delta u = 0, \quad u(x,0) = 0.1 \sin(\pi x), \quad u(x,2) = 0, \quad u(0,y) = 0, \quad u(1,y) = 0.$$

Answer

$u(x,y) = 0.1 \sin(\pi x) \left(\frac{\sinh(\pi(2-y))}{\sinh(2\pi)} \right)$

6.E.10: 4.10: Dirichlet problem in the circle and the Poisson kernel

Exercise $6.E.4.10.1$

Using series solve $\Delta u = 0, u(1,\theta) = |\theta|$ for $-\pi < \theta \leq \pi$.

Exercise $6.E.4.10.2$

Using series solve $\Delta u = 0, u(1,\theta) = g(\theta)$ for the following data. Hint: trig identities.

a. $g(\theta) = 1/2 + 3\sin(\theta) + \cos(3\theta)$
b. $g(\theta) = \cos(3\theta) + 3\sin(3\theta) + \sin(9\theta)$
c. $g(\theta) = 2\cos(\theta + 1)$
d. $g(\theta) = \sin^2(\theta)$

Exercise $6.E.4.10.3$

Using the Poisson kernel, give the solution to $\Delta u = 0$, where $u(1,\theta)$ is zero for θ outside the interval $[-\pi/4, \pi/4]$and $u(1,\theta)$ is 1 for θ on the interval $[-\pi/4, \pi/4]$

Exercise $6.E.4.10.4$

a. Draw a graph for the Poisson kernel as a function of α when $r = 1/2$ and $\theta = 0$.
b. Describe what happens to the graph when you make r bigger (as it approaches 1).
c. Knowing that the solution $u(r,\theta)$ is the weighted average of $g(\theta)$ with Poisson kernel as the weight, explain what your answer to part b means.

Exercise $6.E.4.10.5$

Take the function $g(\theta)$ to be the function $xy = \cos(\theta)\sin(\theta)$ on the boundary. Use the series solution to find a solution to the Dirichlet problem $\Delta u = 0, u(1,\theta) = g(\theta)$. Now convert the solution to Cartesian coordinates x and y. Is this solution surprising? Hint: use your trig identities.

Exercise $6.E.4.10.6$

Carry out the computation we needed in the separation of variables and solve $r^2R'' + rR' - n^2R = 0$, for $n = 0, 1, 2, 3, \dots$.

Exercise $6.E.4.10.7$: (challenging)

Derive the series solution to the Dirichlet problem if the region is a circle of radius ρ rather than 1. That is, solve $\Delta u = 0, u(\rho,\theta) = g(\theta)$.

Exercise $6.E.4.10.8$: (challenging)

1. Find the solution for $\Delta u = 0,\ u(1,\theta) = x^2y^3 + 5x^2$. Write the answer in Cartesian coordinates.
2. Now solve $\Delta u = 0,\ u(1,\theta) = x^ky^l$. Write the solution in Cartesian coordinates.
3. Suppose you have a polynomial $P(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} c_{j,k}x^jy^k$, solve $\Delta u = 0,\ u(1,\theta) = P(x,y)$ (that is, write down the formula for the answer). Write the answer in Cartesian coordinates.

Notice the answer is again a polynomial in x and y. See also Exercise $6.E.5$.

Exercise $6.E.4.10.9$

Using series solve $\Delta u = 0,\ u(1,\theta) = 1 + \sum_{n=1}^{\infty}\frac{1}{n^2}\sin(n\theta)$.

Answer

$$u = 1\sum_{n=1}^{\infty}\frac{1}{n^2}r^n\sin(n\theta)$$

Exercise $6.E.4.10.10$

Using the series solution find the solution to $\Delta u = 0,\ u(1,\theta) = 1 - \cos(\theta)$. Express the solution in Cartesian coordinates (that is, using x and y).

Answer

$$u = 1 - x$$

Exercise $6.E.4.10.11$

a. Try and guess a solution to $\Delta u = -1,\ u(1,\theta) = 0$. Hint: try a solution that only depends on r. Also first, don't worry about the boundary condition.
b. Now solve $\Delta u = -1,\ u(1,\theta) = \sin(2\theta)$ using superposition.

Answer

a. $u = \frac{-1}{4} r^2 + \frac{1}{4}$
b. $u = \frac{-1}{4} r^2 + \frac{1}{4} + r^2 \sin(2\theta)$

? Exercise $6.E.4.10.12$: (challenging)

Derive the Poisson kernel solution if the region is a circle of radius ρ rather than 1. That is, solve $\Delta u = 0,\ u(\rho, \theta) = g(\theta)$.

Answer

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{\rho - 2r\rho \cos(\theta - \alpha) + r^2} g(\alpha) d\alpha$$

CHAPTER OVERVIEW

7: Eigenvalue problems

Contributors and Attributions

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7.1: Sturm-Liouville problems

7.1.1: Boundary Value Problems

In Chapter 4 we have encountered several different eigenvalue problems such as:

$$X''(x) + \lambda X(x) = 0$$

with different boundary conditions

$$\begin{array}{lll} X(0)=0 & X(L)=0 & \text{(Dirichlet), or} \\ X'(0)=0 & X'(L)=0 & \text{(Neumann), or} \\ X'(0)=0 & X(L)=0 & \text{(Mixed), or} \\ X(0)=0 & X'(L)=0 & \text{(Mixed)}, \ldots \end{array}$$

For example for the insulated wire, Dirichlet conditions correspond to applying a zero temperature at the ends, Neumann means insulating the ends, etc…. Other types of endpoint conditions also arise naturally, such as the *Robin boundary conditions*

$$hX(0) - X'(0) = 0 \quad hX(L) + X'(L) = 0,$$

for some constant h. These conditions come up when the ends are immersed in some medium.

Boundary problems came up in the study of the heat equation $u_t = ku_{xx}$ when we were trying to solve the equation by the method of separation of variables in Section 4.6. In the computation we encountered a certain eigenvalue problem and found the eigenfunctions $X_n(x)$. We then found the *eigenfunction decomposition* of the initial temperature $f(x) = u(x,0)$ in terms of the eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x).$$

Once we had this decomposition and found suitable $T_n(t)$ such that $T_n(0) = 1$ and $T_n(t)X(x)$ were solutions, the solution to the original problem including the initial condition could be written as

$$u(x,t) = \sum_{n=1}^{\infty} c_n T_n(t) X_n(x).$$

We will try to solve more general problems using this method. First, we will study second order linear equations of the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x)y = 0. \tag{7.1.1}$$

Essentially any second order linear equation of the form $a(x)y'' + b(x)y' + c(x)y + \lambda d(x)y = 0$ can be written as (7.1.1) after multiplying by a proper factor.

Example 7.1.1: Sturm-Liouville Problem

Put the following equation into the form (7.1.1):

$$x^2y'' + xy' + (\lambda x^2 - n^2)y = 0.$$

Multiply both sides by $\frac{1}{x}$ to obtain

$$\frac{1}{x}(x^2y'' + xy' + (\lambda x^2 - n^2)y) \quad = xy'' + y' + \left(\lambda x - \frac{n^2}{x}\right)y \qquad = \frac{d}{dx}\left(x\frac{dy}{dx}\right) - \frac{n^2}{x}y + \lambda xy = 0. \tag{7.1.2}$$

The Bessel equation turns up for example in the solution of the two-dimensional wave equation. If you want to see how one solves the equation, you can look at subsection 7.3.3.

The so-called *Sturm-Liouville problem*[1] is to seek nontrivial solutions to

$$\begin{aligned} \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x)y &= 0, \quad a < x < b, \\ \alpha_1 y(a) - \alpha_2 y'(a) &= 0, \\ \beta_1 y(b) + \beta_2 y'(b) &= 0. \end{aligned} \tag{7.1.3}$$

In particular, we seek λs that allow for nontrivial solutions. The λs that admit nontrivial solutions are called the *eigenvalues* and the corresponding nontrivial solutions are called *eigenfunctions*. The constants α_1 and α_2 should not be both zero, same for β_1 and β_2.

Theorem 7.1.1

Suppose $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous on $[a, b]$ and suppose $p(x) > 0$ and $r(x) > 0$ for all x in $[a, b]$. Then the Sturm-Liouville problem (5.1.8) has an increasing sequence of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

such that

$$\lim_{n\to\infty} \lambda_n = +\infty$$

and such that to each λ_n there is (up to a constant multiple) a single eigenfunction $y_n(x)$.

Moreover, if $q(x) \geq 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, then $\lambda_n \geq 0$ for all n.

Problems satisfying the hypothesis of the theorem (including the "Moreover") are called *regular Sturm-Liouville problems*, and we will only consider such problems here. That is, a regular problem is one where $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous, $p(x) > 0$, $r(x) > 0$, $q(x) \geq 0$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$. Note: Be careful about the signs. Also be careful about the inequalities for r and p, they must be strict for all x in the interval $[a, b]$, including the endpoints!

When zero is an eigenvalue, we usually start labeling the eigenvalues at 0 rather than at 1 for convenience. That is we label the eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$.

Example 7.1.2

The problem $y'' + \lambda y$, $0 < x < L$, $y(0) = 0$, and $y(L) = 0$ is a regular Sturm-Liouville problem: $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, and we have $p(x)1 > 0$ and $r(x)1 > 0$. We also have $a = 0$, $b = L$, $\alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = 0$. The eigenvalues are $\lambda_n = \frac{n^2\pi^2}{L^2}$ and eigenfunctions are $y_n(x) = \sin(\frac{n\pi}{L}x)$. All eigenvalues are nonnegative as predicted by the theorem.

Exercise 7.1.1

Find eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) = 0.$$

Identify the p, q, r, α_j, β_j. Can you use the theorem to make the search for eigenvalues easier? (Hint: Consider the condition $-y'(0) = 0$)

Example 7.1.3

Find eigenvalues and eigenfunctions of the problem

$$\begin{aligned} y'' + \lambda y &= 0, \quad & 0 < x < 1, \\ hy(0) - y'(0) &= 0, \quad & y'(1) = 0, \quad h > 0. \end{aligned} \tag{7.1.4}$$

These equations give a regular Sturm-Liouville problem.

> **Exercise 7.1.2**
>
> Identify p, q, r, α_j, β_j in the example above.

First note that $\lambda \geq 0$ by Theorem 7.1.1. Therefore, the general solution (without boundary conditions) is

$$\begin{aligned} y(x) &= A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x) && \text{if } \lambda > 0, \\ y(x) &= Ax + B && \text{if } \lambda = 0. \end{aligned} \tag{7.1.5}$$

Let us see if $\lambda = 0$ is an eigenvalue: We must satisfy $0 = hB - A$ and $A = 0$, hence $B = 0$ (as $h > 0$), therefore, 0 is not an eigenvalue (no nonzero solution, so no eigenfunction).

Now let us try $h > 0$. We plug in the boundary conditions.

$$\begin{aligned} 0 &= hA - \sqrt{\lambda}B, \\ 0 &= -A\sqrt{\lambda}\sin(\sqrt{\lambda}) + B\sqrt{\lambda}\cos(\sqrt{\lambda}). \end{aligned} \tag{7.1.6}$$

If $A = 0$, then $B = 0$ and vice-versa, hence both are nonzero. So $B = \frac{hA}{\sqrt{\lambda}}$, and $0 = -A\sqrt{\lambda}\sin(\sqrt{\lambda}) + \frac{hA}{\sqrt{\lambda}}\sqrt{\lambda}\cos(\sqrt{\lambda})$. As $A \neq 0$ we get

$$0 = -\sqrt{\lambda}\sin(\sqrt{\lambda}) + h\cos(\sqrt{\lambda}),$$

or

$$\frac{h}{\sqrt{\lambda}} = \tan\sqrt{\lambda}.$$

Now use a computer to find λ_n. There are tables available, though using a computer or a graphing calculator is far more convenient nowadays. Easiest method is to plot the functions $\frac{h}{x}$ and $\tan(x)$ and see for which x they intersect. There is an infinite number of intersections. Denote the first intersection by $\sqrt{\lambda_1}$ the first intersection, by $\sqrt{\lambda_2}$ the second intersection, etc.... For example, when $h = 1$, we get that $\sqrt{\lambda_1} \approx 0.86$, $\sqrt{\lambda_2} \approx 3.43, \ldots$. That is $\lambda_1 \approx 0.74$, $\lambda_2 \approx 11.73, \ldots$, ... A plot for $h = 1$ is given in Figure 7.1.1. The appropriate eigenfunction (let $A = 1$ for convenience, then $B = \frac{h}{\sqrt{\lambda}}$) is

$$y_n(x) = \cos(\sqrt{\lambda_n}x) + \frac{h}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}x).$$

When $h = 1$ we get (approximately)

$$y_1(x) \approx \cos(0.86x) + \frac{1}{0.86}\sin(0.86x), \quad y_2(x) \approx \cos(3.43x) + \frac{1}{3.43}\sin(3.43x), \quad \ldots.$$

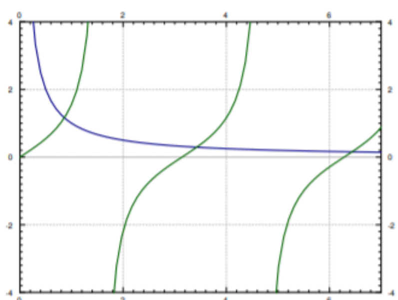

Figure 7.1.1: Plot of $\frac{1}{x}$ and $\tan x$.

7.1.2: Orthogonality

We have seen the notion of orthogonality before. For example, we have shown that $\sin(nx)$ are orthogonal for distinct n on $[0, \pi]$. For general Sturm-Liouville problems we will need a more general setup. Let $r(x)$ be a *weight function* (any function, though generally we will assume it is positive) on $[a, b]$. Two functions $f(x)$, $g(x)$ are said to be *orthogonal* with respect to the weight function $r(x)$ when

$$\int_a^b f(x)g(x)r(x)dx = 0.$$

In this setting, we define the inner product as

$$\langle f, g\rangle \stackrel{\text{def}}{=} \int_a^b f(x)g(x)r(x)dx,$$

and then say f and g are orthogonal whenever $\langle f, g\rangle = 0$. The results and concepts are again analogous to finite dimensional linear algebra.

The idea of the given inner product is that those x where $r(x)$ is greater have more weight. Nontrivial (nonconstant) $r(x)$ arise naturally, for example from a change of variables. Hence, you could think of a change of variables such that $d\xi = r(x)dx$.

Eigenfunctions of a regular Sturm–Liouville problem satisfy an orthogonality property, just like the eigenfunctions in Section 4.1. Its proof is very similar to the analogous Theorem 4.1.1.

Theorem 7.1.2

Suppose we have a regular Sturm-Liouville problem

$$\begin{aligned} \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x)y &= 0, \\ \alpha_1 y(a) - \alpha_2 y'(a) &= 0, \\ \beta_1 y(b) + \beta_2 y'(b) &= 0. \end{aligned} \tag{7.1.7}$$

Let y_j and y_k be two distinct eigenfunctions for two distinct eigenvalues λ_j and λ_k. Then

$$\int_a^b y_j(x)y_k(x)r(x)dx = 0,$$

that is, y_j and y_k are orthogonal with respect to the weight function r.

7.1.3: Fredholm Alternative

We also have the *Fredholm alternative* theorem we talked about before (Theorem 4.1.2) for all regular Sturm-Liouville problems. We state it here for completeness.

Theorem 7.1.3

Fredholm Alternative

Suppose that we have a regular Sturm-Liouville problem. Then either

$$\begin{aligned} \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x)y &= 0, \\ \alpha_1 y(a) - \alpha_2 y'(a) &= 0, \\ \beta_1 y(b) + \beta_2 y'(b) &= 0, \end{aligned} \tag{7.1.8}$$

has a nonzero solution, or

$$\begin{aligned} \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x)y &= f(x), \\ \alpha_1 y(a) - \alpha_2 y'(a) &= 0, \\ \beta_1 y(b) + \beta_2 y'(b) &= 0, \end{aligned} \tag{7.1.9}$$

has a unique solution for any $f(x)$ continuous on $[a, b]$.

This theorem is used in much the same way as we did before in Section 4.4. It is used when solving more general nonhomogeneous boundary value problems. The theorem does not help us solve the problem, but it tells us when a unique solution exists, so that we know when to spend time looking for it. To solve the problem we decompose $f(x)$ and $y(x)$ in terms of the eigenfunctions of the homogeneous problem, and then solve for the coefficients of the series for $y(x)$.

7.1.4: Eigenfunction Series

What we want to do with the eigenfunctions once we have them is to compute the *eigenfunction decomposition* of an arbitrary function $f(x)$. That is, we wish to write

$$f(x)=\sum_{n=1}^{\infty} c_n y_n(x), \tag{7.1.10}$$

where $y_n(x)$ the eigenfunctions. We wish to find out if we can represent any function $f(x)$ in this way, and if so, we wish to calculate c_n (and of course we would want to know if the sum converges). OK, so imagine we could write $f(x)$ as (7.1.10). We will assume convergence and the ability to integrate the series term by term. Because of orthogonality we have

$$\begin{aligned}\langle f, y_m\rangle &= \int_a^b f(x) y_m(x) r(x) dx \\ &= \sum_{n=1}^{\infty} c_n \int_a^b y_n(x) y_m(x) r(x) dx \\ &= c_m \int_a^b y_m(x) y_m(x) r(x) dx = c_m \langle y_m, y_m\rangle.\end{aligned} \tag{7.1.11}$$

Hence,

$$c_m=\frac{\langle f, y_m\rangle}{\langle y_m, y_m\rangle}=\frac{\int_a^b f(x) y_m(x) r(x) dx}{\int_a^b (y_m(x))^2 r(x) dx}. \tag{7.1.12}$$

Note that y_m are known up to a constant multiple, so we could have picked a scalar multiple of an eigenfunction such that $\langle y_m, y_m\rangle=1$ (if we had an arbitrary eigenfunction $\tilde{y}_m$, divide it by $\sqrt{\langle\tilde{y}_m, \tilde{y}_m\rangle}$). When $\langle y_m, y_m\rangle=1$ we have the simpler form $c_m=\langle f, y_m\rangle$ as we did for the Fourier series. The following theorem holds more generally, but the statement given is enough for our purposes.

Theorem 7.1.4

Suppose f is a piecewise smooth continuous function on $[a, b]$. If $y_1, y_2, \ldots$ are the eigenfunctions of a regular Sturm-Liouville problem, then there exist real constants $c_1, c_2, \ldots$ given by (7.1.12) such that (7.1.10) converges and holds for $a<x<b$.

Example 7.1.4

Take the simple Sturm-Liouville problem

$$\begin{aligned}& y''+\lambda y=0, \quad 0<x<\frac{\pi}{2}, \\ & y(0)=0, \quad y'(\frac{\pi}{2})=0.\end{aligned} \tag{7.1.13}$$

The above is a regular problem and furthermore we know by Theorem 7.1.1 that $\lambda \geq 0$.

Suppose $\lambda=0$, then the general solution is $y(x) A x+B$, we plug in the initial conditions to get $0=y(0)=B$, and $0=y'(\pi/2)=A$, hence $\lambda=0$ is not an eigenvalue. The general solution, therefore, is

$$y(x)=A \cos(\sqrt{\lambda} x)+B \sin(\sqrt{\lambda} x).$$

Plugging in the boundary conditions we get $0=y(0)=A$ and $0=y'(\pi/2)=\sqrt{\lambda} B \cos(\sqrt{\lambda} \frac{\pi}{2})$. B cannot be zero and hence $\cos(\sqrt{\lambda} \frac{\pi}{2}=0)$. This means that $\sqrt{\lambda} \frac{\pi}{2}$ must be an odd integral multiple of $\frac{\pi}{2}$, i.e. $(2n-1)\frac{\pi}{2}=\sqrt{\lambda_n} \frac{\pi}{2}$. Hence

$$\lambda_n=(2n-1)^2.$$

We can take $B=1$. Hence our eigenfunctions are

$$y_n(x) = \sin((2n-1)x).$$

Finally we compute

$$\int_0^{\frac{\pi}{2}} (\sin((2n-1)x))^2 \, dx = \frac{\pi}{4}.$$

So any piecewise smooth function on $[0, \pi/2]$can be written as

$$f(x) = \sum_{n=1}^{\infty} c_n \sin((2n-1)x),$$

where

$$c_n = \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} = \frac{\int_0^{\frac{\pi}{2}} \sin((2n-1)x) dx}{\int_0^{\frac{\pi}{2}} (\sin((2n-1)x))^2 dx} = \frac{4}{\pi} \int f(x)_0^{\frac{\pi}{2}} \sin((2n-1)x) dx.$$

Note that the series converges to an odd 2π-periodic (not $\pi\pi$-periodic!) extension of $f(x)$.

? Exercise 7.1.3

In the above example, the function is defined on $0 < x < \pi/2$, yet the series with respect to the eigenfunctions $\sin((2n-1)x)$ converges to an odd 2π-periodic extension of $f(x)$. Find out how is the extension defined for $\pi/2 < x < \pi$.

Let us compute an example. Consider $f(x) = x$ for $0 < x < \frac{\pi}{2}$. Some calculus later we find

$$c_n = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} f(x) \, \sin\big((2n-1)x\big) \, dx = \frac{4(-1)^{n+1}}{\pi(2n-1)^2},$$

and so for x in $[0, \frac{\pi}{2}]$,

$$f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi(2n-1)^2} \sin\big((2n-1)x\big).$$

This is different from the π-periodic regular sine series which can be computed to be

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2nx).$$

Both sums converge are equal to $f(x)$ for $0 < x < \frac{\pi}{2}$, but the eigenfunctions involved come from different eigenvalue problems.

7.1.5: Footnotes

[1] Named after the French mathematicians Jacques Charles François Sturm (1803–1855) and Joseph Liouville (1809–1882).

7.2: Application of Eigenfunction Series

The eigenfunction series can arise even from higher order equations. Consider an elastic beam (say made of steel). We will study the transversal vibrations of the beam. That is, suppose the beam lies along the x-axis and let $y(x,t)$ measure the displacement of the point x on the beam at time t. See Figure $7.2.1$.

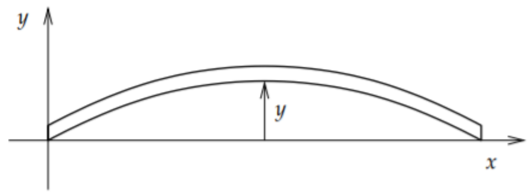

Figure $7.2.1$: Transversal vibrations of a beam.

The equation that governs this setup is

$$a^4 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0,$$

for some constant $a > 0$, let us not worry about the physics[1].

Suppose the beam is of length 1 simply supported (hinged) at the ends. The beam is displaced by some function $f(x)$ at time $t = 0$ and then let go (initial velocity is 0). Then y satisfies:

$$\begin{aligned}
&a^4 y_{xxxx} + y_{tt} = 0 \qquad (0 < x < 1, \;\; t > 0), \\
&y(0,t) = y_{xx}(0,t) = 0, \qquad (7.2.1) \\
&y(1,t) = y_{xx}(1,t) = 0, \qquad (7.2.2) \\
&y(x,0) = f(x), \qquad y_t(x,0) = 0.
\end{aligned}$$

Again we try $y(x,t) = X(x)T(t)$ and plug in to get $a^4 X^{(4)} T + XT'' = 0$ or

$$\frac{X^{(4)}}{X} = \frac{-T''}{a^4 T} = \lambda.$$

The equations are

$$T'' + \lambda a^4 T = 0, \qquad X^{(4)} - \lambda X = 0.$$

The boundary conditions $y(0,t) = y_{xx}(0,t) = 0$ and $y(1,t) = y_{xx}(1,t) = 0$ imply

$$X(0) = X''(0) = 0, \qquad \text{and} \qquad X(1) = X''(1) = 0.$$

The initial homogeneous condition $y_t(x,0) = 0$ implies

$$T'(0) = 0.$$

As usual, we leave the nonhomogeneous $y(x,0) = f(x)$ for later.

Considering the equation for T, that is, $T'' + \lambda a^4 T = 0$, and physical intuition leads us to the fact that if λ is an eigenvalue then $\lambda > 0$: We expect vibration and not exponential growth nor decay in the t direction (there is no friction in our model for instance). So there are no negative eigenvalues. Similarly $\lambda = 0$ is not an eigenvalue.

? Exercise $7.2.1$

Justify $\lambda > 0$ just from the equation for X and the boundary conditions.

Write $\omega^4 = \lambda$, so that we do not need to write the fourth root all the time. For X we get the equation $X^{(4)} - \omega^4 X = 0$. The general solution is

$$X(x) = Ae^{\omega x} + Be^{-\omega x} + C\sin(\omega x) + D\cos(\omega x).$$

Now $0 = X(0)A + B + D, 0 = X''(0) = \omega^2(A + B - D)$. Hence, $D = 0$ and $A + B = 0$, or $B = -A$. So we have

$$X(x) = Ae^{\omega x} - Ae^{-\omega x} + C\sin(\omega x).$$

Also $0 = X(1) = A(e^{\omega} - e^{-\omega}) + C\sin\omega$, and $0 = X''(1) = A\omega^2(e^{\omega} - e^{-\omega}) - C\omega^2\sin\omega$. This means that $C\sin\omega = 0$ and $A(e^{\omega} - e^{-\omega}) = 2A\sinh\omega = 0$. If $\omega > 0$, then $\omega \neq 0$ and so $A = 0$. This means that $C \neq 0$ otherwise λ is not an eigenvalue. Also ω must be an integer multiple of π. Hence $\omega = n\pi$ and $n \geq 1$ (as $\omega > 0$). We can take $C = 1$. So the eigenvalues are $\lambda_n = n^4\pi^4$ and the eigenfunctions are $\sin(n\pi x)$.

Now $T'' + n^4\pi^4a^4T = 0$. The general solution is $T(t) = A\sin(n^2\pi^2a^2t) + B\cos(n^2\pi^2a^2t)$. But $T'(0) = 0$ and hence we must have $A = 0$ and we can take $B = 1$ to make $T(0) = 1$ for convenience. So our solutions are $T_n(t) = \cos(n^2\pi^2a^2t)$.

As the eigenfunctions are just sines again, we can decompose the function $f(x)$ on $0 < x < 1$ using the sine series. We find numbers b_n such that for $0 < x < 1$ we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

Then the solution to (7.2.1) is

$$y(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\cos(n^2\pi^2a^2t).$$

The point is that X_nT_n is a solution that satisfies all the homogeneous conditions (that is, all conditions except the initial position). And since and $T_n(0) = 1$, we have

$$y(x,0) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(0) = \sum_{n=1}^{\infty} b_n X_n(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = f(x).$$

So $y(x,t)$ solves (7.2.1).

The natural (circular) frequencies of the system are $n^2\pi^2a^2$. These frequencies are all integer multiples of the fundamental frequency π^2a^2, so we get a nice musical note. The exact frequencies and their amplitude are what we call the *timbre* of the note.

The timbre of a beam is different than for a vibrating string where we get “more” of the lower frequencies since we get all integer multiples, $1, 2, 3, 4, 5, \ldots$ For a steel beam we get only the square multiples $1, 4, 9, 16, 25, \ldots$ That is why when you hit a steel beam you hear a very pure sound. The sound of a xylophone or vibraphone is, therefore, very different from a guitar or piano.

Example 7.2.1

Let us assume that $f(x) = \frac{x(x-1)}{10}$. On $0 < x < 1$ we have (you know how to do this by now)

$$f(x) = \sum_{\substack{n=1 \\ \text{n odd}}}^{\infty} \frac{4}{5\pi^3n^3}\sin(n\pi x).$$

Hence, the solution to (7.2.1) with the given initial position $f(x)$ is

$$y(x,t) = \sum_{\substack{n=1 \\ \text{n odd}}}^{\infty} \frac{4}{5\pi^3n^3}\sin(n\pi x)\cos(n^2\pi^2a^2t).$$

There are other boundary conditions than just hinged ends. There are three basic possibilities: hinged, free, or fixed. Let us consider the end at $x = 0$. For the other end, it is the same idea. If the end is *hinged*, then

$$u(0,t) = u_{xx}(0,t) = 0.$$

If the end is *free*, that is, it is just floating in air, then

$$u_{xx}(0,t) = u_{xxx}(0,t) = 0.$$

And finally, if the end is *clamped* or *fixed*, for example it is welded to a wall, then

$$u(0,t) = u_x(0,t) = 0.$$

7.2.1: Footnotes

[1] If you are interested, $a^4 = \frac{EI}{\rho}$, where E is the elastic modulus, I is the second moment of area of the cross section, and ρ is linear density.

7.3: Steady Periodic Solutions

7.3.1: Forced Vibrating String

Consider a guitar string of length L. We studied this setup in Section 4.7. Let x be the position on the string, t the time, and y the displacement of the string. See Figure 7.3.1.

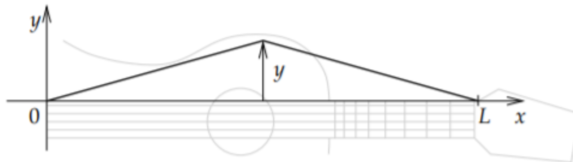

Figure 7.3.1: Vibrating string.

The problem is governed by the equations

$$\begin{aligned} &y_{tt} = a^2 y_{xx}, \\ &y(0,t) = 0, \qquad y(L,t) = 0, \\ &y(x,0) = f(x), \quad y_t(x,0) = g(x). \end{aligned} \tag{7.3.1}$$

We saw previously that the solution is of the form

$$y = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi a}{L} t\right) + B_n \sin\left(\frac{n\pi a}{L} t\right) \right) \sin\left(\frac{n\pi}{L} x\right),$$

where A_n and B_n were determined by the initial conditions. The natural frequencies of the system are the (circular) frequencies $\frac{n\pi a}{L}$ for integers $n \geq 1$.

But these are free vibrations. What if there is an external force acting on the string. Let us assume say air vibrations (noise), for example a second string. Or perhaps a jet engine. For simplicity, assume nice pure sound and assume the force is uniform at every position on the string. Let us say $F(t) = F_0 \cos(\omega t)$ as force per unit mass. Then our wave equation becomes (remember force is mass times acceleration)

$$y_{tt} = a^2 y_{xx} + F_0 \cos(\omega t), \tag{7.3.2}$$

with the same boundary conditions of course.

We want to find the solution here that satisfies the above equation and

$$y(0,t) = 0, \quad y(L,t) = 0, \quad y(x,0) = 0, \quad y_t(x,0) = 0. \tag{7.3.3}$$

That is, the string is initially at rest. First we find a particular solution y_p of (7.3.2) that satisfies $y(0,t) = y(L,t) = 0$. We define the functions f and g as

$$f(x) = -y_p(x,0), \quad g(x) = -\frac{\partial y_p}{\partial t}(x,0).$$

We then find solution y_c of (7.3.1). If we add the two solutions, we find that $y = y_c + y_p$ solves (7.3.2) with the initial conditions.

Exercise 7.3.1

Check that $y = y_c + y_p$ solves (7.3.2) and the side conditions (7.3.3).

So the big issue here is to find the particular solution y_p. We look at the equation and we make an educated guess

$$y_p(x,t) = X(x)\cos(\omega t).$$

We plug in to get

$$-\omega^2 X \cos(\omega t) = a^2 X'' \cos(\omega t),$$

or $-\omega X = a^2 X'' + F_0$ after canceling the cosine. We know how to find a general solution to this equation (it is a nonhomogeneous constant coefficient equation). The general solution is

$$X(x) = A\cos\left(\frac{\omega}{a}x\right) + B\sin\left(\frac{\omega}{a}x\right) - \frac{F_0}{\omega^2}.$$

The endpoint conditions imply $X(0) = X(L) = 0$. So

$$0 = X(0) = A - \frac{F_0}{\omega^2},$$

or $A = \frac{F_0}{\omega^2}$, and also

$$0 = X(L) = \frac{F_0}{\omega^2}\cos\left(\frac{\omega L}{a}\right) + B\sin\left(\frac{\omega L}{a}\right) - \frac{F_0}{\omega^2}.$$

Assuming that $\sin\left(\frac{\omega L}{a}\right)$ is not zero we can solve for B to get

$$B = \frac{-F_0\left(\cos\left(\frac{\omega L}{a}\right) - 1\right)}{-\omega^2 \sin\left(\frac{\omega L}{a}\right)}. \tag{7.3.4}$$

Therefore,

$$X(x) = \frac{F_0}{\omega^2}\left(\cos\left(\frac{\omega}{a}x\right) - \frac{\cos\left(\frac{\omega L}{a}\right) - 1}{\sin\left(\frac{\omega L}{a}\right)}\sin\left(\frac{\omega}{a}x\right) - 1\right).$$

The particular solution y_p we are looking for is

$$y_p(x,t) = \frac{F_0}{\omega^2}\left(\cos\left(\frac{\omega}{a}x\right) - \frac{\cos\left(\frac{\omega L}{a}\right) - 1}{\sin\left(\frac{\omega L}{a}\right)}\sin\left(\frac{\omega}{a}x\right) - 1\right)\cos(\omega t).$$

? Exercise 7.3.2

Check that y_p works.

Now we get to the point that we skipped. Suppose that $\sin\left(\frac{\omega L}{a}\right) = 0$. What this means is that ω is equal to one of the natural frequencies of the system, i.e. a multiple of $\frac{\pi a}{L}$. We notice that if ω is not equal to a multiple of the base frequency, but is very close, then the coefficient B in (7.3.4) seems to become very large. But let us not jump to conclusions just yet. When $\omega = \frac{n\pi a}{L}$ for n even, then $\cos\left(\frac{\omega L}{a}\right) = 1$ and hence we really get that $B = 0$. So resonance occurs only when both $\cos\left(\frac{\omega L}{a}\right) = -1$ and $\sin\left(\frac{\omega L}{a}\right) = 0$. That is when $\omega = \frac{n\pi a}{L}$ for odd n.

We could again solve for the resonance solution if we wanted to, but it is, in the right sense, the limit of the solutions as ω gets close to a resonance frequency. In real life, pure resonance never occurs anyway.

The above calculation explains why a string will begin to vibrate if the identical string is plucked close by. In the absence of friction this vibration would get louder and louder as time goes on. On the other hand, you are unlikely to get large vibration if the forcing frequency is not close to a resonance frequency even if you have a jet engine running close to the string. That is, the amplitude will not keep increasing unless you tune to just the right frequency.

Similar resonance phenomena occur when you break a wine glass using human voice (yes this is possible, but not easy[1]) if you happen to hit just the right frequency. Remember a glass has much purer sound, i.e. it is more like a vibraphone, so there are far fewer resonance frequencies to hit.

When the forcing function is more complicated, you decompose it in terms of the Fourier series and apply the above result. You may also need to solve the above problem if the forcing function is a sine rather than a cosine, but if you think about it, the solution is almost the same.

Example 7.3.1

Let us do the computation for specific values. Suppose $F_0 = 1$ and $\omega = 1$ and $L = 1$ and $a = 1$. Then

$$y_p(x,t) = \left(\cos(x) - \frac{\cos(1)-1}{\sin(1)}\sin(x) - 1\right)\cos(t).$$

Write $B = \frac{\cos(1)-1}{\sin(1)}$ for simplicity.

Then plug in $t = 0$ to get

$$f(x) = -y_p(x,0) = -\cos x + B\sin x + 1,$$

and after differentiating in t we see that $g(x) = -\frac{\partial y_P}{\partial t}(x,0) = 0$.

Hence to find y_c we need to solve the problem

$$\begin{aligned} &y_{tt} = y_{xx}, \\ &y(0,t) = 0, \quad y(1,t) = 0, \\ &y(x,0) = -\cos x + B\sin x + 1, \\ &y_t(x,0) = 0. \end{aligned} \tag{7.3.5}$$

Note that the formula that we use to define $y(x,0)$ is not odd, hence it is not a simple matter of plugging in to apply the D'Alembert formula directly! You must define F to be the odd, 2-periodic extension of $y(x,0)$. Then our solution would look like

$$y(x,t) = \frac{F(x+t)+F(x-t)}{2} + \left(\cos(x) - \frac{\cos(1)-1}{\sin(1)}\sin(x) - 1\right)\cos(t). \tag{7.3.6}$$

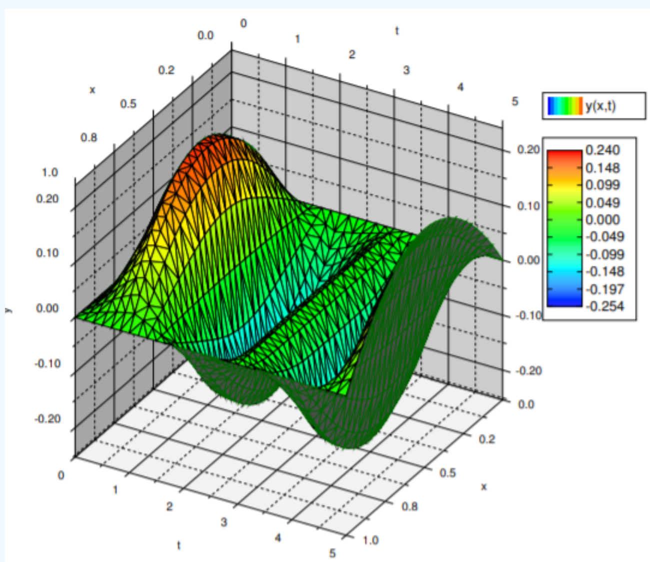

Figure 7.3.2: Plot of $y(x,t) = \frac{F(x+t)+F(x-t)}{2} + \left(\cos(x) - \frac{\cos(1)-1}{\sin(1)}\sin(x) - 1\right)\cos(t)$.

It is not hard to compute specific values for an odd extension of a function and hence (7.3.6) is a wonderful solution to the problem. For example it is very easy to have a computer do it, unlike a series solution. A plot is given in Figure 7.3.2.

7.3.2: Underground Temperature Oscillations

Let $u(x,t)$ be the temperature at a certain location at depth x underground at time t. See Figure 7.3.3.

The temperature u satisfies the heat equation $u_t = ku_{xx}$, where k is the diffusivity of the soil. We know the temperature at the surface $u(0,t)$ from weather records. Let us assume for simplicity that

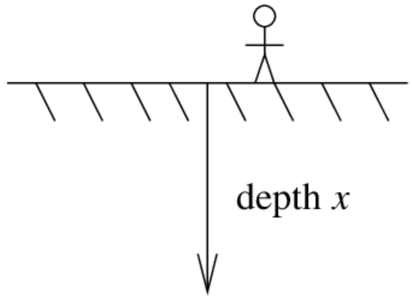

Figure 7.3.3: Underground temperature.

$$u(0,t) = T_0 + A_0 \cos(\omega t),$$

where T_0 is the yearly mean temperature, and $t=0$ is midsummer (you can put negative sign above to make it midwinter if you wish). A_0 gives the typical variation for the year. That is, the hottest temperature is $T_0 + A_0$ and the coldest is $T_0 - A_0$. For simplicity, we will assume that $T_0 = 0$. The frequency ω is picked depending on the units of t, such that when $t=1$, then $\omega t = 2\pi$. For example if t is in years, then $\omega = 2\pi$.

It seems reasonable that the temperature at depth x will also oscillate with the same frequency. This, in fact, will be the steady periodic solution, independent of the initial conditions. So we are looking for a solution of the form

$$u(x,t) = V(x)\cos(\omega t) + W(x)\sin(\omega t).$$

for the problem

$$u_t = k u_{xx}, \qquad u(0,t) = A_0 \cos(\omega t). \tag{7.3.7}$$

We will employ the complex exponential here to make calculations simpler. Suppose we have a complex valued function

$$h(x,t) = X(x)e^{i\omega t}.$$

We will look for an h such that $\text{Re} h = u$. To find an h, whose real part satisfies (7.3.7), we look for an h such that

$$h_t = k h_{xx}, \qquad h(0,t) = A_0 e^{i\omega t}. \tag{7.3.8}$$

? Exercise 7.3.3

Suppose h satisfies (7.3.8). Use Euler's formula for the complex exponential to check that $u = \text{Re}\, h$ satisfies (7.3.7).

Substitute h into (7.3.8).

$$i\omega X e^{i\omega t} = kX'' e^{i\omega t}.$$

Hence,

$$kX'' - i\omega X = 0,$$

or

$$X'' - \alpha^2 X = 0,$$

where $\alpha = \pm\sqrt{\frac{i\omega}{k}}$. Note that $\pm\sqrt{i} = \pm\frac{1+i}{\sqrt{2}}$ so you could simplify to $\alpha = \pm(1+i)\sqrt{\frac{\omega}{2k}}$. Hence the general solution is

$$X(x) = Ae^{-(1+i)\sqrt{\frac{\omega}{2k}}x} + Be^{(1+i)\sqrt{\frac{\omega}{2k}}x}.$$

We assume that an $X(x)$ that solves the problem must be bounded as $x \to \infty$ since $u(x,t)$ should be bounded (we are not worrying about the earth core!). If you use Euler's formula to expand the complex exponentials, you will note that the second term will be unbounded (if $B \neq 0$), while the first term is always bounded. Hence $B = 0$.

? Exercise 7.3.4

Use Euler's formula to show that $e^{(1+i)\sqrt{\frac{\omega}{2k}}x}$ is unbounded as $x \to \infty$, while $e^{-(1+i)\sqrt{\frac{\omega}{2k}}x}$ is bounded as $x \to \infty$.

Furthermore, $X(0) = A_0$ since $h(0,t) = A_0 e^{i\omega t}$. Thus $A = A_0$. This means that

$$h(x,t) = A_0 e^{-(1+i)\sqrt{\frac{\omega}{2k}}x} e^{i\omega t} = A_0 e^{-(1+i)\sqrt{\frac{\omega}{2k}}x + i\omega t} = A_0 e^{-\sqrt{\frac{\omega}{2k}}x} e^{i(\omega t - \sqrt{\frac{\omega}{2k}}x)}.$$

We will need to get the real part of h, so we apply Euler's formula to get

$$h(x,t) = A_0 e^{-\sqrt{\frac{\omega}{2k}}x} \left(\cos\left(\omega t - \sqrt{\frac{\omega}{2k}}x\right) + i \sin\left(\omega t - \sqrt{\frac{\omega}{2k}}x\right) \right).$$

Then finally

$$u(x,t) = \operatorname{Re} h(x,t) = A_0 e^{-\sqrt{\frac{\omega}{2k}}x} \cos\left(\omega t - \sqrt{\frac{\omega}{2k}}x\right).$$

Yay!

Notice the phase is different at different depths. At depth x the phase is delayed by $x\sqrt{\frac{\omega}{2k}}$. For example in cgs units (centimeters-grams-seconds) we have $k = 0.005$ (typical value for soil), $\omega = \frac{2\pi}{\text{seconds in a year}} = \frac{2\pi}{31{,}557{,}341} \approx 1.99 \times 10^{-7}$. Then if we compute where the phase shift $x\sqrt{\frac{\omega}{2k}} = \pi$ we find the depth in centimeters where the seasons are reversed. That is, we get the depth at which summer is the coldest and winter is the warmest. We get approximately 700 centimeters, which is approximately 23 feet below ground.

Be careful not to jump to conclusions. The temperature swings decay rapidly as you dig deeper. The amplitude of the temperature swings is $A_0 e^{-\sqrt{\frac{\omega}{2k}}x}$. This function decays very quickly as x (the depth) grows. Let us again take typical parameters as above. We will also assume that our surface temperature swing is $\pm 15^\circ$ Celsius, that is, $A_0 = 15$. Then the maximum temperature variation at 700 centimeters is only $\pm 0.66^\circ$ Celsius.

You need not dig very deep to get an effective "refrigerator," with nearly constant temperature. That is why wines are kept in a cellar; you need consistent temperature. The temperature differential could also be used for energy. A home could be heated or cooled by taking advantage of the above fact. Even without the earth core you could heat a home in the winter and cool it in the summer. The earth core makes the temperature higher the deeper you dig, although you need to dig somewhat deep to feel a difference. We did not take that into account above.

7.3.3: Footnotes

[1] Mythbusters, episode 31, Discovery Channel, originally aired may 18th 2005.

7.E: Eigenvalue Problems (Exercises)

These are homework exercises to accompany Libl's "Differential Equations for Engineering" Textmap. This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Prerequisite for the course is the basic calculus sequence.

7.E.1: 5.1: Sturm-Liouville problems

Exercise $7.E.5.1.1$

Find eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) - y'(0) = 0, \quad y(1) = 0. \tag{7.E.1}$$

Exercise $7.E.5.1.2$

Expand the function $f(x) = x$ on $0 \leq x \leq 1$ using the eigenfunctions of the system

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0. \tag{7.E.2}$$

Exercise $7.E.5.1.3$

Suppose that you had a Sturm-Liouville problem on the interval $[0,1]$ and came up with $y_n(x) = \sin(\gamma n x)$, where $\gamma > 0$ is some constant. Decompose $f(x) = x, 0 < x < 1$, in terms of these eigenfunctions.

Exercise $7.E.5.1.4$

Find eigenvalues and eigenfunctions of

$$y'^{(4)} + \lambda y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y(1) = 0 \quad y'(1) = 0. \tag{7.E.3}$$

This problem is not a Sturm-Liouville problem, but the idea is the same.

Exercise $7.E.5.1.5$: (more challenging)

Find eigenvalues and eigenfunctions for

$$\frac{d}{dx}(e^x y') + \lambda e^x y = 0, \quad y(0) = 0, \quad y(1) = 0. \tag{7.E.4}$$

Hint: First write the system as a constant coefficient system to find general solutions. Do note that Theorem 5.1.1 guarantees $\lambda \geq 0$.

Exercise $7.E.5.1.6$

Find eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \quad y(-1) = 0, \quad y(1) = 0. \tag{7.E.5}$$

Answer

$\lambda_n = \frac{(2n-1)\pi}{2}, \ n = 1, \ 2, \ 3, \cdots, \ y_n = \cos\left(\frac{(2n-1)\pi}{2} x\right)$

Exercise $7.E.5.1.7$

Put the following problems into the standard form for Sturm-Liouville problems, that is, find $p(x), q(x), r(x), \alpha_1, \alpha_, \beta_1, \beta_1$, and decide if the problems are regular or not.

a. $xy'' + \lambda y = 0$ for $0 < x < 1,\ y(0) = 0,\ y(1) = 0,$
b. $(1+x^2)y'' + 2xy' + (\lambda - x^2)y = 0 \quad$ for $-1 < x < 1,\ y(-1) = 0,\ y(1) + y'(1) = 0$

Answer

a. $p(x) = 1,\ q(x) = 0,\ r(x) = \frac{1}{x},\ \alpha_1 = 1,\ \alpha_2 = 0,\ \beta_1 = 1,\ \beta_2 = 0$. The problem is not regular.
b. $p(x) = 1 + x^2,\ q(x) = x^2,\ r(x) = 1,\ \alpha_1 = 1,\ \alpha_2 = 0,\ \beta_1 = 1,\ \beta_2 = 1$. The problem is regular.

7.E.2: 5.2: Application of eigenfunction series

Exercise $7.E.5.2.1$

Suppose you have a beam of length 5 with free ends. Let y be the transverse deviation of the beam at position x on the beam $(0 < x < 5)$. You know that the constants are such that this satisfies the equation $y_{tt} + 4y_{xxxx} = 0$. Suppose you know that the initial shape of the beam is the graph of $x(5 - x)$, and the initial velocity is uniformly equal to 2 (same for each x) in the positive y direction. Set up the equation together with the boundary and initial conditions. Just set up, do not solve.

Exercise $7.E.5.2.2$

Suppose you have a beam of length 5 with one end free and one end fixed (the fixed end is at $x = 5$). Let u be the longitudinal deviation of the beam at position x on the beam $(0 < x < 5)$. You know that the constants are such that this satisfies the equation $u_{tt} = 4u_{xx}$. Suppose you know that the initial displacement of the beam is $\frac{x-5}{50}$, and the initial velocity is $\frac{-(x-5)}{100}$ in the positive u direction. Set up the equation together with the boundary and initial conditions. Just set up, do not solve.

Exercise $7.E.5.2.3$

Suppose the beam is L units long, everything else kept the same as in (5.2.2). What is the equation and the series solution?

Exercise $7.E.5.2.4$

Suppose you have

$$\begin{aligned} &a^4 y_{xxxx} + y_{tt} = 0 \quad (0 < x < 1, t > 0), \\ &y(0,t) = y_{xx}(0,t) = 0, \\ &y(1,t) = y_{xx}(1,t) = 0, \\ &y(x,0) = f(x), \quad y_t(x,0) = g(x). \end{aligned} \tag{7.E.6}$$

That is, you have also an initial velocity. Find a series solution. Hint: Use the same idea as we did for the wave equation.

Exercise $7.E.5.2.5$

Suppose you have a beam of length 1 with hinged ends. Let y be the transverse deviation of the beam at position x on the beam $(0 < x < 1)$. You know that the constants are such that this satisfies the equation $y_{tt} + 4y_{xxxx} = 0$. Suppose you know that the initial shape of the beam is the graph of $\sin(\pi x)$, and the initial velocity is 0. Solve for y.

Answer

$y(x,t) = \sin(\pi x)\cos(2\pi^2 t)$

Exercise $7.E.5.2.6$

Suppose you have a beam of length 10 with two fixed ends. Let y be the transverse deviation of the beam at position x on the beam ($0 < x < 10$). You know that the constants are such that this satisfies the equation $y_{tt} + 9y_{xxxx} = 0$. Suppose you know that the initial shape of the beam is the graph of $\sin(\pi x)$, and the initial velocity is uniformly equal to $x(10-x)$. Set up the equation together with the boundary and initial conditions. Just set up, do not solve.

Answer

$$9y_{xxxx} + y_{tt} = 0 \quad (0 < x < 10,\ t > 0)\,, \qquad y(0,t) = y_x(0,t) = 0, \qquad y(10,t) = y_x(10,t) = 0, \qquad y(x,0) = \sin(\pi x)\,, \quad y_t(x,0) = x(10-x)\,.$$

7.E.3: 5.3: Steady periodic solutions

Exercise $7.E.5.3.1$

Suppose that the forcing function for the vibrating string is $F_0 \sin(\omega t)$. Derive the particular solution y_p.

Exercise $7.E.5.3.2$

Take the forced vibrating string. Suppose that $L = 1, a = 1$. Suppose that the forcing function is the square wave that is 1 on the interval $0 < x < 1$ and -1on the interval $-1 < x < 0$. Find the particular solution. Hint: You may want to use result of Exercise $7.E.5.3.1$.

Exercise $7.E.5.3.3$

The units are cgs (centimeters-grams-seconds). For $k = 0.005, \omega = 1.991 \times 10^{-7}, A_0 = 20$. Find the depth at which the temperature variation is half (± 10 degrees) of what it is on the surface.

Exercise $7.E.5.3.4$

Derive the solution for underground temperature oscillation without assuming that $T_0 = 0$.

Exercise $7.E.5.3.5$

Take the forced vibrating string. Suppose that $L = 1, a = 1$. Suppose that the forcing function is a sawtooth, that is $|x| - \frac{1}{2}$ on $-1 < x < 1$ extended periodically. Find the particular solution.

Answer

$$y_p(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^4\pi^4} \left(\cos(n\pi x) - \frac{\cos(n\pi) - 1}{\sin(n\pi)} \sin(n\pi x) - 1 \right) \cos(n\pi t).$$

Exercise $7.E.5.3.6$

The units are cgs (centimeters-grams-seconds). For $k = 0.01, \omega = 1.991 \times 10^{-7}, A_0 = 25$. Find the depth at which the summer is again the hottest point.

Answer

Approximately 1991 centimeters

- **5.E: Eigenvalue Problems (Exercises)** has no license indicated.

CHAPTER OVERVIEW

8: The Laplace Transform

The Laplace transform can also be used to solve differential equations and reduces a linear differential equation to an algebraic equation, which can then be solved by the formal rules of algebra.

Contributors and Attributions

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8.1: The Laplace Transform

8.1.1: Transform

In this chapter we will discuss the Laplace transform[1]. The Laplace transform turns out to be a very efficient method to solve certain ODE problems. In particular, the transform can take a differential equation and turn it into an algebraic equation. If the algebraic equation can be solved, applying the inverse transform gives us our desired solution. The Laplace transform also has applications in the analysis of electrical circuits, NMR spectroscopy, signal processing, and elsewhere. Finally, understanding the Laplace transform will also help with understanding the related Fourier transform, which, however, requires more understanding of complex numbers.

The Laplace transform also gives a lot of insight into the nature of the equations we are dealing with. It can be seen as converting between the time and the frequency domain. For example, take the standard equation

$$mx''(t) = cx'(t) + kx(t) = f(t).$$

We can think of t as time and $f(t)$ as incoming signal. The Laplace transform will convert the equation from a differential equation in time to an algebraic (no derivatives) equation, where the new independent variable s is the frequency.

We can think of the *Laplace transform* as a black box that eats functions and spits out functions in a new variable. We write $\mathcal{L}\{f(t)\} = F(s)$ for the Laplace transform of $f(t)$. It is common to write lower case letters for functions in the time domain and upper case letters for functions in the frequency domain. We use the same letter to denote that one function is the Laplace transform of the other. For example $F(s)$ is the Laplace transform of $f(t)$. Let us define the transform.

$$\mathcal{L}\{f(t)\} = F(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} f(t)\, dt.$$

We note that we are only considering $t \geq 0$ in the transform. Of course, if we think of t as time there is no problem, we are generally interested in finding out what will happen in the future (Laplace transform is one place where it is safe to ignore the past).

Below is a video that introduces Laplace Transforms.

Let us compute some simple transforms.

> **Example 8.1.1**
>
> Suppose $f(t) = 1$, then
>
> $$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty} = \lim_{h\to\infty} \left[\frac{e^{-st}}{-s}\right]_{t=0}^{h} = \lim_{h\to\infty} \left(\frac{e^{-sh}}{-s} - \frac{1}{-s}\right) = \frac{1}{s}.$$
>
> The limit (the improper integral) only exists if $s > 0$. So $\mathcal{L}\{1\}$ is only defined for $s > 0$.

Below is a video on finding the Laplace Transform of a constant using the definition.

> **Example 8.1.2**
>
> Suppose $f(t) = e^{-at})$, then
>
> $$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} dt = \int_0^\infty e^{-(s+a)t} dt = \left[\frac{e^{-(s+a)t}}{-s(s+a)}\right]_{t=0}^\infty = \frac{1}{s+a}.$$
>
> The limit only exists if $s + a > 0$. So $\mathcal{L}\{e^{-at}\}$ is only defined for $s + a > 0$.

Below is a video on finding the Laplace Transform of an exponential.

Below is a video on finding the Laplace Transform of a cubic.

Below is a video on finding the Laplace Transform of a sin function.

Example 8.1.3

Suppose $f(t) = t$, then using integration by parts

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^\infty e^{-st} t dt \\ &= \left[\frac{-te^{-st}}{s}\right]_{t=0}^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt \\ &= 0 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_{t=0}^\infty \\ &= \frac{1}{s^2}.\end{aligned} \tag{8.1.1}$$

Again, the limit only exists if $s > 0$.

Below is a video on finding the Laplace Transform of a basic step function.

Below is a video on writing as step fuction in terms of the unit step function.

Below is a video on writing a ramp function in terms of the unit step function.

Below is a video on finding the Laplace Transform of a step function.

Below is another video on finding the Laplace Transform of a step function.

Below is a video on finding the Laplace Transform of a piecewise defined function.

Below is a video on finding the Laplace Transform of a function times the unit step function.

Example 8.1.4

A common function is the unit step function, which is sometimes called the *Heaviside function*[2]. This function is generally given as

$$u(t)=\begin{cases}0 & \text{if } t<0,\\ 1 & \text{if } t\geq 0.\end{cases}$$

Let us find the Laplace transform of $u(t-a)$, where $a\geq 0$ is some constant. That is, the function that is 0 for $t<a$ and 1 for $t\geq a$.

$$\mathcal{L}\{u(t-a)\}=\int_0^\infty e^{-st}u(t-a)dt=\int_a^\infty e^{-st}dt=\left[\frac{e^{-st}}{-s}\right]_{t=a}^{e^{-as}},$$

where of course $s>0$ (and $a\geq 0$ as we said before).

By applying similar procedures we can compute the transforms of many elementary functions. Many basic transforms are listed in Table 8.1.1.

Table 8.1.1: Some Laplace transforms (C, ω, and a are constants).

$f(t)$	$\{f(t)\}$
C	$\frac{C}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^3	$\frac{6}{s^4}$
t^n	$\frac{n!}{s^{s+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$

Exercise 8.1.1

Verify all the entries in Table 8.1.1.

Since the transform is defined by an integral. We can use the linearity properties of the integral. For example, suppose C is a constant, then

$$\mathcal{L}\{f(t)\} = \int_o^\infty e^{-st} C f(t)\, dt = C \int_0^\infty e^{-st} f(t)\, dt = C\mathcal{L}\{f(t)\}.$$

So we can "pull out" a constant out of the transform. Similarly we have linearity. Since linearity is very important we state it as a theorem.

Theorem 8.1.1

Linearity of the Laplace Transform

Suppose that A, B, and C are constants, then

$$\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$$

and in particular

$$\mathcal{L}\{Cf(t)\} = C\mathcal{L}\{f(t)\}.$$

Exercise 8.1.2

Verify theorem 8.1.1. That is, show that $\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$.

These rules together with Table 8.1.1 make it easy to find the Laplace transform of a whole lot of functions already. But be careful. It is a common mistake to think that the Laplace transform of a product is the product of the transforms. In general

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

It must also be noted that not all functions have a Laplace transform. For example, the function $1/t$ does not have a Laplace transform as the integral diverges for all s. Similarly, $\tan t$ or e^{t^2} do not have Laplace transforms.

8.1.2: Existence and Uniqueness

Let us consider when does the Laplace transform exist in more detail. First let us consider functions of exponential order. The function $f(t)$ is of *exponential order* as t goes to infinity if

$$|f(t)| \leq Me^{ct},$$

for some constants M and c, for sufficiently large t (say for all $t > t_o$ for some t_o). The simplest way to check this condition is to try and compute

$$\lim_{t\to\infty} \frac{f(t)}{e^{ct}}$$

If the limit exists and is finite (usually zero), then $f(t)$ is of exponential order.

Exercise 8.1.3

Use L'Hopital's rule from calculus to show that a polynomial is of exponential order. Hint: Note that a sum of two exponential order functions is also of exponential order. Then show that t^n is of exponential order for any n.

For an exponential order function we have existence and uniqueness of the Laplace transform.

Theorem 8.1.2

Existence

Let $f(t)$ be continuous and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

The existence is not difficult to see. Let $f(t)$ be of exponential order, that is $|f(t)| \leq Me^{ct}$ for all $t > 0$ (for simplicity $t_0 = 0$). Let $s > c$, or in other words $(c - s) < 0$. By the comparison theorem from calculus, the improper integral defining $\mathcal{L}\{f(t)\}$ exists if the following integral exists

$$\int_0^\infty e^{-st}(Me^{ct})dt = M\int_0^\infty e^{(c-s)t}dt = M\left[\frac{e^{(c-s)t}}{c-s}\right]_{t=0}^\infty = \frac{M}{c-s}.$$

The transform also exists for some other functions that are not of exponential order, but that will not be relevant to us. Before dealing with uniqueness, let us note that for exponential order functions we obtain that their Laplace transform decays at infinity:

$$\lim_{s\to\infty} F(s) = 0$$

Theorem 8.1.3

Uniqueness

Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.

Both theorems hold for piecewise continuous functions as well. Recall that piecewise continuous means that the function is continuous except perhaps at a discrete set of points where it has jump discontinuities like the Heaviside function. Uniqueness however does not “see” values at the discontinuities. So we can only conclude that $F(s) = G(s)$ outside of discontinuities. For example, the unit step function is sometimes defined using $u(0) = 1/2$. This new step function, however, has the exact same Laplace transform as the one we defined earlier where $u(0) = 1$.

8.1.3: 6.1.3Inverse Transform

As we said, the Laplace transform will allow us to convert a differential equation into an algebraic equation. Once we solve the algebraic equation in the frequency domain we will want to get back to the time domain, as that is what we are interested in. If we have a function $F(s)$, to be able to find $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$, we need to first know if such a function is unique. It turns out we are in luck by Theorem 8.1.3. So we can without fear make the following definition.

If $F(s) = \mathcal{L}\{f(t)\}$ for some function $f(t)$. We define the *inverse Laplace transform* as

$$\mathcal{L}^{-1}\{F(s)\} \overset{\text{def}}{=} f(t)$$

There is an integral formula for the inverse, but it is not as simple as the transform itself—it requires complex numbers and path integrals. For us it will suffice to compute the inverse using Table 8.1.1.

Below is a video on finding the inverse Laplace Transform.

✓ Example 8.1.5

Find the inverse Laplace transform of $F(s) = \dfrac{1}{s+1}$

Solution

We look at the table to find

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

As the Laplace transform is linear, the inverse Laplace transform is also linear. That is,

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}$$

Of course, we also have $\mathcal{L}^{-1}\{AF(s) = A\mathcal{L}^{-1}\{F(s)\}.$ Let us demonstrate how linearity can be used.

Below is a video on finding the inverse Laplace Transform.

Below is another video on finding the inverse Laplace Transform.

Below is a video on finding the inverse Laplace Transform that gives sinh and an exponential.

Example 8.1.6

Find the inverse Laplace transform of

$$F(s) = \frac{s^2+s+1}{s^3+s}.$$

Solution

First we use the *method of partial fractions* to write F in a form where we can use Table 8.1.1. We factor the denominator as $s(s^2+1)$ and write

$$\frac{s^2+s+1}{s^3+s} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

Putting the right hand side over a common denominator and equating the numerators we get $A(s^2+1)+s(Bs+C)=s^2+s+1$. Expanding and equating coefficients we obtain $A+B=1$, $C=1$, $A=1$ and thus $B=0$. In other words,

$$F(s) = \frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}$$

By linearity of the inverse Laplace transform we get

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^3+s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = 1+\sin t.$$

8.1.4: Shifting Property of Laplace Transforms

Another useful property is the so-called *shifting property* or the *first shifting property*

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

where $F(s)$ is the Laplace transform of $f(t)$ and a is a constant.

Exercise 8.1.4

Derive the first shifting property from the definition of the Laplace transform.

The shifting property can be used, for example, when the denominator is a more complicated quadratic that may come up in the method of partial fractions. We complete the square and write such quadratics as $(s+a)^2+b$ and then use the shifting property.

Example 8.1.7

Find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\}.$$

Solution

First we complete the square to make the denominator $(s+2)^2+4$. Next we find

$$\mathcal{L}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin(2t).$$

Putting it all together with the shifting property, we find

$$\mathcal{L}\left\{\frac{1}{s^2+4s+8}\right\} = \mathcal{L}\left\{\frac{1}{(s+2)^2+4}\right\} = \frac{1}{2}e^{-2t}\sin(2t).$$

In general, we want to be able to apply the Laplace transform to *rational functions*, that is functions of the form

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials. Since normally, for the functions that we are considering, the Laplace transform goes to zero as $s\to\infty$, it is not hard to see that the degree of $F(s)$ must be smaller than that of $G(s)$. Such rational functions are called *proper rational functions* and we can always apply the method of partial fractions. Of course this means we need to be able to factor the denominator into linear and quadratic terms, which involves finding the roots of the denominator,

8.1.5: Footnotes

[1] Just like the Laplace equation and the Laplacian, the Laplace transform is also named after Pierre-Simon, marquis de Laplace (1749 – 1827).

[2] The function is named after the English mathematician, engineer, and physicist Oliver Heaviside (1850–1925). Only by coincidence is the function “heavy” on “one side.”

8.2: Transforms of derivatives and ODEs

8.2.1: Transforms of derivatives

Let us see how the Laplace transform is used for differential equations. First let us try to find the Laplace transform of a function that is a derivative. Suppose $g(t)$ is a differentiable function of exponential order, that is, $|g(t)| \leq Me^{ct}$ for some M and c. So $\mathcal{L}\{g(t)\}$ exists, and what is more, $\lim_{t\to\infty} e^{-st}g(t) = 0$ when $s > c$. Then

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st}g'(t)dt = \left[e^{-st}g(t)\right]_{t=0}^{\infty} - \int_0^\infty (-s)e^{-st}g(t)dt = -g(0) + s\mathcal{L}\{g(t)\}.$$

We repeat this procedure for higher derivatives. The results are listed in Table 8.2.1. The procedure also works for piecewise smooth functions, that is functions that are piecewise continuous with a piecewise continuous derivative. The fact that the function is of exponential order is used to show that the limits appearing above exist. We will not worry much about this fact.

Table 8.2.1: Laplace transforms of derivatives ($G(s) = \mathcal{L}\{g(t)\}$ as usual).

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$

8.2.1.1: Solving ODEs with the Laplace Transform

Notice that the Laplace transform turns differentiation into multiplication by s. Let us see how to apply this fact to differential equations.

Below is a video on finding the Laplace Transform of a homogeneous differential equation.

Example 8.2.1

Take the equation

$$x''(t) + x(t) = \cos(2t), \qquad x(0) = 0, \qquad x'(0) = 1.$$

We will take the Laplace transform of both sides. By $X(s)$ we will, as usual, denote the Laplace transform of $x(t)$.

$$\begin{aligned} \mathcal{L}\{x''(t) + x(t)\} &= \mathcal{L}\{\cos(2t)\}, \\ s^2X(x) - sx(0) + x'(0) + X(s) &= \frac{s}{s^2+4}. \end{aligned} \tag{8.2.1}$$

We plug in the initial conditions now—this makes the computations more streamlined—to obtain

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}.$$

We solve for $X(s)$,

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}.$$

We use partial fractions (exercise) to write

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}.$$

Now take the inverse Laplace transform to obtain

$$x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t).$$

Below is a video on finding the Laplace Transform of a nonhomogeneous differential equation.

The procedure for linear constant coefficient equations is as follows. We take an ordinary differential equation in the time variable t. We apply the Laplace transform to transform the equation into an algebraic (non differential) equation in the frequency domain. All the $x(t)$, $x'(t)$, $x''(t)$, and so on, will be converted to $X(s)$, $sX(s) - x(0)$, $s^2X(s) - sx(0) - x'(0)$, and so on. We solve the equation for $X(s)$. Then taking the inverse transform, if possible, we find $x(t)$.

It should be noted that since not every function has a Laplace transform, not every equation can be solved in this manner. Also if the equation is not a linear constant coefficient ODE, then by applying the Laplace transform we may not obtain an algebraic equation.

Below is a video on finding the inverse Laplace Transform.

Below is a video on finding the inverse Laplace Transform using partial fractions.

Below is a video on using the Laplace Transform to solve a homogeneous differential equation.

Below is another video on using the Laplace Transform to solve a homogeneous differential equation.

8.2.1.2: Using the Heaviside Function

Before we move on to more general equations than those we could solve before, we want to consider the Heaviside function. See Figure $8.2.1$ for the graph.

$$u(t)=\begin{cases}0 & \text{if } t<0,\\ 1 & \text{if } t\geq 0.\end{cases}$$

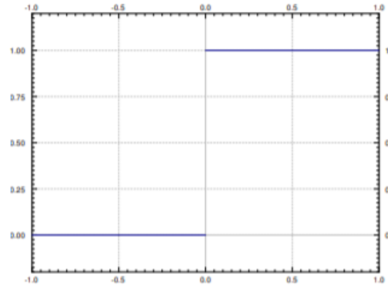


Figure 8.2.1: Plot of the Heaviside (unit step) function $u(t)$.

This function is useful for putting together functions, or cutting functions off. Most commonly it is used as $u(t-a)$ for some constant a. This just shifts the graph to the right by a. That is, it is a function that is 0 when $< a$ and 1 when $t \geq a$. Suppose for example that $f(t)$ is a “signal” and you started receiving the signal $\sin t$ at time $t = \pi$. The function $f(t)$ should then be defined as

$$f(t) = \begin{cases} 0 & \text{if } t < \pi, \\ \sin t & \text{if } t \geq \pi. \end{cases}$$

Using the Heaviside function, $f(t)$ can be written as

$$f(t) = u(t-\pi)\sin t$$

Similarly the step function that is 1 on the interval $[1, 2)$ and zero everywhere else can be written as

$$u(t-1) - u(t-2).$$

The Heaviside function is useful to define functions defined piecewise. If you want to define $f(t)$ such that $f(t) = t$ when t is in $[0, 1]$, $f(t) = -t + 2$ when t is in $[1, 2)$ and $f(t) = 0$ otherwise, you can use the expression

$$f(t) = t\left(u(t) - u(t-1)\right) + (-t+2)\left(u(t-1) - u(t-2)\right).$$

Hence it is useful to know how the Heaviside function interacts with the Laplace transform. We have already seen that

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{2}.$$

8.2.1.2.1: Shifting Property

This can be generalized into a shifting property or second shifting property.

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}. \tag{8.2.2}$$

Example 8.2.2

Suppose that the forcing function is not periodic. For example, suppose that we had a mass-spring system

$$x''(t) + x(t) = f(t), \qquad x(0) = 0, \qquad x'(0) = 0,$$

where $f(t) = 1$ if $1 \leq t < 5$ and zero otherwise. We could imagine a mass-spring system, where a rocket is fired for 4 seconds starting at $t = 1$. Or perhaps an RLC circuit, where the voltage is raised at a constant rate for 4 seconds starting at $t = 1$, and then held steady again starting at $t = 5$.

We can write $f(t) = u(t-1) - u(t-5)$. We transform the equation and we plug in the initial conditions as before to obtain

$$s^2 X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

We solve for $X(s)$ to obtain

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

We leave it as an exercise to the reader to show that

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1-\cos t.$$

In other words $\mathcal{L}\{1-\cos t\} = \frac{1}{s(s^2+1)}$. So using (8.2.2) we find

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{e^{-s}\mathcal{L}\{1-\cos t\}\} = (1-\cos(t-1))u(t-1).$$

Similarly

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{e^{-5s}\mathcal{L}\{1-\cos t\}\} = (1-\cos(t-5))u(t-5).$$

Hence, the solution is

$$x(t) = (1-\cos(t-1))\,u(t-1) - (1-\cos(t-5))\,u(t-5).$$

The plot of this solution is given in Figure $8.2.2$.

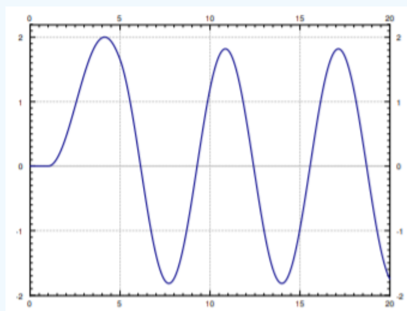


Figure $8.2.2$: Plot of $x(t)$.

8.2.1.3: Transfer Functions

Laplace transform leads to the following useful concept for studying the steady state behavior of a linear system. Suppose we have an equation of the form

$$Lx = f(t),$$

where L is a linear constant coefficient differential operator. Then $f(t)$ is usually thought of as input of the system and $x(t)$ is thought of as the output of the system. For example, for a mass-spring system the input is the forcing function and output is the behavior of the mass. We would like to have an convenient way to study the behavior of the system for different inputs.

Let us suppose that all the initial conditions are zero and take the Laplace transform of the equation, we obtain the equation

$$A(s)X(s) = F(s).$$

Solving for the ratio $\frac{X(s)}{F(s)}$ we obtain the so-called *transfer function* $H(s) = \frac{1}{A(s)}$.

$$H(s) = \frac{X(s)}{F(s)}$$

In other words, $X(s) = H(s)F(s)$. We obtain an algebraic dependence of the output of the system based on the input. We can now easily study the steady state behavior of the system given different inputs by simply multiplying by the transfer function.

Example $8.2.3$

Given $x'' + \omega_0^2 x = f(t)$, let us find the transfer function (assuming the initial conditions are zero).

First, we take the Laplace transform of the equation.

$$s^2X(s) + \omega_0^2 X(s) = F(s).$$

Now we solve for the transfer function $\frac{X(s)}{F(s)}$.

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + \omega_0^2}.$$

Let us see how to use the transfer function. Suppose we have the constant input $f(t) = 1$. Hence $F(s) = \frac{1}{s}$, and

$$X(s) = H(s)F(s) = \frac{1}{s^2 + \omega_0^2} \frac{1}{s}.$$

Taking the inverse Laplace transform of $X(s)$ we obtain

$$x(t) = \frac{1 - \cos(\omega_0 t)}{\omega_0^2}.$$

8.2.1.4: Transforms of Integrals

A feature of Laplace transforms is that it is also able to easily deal with integral equations. That is, equations in which integrals rather than derivatives of functions appear. The basic property, which can be proved by applying the definition and doing integration by parts, is

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s).$$

It is sometimes useful (e.g. for computing the inverse transform) to write this as

$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

Example 8.2.4

To compute $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ we could proceed by applying this integration rule.

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s^2+1}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \int_0^t \sin\tau\,d\tau = 1 - \cos t.$$

Example 8.2.5

An equation containing an integral of the unknown function is called an *integral equation*. For example, take

$$t^2 = \int_0^t e^\tau x(\tau)\,d\tau$$

where we wish to solve for $x(t)$. We apply the Laplace transform and the shifting property to get

$$\frac{2}{s^3} = \frac{1}{s}\mathcal{L}\{e^t x(t)\} = \frac{1}{s}X(s-1),$$

where $X(s) = \mathcal{L}\{x(t)\}$. Thus

$$X(s-1) = \frac{2}{s^2} \quad \text{or} \quad X(s) = \frac{2}{(s+1)^2}.$$

We use the shifting property again to get

$$x(t) = 2e^{-t}t.$$

- **6.2: Transforms of derivatives and ODEs** by Jiří Lebl is licensed CC BY-SA 4.0. Original source: https://www.jirka.org/diffyqs.

8.3: Convolution

8.3.1: Convolution

We said that the Laplace transformation of a product is not the product of the transforms. All hope is not lost however. We simply have to use a different type of a "product." Take two functions $f(t)$ and $g(t)$ defined for $t \geq 0$, and define the *convolution*[1] of $f(t)$ and $g(t)$ as

$$(f * g)(t) \stackrel{\text{def}}{=} \int_0^t f(\tau)g(t-\tau)d\tau. \tag{8.3.1}$$

As you can see, the convolution of two functions of t is another function of t.

Below is a video on finding the convolution of two exponential functions.

Example 8.3.1

Take $f(t) = e^t$ and $g(t) = t$ for $t \geq 0$. Then

$$(f * g)(t) = \int_0^t e^\tau (t-\tau)d\tau = e^t - t - 1.$$

To solve the integral we did one integration by parts.

Example 8.3.2

Take $f(t) = \sin(\omega t)$ and $g(t) = \cos(\omega t)$ for $t \geq 0$. Then

$$(f * g)(t) = \int_0^t \sin(\omega\tau)\cos(\omega(t-\tau))d\tau.$$

We apply the identity

$$\cos(\theta)\sin(\psi) = \frac{1}{2}(\sin(\theta+\psi) - \sin(\theta-\psi))$$

Hence,

$$\begin{aligned}(f * g)(t) &= \int_0^t \frac{1}{2}(\sin(\omega t) - \sin(\omega t - 2\omega\tau))d\tau \\ &= \left[\frac{1}{2}\tau\sin(\omega t) + \frac{1}{4\omega}\cos(2\omega\tau - \omega t)\right]_{\tau=0}^t \\ &= \frac{1}{2}t\sin(\omega t).\end{aligned} \tag{8.3.2}$$

The formula holds only for $t \geq 0$. We assumed that f and g are zero (or simply not defined) for negative t.

The convolution has many properties that make it behave like a product. Let c be a constant and f, g, and h be functions then

$$\begin{aligned} f * g &= g * f \\ (c f) * g &= f * (c g) = c (f * g) \\ (f * g) * h &= f * (g * h) \end{aligned} \tag{8.3.3}$$

The most interesting property for us, and the main result of this section is the following theorem.

Theorem 8.3.1

Let $f(t)$ and $g(t)$ be of exponential type, then

$$\{(f * g)(t)\} = \mathcal{L}\left\{\int_0^t f(\tau) g(t-\tau) d\tau\right\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

In other words, the Laplace transform of a convolution is the product of the Laplace transforms. The simplest way to use this result is in reverse.

Example 8.3.3

Suppose we have the function of s defined by

$$\frac{1}{(s+1)s^2} = \frac{1}{s+1}\frac{1}{s^2}.$$

We recognize the two entries of Table 6.1.2. That is

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\frac{1}{s^2}\right\} = \int_0^t \tau e^{-(t-\tau)} d\tau = e^{-t} + t - 1.$$

The calculation of the integral involved an integration by parts.

Below is a video on finding the Laplace Transform of the convolution integral.

8.3.2: Solving ODEs

The next example demonstrates the full power of the convolution and the Laplace transform. We can give the solution to the forced oscillation problem for any forcing function as a definite integral.

Example 8.3.4

Find the solution to

$$x''+\omega_0^2 x=f(t), \quad x(0)=0, \quad x'(0)=0,$$

for an arbitrary function $f(t)$.

We first apply the Laplace transform to the equation. Denote the transform of $x(t)$ by $X(s)$ and the transform of $f(t)$ by $F(s)$ as usual.

$$s^2X(s)+\omega_0^2X(s)=F(s),$$

or in other words

$$X(s)=F(s)\frac{1}{s^2+\omega_0^2}.$$

We know

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega_0^2}\right\}=\frac{\sin(\omega_0 t)}{\omega_0}.$$

Therefore,

$$x(t)=\int_0^t f(\tau)\frac{\sin(\omega_0(t-\tau))}{\omega_0}d\tau,$$

or if we reverse the order

$$x(t)=\int_0^t \frac{\sin(\omega_0\tau)}{\omega_0}f(t-\tau)d\tau.$$

Let us notice one more feature of this example. We can now see how Laplace transform handles resonance. Suppose that $f(t)=\cos(\omega_0 t)$. Then

$$x(t)=\int_0^t \frac{\sin(\omega_0\tau)}{\omega_0}\cos(\omega_0(t-\tau))d\tau=\frac{1}{\omega_0}\int_0^t \sin(\omega_0\tau)\cos(\omega_0(t-\tau))d\tau.$$

We have computed the convolution of sine and cosine in Example 6.3.2. Hence

$$x(t)=\left(\frac{1}{\omega_0}\right)\left(\frac{1}{2}t\sin(\omega_0 t)\right)=\frac{1}{2\omega_0}\sin(\omega_0 t).$$

Note the t in front of the sine. The solution, therefore, grows without bound as t gets large, meaning we get resonance.

Similarly, we can solve any constant coefficient equation with an arbitrary forcing function $f(t)$ as a definite integral using convolution. A definite integral, rather than a closed form solution, is usually enough for most practical purposes. It is not hard to numerically evaluate a definite integral.

Below is a video on using the inverse Laplace Transform of a convolution.

8.3.3: Volterra Integral Equation

A common integral equation is the Volterra integral equation[2]

$$x(t)=f(t)+\int_0^t g(t-\tau)x(\tau)\,d\tau$$

where $f(t)$ and $g(t)$ are known functions and $x(t)$ is an unknown we wish to solve for. To find $x(t)$, we apply the Laplace transform to the equation to obtain

$$X(s)=F(s)+G(s)X(s),$$

where $X(s)$, $F(s)$, and $G(s)$ are the Laplace transforms of $x(t)$, $f(t)$, and $g(t)$, respectively. We find

$$X(s)=\frac{F(s)}{1-G(s)}.$$

To find $x(t)$ we now need to find the inverse Laplace transform of $X(s)$.

Example 8.3.5

Solve

$$x(t)=e^{-t}+\int_0^t \sinh(t-\tau)x(\tau)\,d\tau$$

We apply Laplace transform to obtain

$$X(s)=\frac{1}{s+1}+\frac{1}{s^2-1}X(s),$$

or

$$X(s)=\frac{\frac{1}{s+1}}{1-\frac{1}{s^2-1}}=\frac{s-1}{s^2-2}=\frac{s}{s^2-2}-\frac{1}{s^2-2}.$$

It is not hard to apply Table 6.1.1 to find

$$x(t)=\cosh\left(\sqrt{2}\,t\right)-\frac{1}{\sqrt{2}}\sinh\left(\sqrt{2}\,t\right).$$

8.3.4: Footnotes

[1] For those that have seen convolution defined before, you may have seen it defined as $f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$. This definition agrees with (8.3.1) if you define $f(t)$ and $g(t)$ to be zero for $t < 0$. When discussing the Laplace transform the definition we gave is sufficient. Convolution does occur in many other applications, however, where you may have to use the more general definition with infinities.

[2] Named for the Italian mathematician Vito Volterra (1860–1940).

8.4: Dirac Delta and Impulse Response

8.4.1: Rectangular Pulse

Often in applications we study a physical system by putting in a short pulse and then seeing what the system does. The resulting behavior is often called *impulse response*. Let us see what we mean by a pulse. The simplest kind of a pulse is a simple rectangular pulse defined by

$$\varphi(t)=\begin{cases}0 & \text{if} \quad t<a, \\ M & \text{if } a \leq t<b, \\ 0 & \text{if } b \leq t .\end{cases}$$

Notice that

$$\varphi(t)=M(u(t-a)-u(t-b)),$$

where $u(t)$ is the unit step function (see Figure $8.4.1$ for a graph).

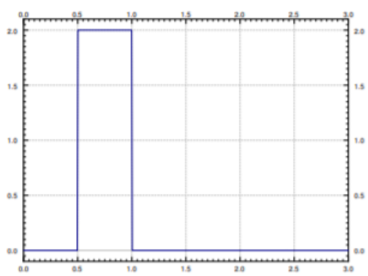


Figure $8.4.1$: Sample square pulse with $a=0.5$, $b=1$, and $M=2$.

Let us take the Laplace transform of a square pulse,

$$\mathcal{L}\{\varphi(t)\} \; =\mathcal{L}\{M(u(t-a)-u(t-b))\} \qquad =M \frac{e^{-a s}-e^{-b s}}{s}. \tag{8.4.1}$$

For simplicity we let $a=0$ and it is convenient to set $M=\frac{1}{b}$ to have

$$\int_0^{\infty} \varphi(t)\, d t=1$$

That is, to have the pulse have "unit mass." For such a pulse we compute

$$\mathcal{L}\{\varphi(t)\}=\mathcal{L}\left\{\frac{u(t)-u(t-b)}{b}\right\}=\frac{1-e^{-b s}}{b s}.$$

We generally want b to be very small. That is, we wish to have the pulse be very short and very tall. By letting b go to zero we arrive at the concept of the *Dirac delta function*.

8.4.2: 6.4.2Delta Function

The *Dirac delta function*[1] is not exactly a function; it is sometimes called a generalized function. We avoid unnecessary details and simply say that it is an object that does not really make sense unless we integrate it. The motivation is that we would like a "function" $\delta(t)$ such that for any continuous function $f(t)$ we have

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, d t=f(0)$$

The formula should hold if we integrate over any interval that contains 0, not just $(-\infty, \infty)$. So $\delta(t)$ is a "function" with all its "mass" at the single point $t=0$. In other words, for any interval $[c, d]$

$$\int_c^d \delta(t)=\begin{cases}1 & \text{if the interval } [c, d] \text{ contains } 0, \text{ i.e. } c \leq 0 \leq d, \\ 0 & \text{otherwise.}\end{cases}$$

Unfortunately there is no such function in the classical sense. You could informally think that $\delta(t)$ is zero for $t \neq 0$ and somehow infinite at $t = 0$.

A good way to think about $\delta(t)$ is as a limit of short pulses whose integral is 1. For example, suppose that we have a square pulse $\varphi(t)$ as above with $a = 0$, $M = \frac{1}{b}$, that is $\varphi(t) = \frac{u(t) - u(t-b)}{b}$.

Compute

$$\int_{-\infty}^{\infty} \varphi(t) f(t)\, dt = \int_{-\infty}^{\infty} \frac{u(t) - u(t-b)}{b} f(t)\, dt = \frac{1}{b} \int_0^b f(t)\, dt.$$

If $f(t)$ is continuous at $t = 0$, then for very small b, the function $f(t)$ is approximately equal to $f(0)$ on the interval $[0, b)$. We approximate the integral

$$\frac{1}{b} \int_0^b f(t)\, dt \approx \frac{1}{b} \int_0^b f(t)\, dt = f(0).$$

Therefore,

$$\lim_{b \to 0} \int_{-\infty}^{\infty} \varphi(t) f(t)\, dt = \lim_{b \to 0} \frac{1}{b} \int_0^b f(t)\, dt = f(0)$$

Let us therefore accept $\delta(t)$ as an object that is possible to integrate. We often want to shift δ to another point, for example $\delta(t-a)$. In that case we have

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = f(a).$$

Note that $\delta(a-t)$ is the same object as $\delta(t-a)$. In other words, the convolution of $\delta(t)$ with $f(t)$ is again $f(t)$,

$$(f * \delta)(t) = \int_0^t \delta(t-s) f(s)\, ds = f(t)$$

As we can integrate $\delta(t)$, let us compute its Laplace transform.

$$\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} e^{-st} \delta(t-a)\, dt = e^{-as}.$$

In particular,

$$\mathcal{L}\{\delta(t)\} = 1.$$

Note

Notice that the Laplace transform of $\delta(t-a)$ looks like the Laplace transform of the derivative of the Heaviside function $u(t-a)$, if we could differentiate the Heaviside function. First notice

$$\mathcal{L}\{\delta(t-a)\} = \frac{e^{-as}}{s}.$$

To obtain what the Laplace transform of the derivative would be we multiply by s, to obtain e^{-as}, which is the Laplace transform of $\delta(t-a)$. We see the same thing using integration,

$$\int_0^t \delta(s-a)\, ds = u(t-a)$$

So in a certain sense

$$\frac{d}{dt}[u(t-a)] = \delta(t-a)$$

This line of reasoning allows us to talk about derivatives of functions with jump discontinuities. We can think of the derivative of the Heaviside function $u(t-a)$ as being somehow infinite at a, which is precisely our intuitive understanding of the delta function.

Example 8.4.1

Compute

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s}\right\}.$$

So far we have always looked at proper rational functions in the s variable. That is, the numerator was always of lower degree than the denominator. Not so with $\frac{s+1}{s}$. We write,

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s}\right\}=\mathcal{L}^{-1}\left\{1+\frac{1}{s}\right\}=\mathcal{L}^{-1}\{1\}+\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=\delta(t)+1.$$

The resulting object is a generalized function and only makes sense when put underneath an integral.

8.4.3: Impulse Response

As we said before, in the differential equation $Lx=f(t)$, we think of $f(t)$ as input, and $x(t)$ as the output. Often it is important to find the response to an impulse, and then we use the delta function in place of $f(t)$. The solution to

$$Lx=\delta(t)$$

is called the *impulse response*.

Example 8.4.2

Solve (find the impulse response)

$$x''+\omega_0^2 x=\delta(t), \quad x(0)=0, \quad x'(0)=0. \tag{8.4.2}$$

We first apply the Laplace transform to the equation. Denote the transform of $x(t)$ by $X(s)$.

$$s^2X(s)+\omega_0^2X(s)=1, \quad \text{and so} \quad X(s)=\frac{1}{s^2+\omega_0^2}.$$

Taking the inverse Laplace transform we obtain

$$x(t)=\frac{\sin(\omega_0 t)}{\omega_0}.$$

Let us notice something about the above example. In Example 6.3.4, we found that when the input was $f(t)$, then the solution to $Lx=f(t)$ was given by

$$x(t)=\int_0^t f(\tau)\frac{\sin(\omega_0(t-\tau))}{\omega_0}\,d\tau.$$

Notice that *the solution for an arbitrary input is given as convolution with the impulse response*. Let us see why. The key is to notice that for functions $x(t)$ and $f(t)$,

$$(x*f)''(t)=\frac{d^2}{dt^2}\left[\int_0^t f(\tau)x(t-\tau)\,d\tau\right]=\int_0^t f(\tau)x''(t-\tau)\,d\tau=(x''*f)(t).$$

We simply differentiate twice under the integral,[2] the details are left as an exercise. And so if we convolve the entire equation (8.4.2), the left hand side becomes

$$(x''+\omega_0^2x)*f=(x''*f)+\omega_0^2(x*f)=(x*f)''+\omega_0^2(x*f).$$

The right hand side becomes

$$(\delta*f)(t)=f(t).$$

Therefore $y(t) = (x * f)(t)$ is the solution to

$$y'' + \omega_0^2 y = f(t).$$

This procedure works in general for other linear equations $Lx = f(t)$. If you determine the impulse response, you also know how to obtain the output $x(t)$ for any input $f(t)$ by simply convolving the impulse response and the input $f(t)$.

Below is a video on the Laplace Transform and the Dirac delta function.

8.4.4: Three-Point Beam Bending

Let us give another quite different example where the delta function turns up: Representing point loads on a steel beam. Consider a beam of length L, resting on two simple supports at the ends. Let x denote the position on the beam, and let $y(x)$ denote the deflection of the beam in the vertical direction. The deflection $y(x)$ satisfies the *Euler-Bernoulli equation,*[3]

$$EI \frac{d^4y}{dx^4} = F(x),$$

where E and I are constants[4] and $F(x)$ is the force applied per unit length at position x. The situation we are interested in is when the force is applied at a single point as in Figure $8.4.2$.

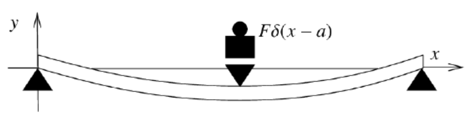


Figure $8.4.2$: Three-point bending.

In this case the equation becomes

$$EI \frac{d^4y}{dx^4} = -F\delta(x-a)$$

where $x = a$ is the point where the mass is applied. F is the force applied and the minus sign indicates that the force is downward, that is, in the negative y direction. The end points of the beam satisfy the conditions,

$$\begin{aligned} y(0) &= 0, \qquad y''(0) = 0, \\ y(L) &= 0, \qquad y''(L) = 0. \end{aligned} \tag{8.4.3}$$

See Section 5.2, for further information about endpoint conditions applied to beams.

Example $8.4.3$

Suppose that length of the beam is 2, and suppose that $EI = 1$ for simplicity. Further suppose that the force $F = 1$ is applied at $x = 1$. That is, we have the equation

$$\frac{d^4y}{dx^4} = -\delta(x-1)$$

and the endpoint conditions are

$$y(0)=0, \quad y''(0)=0, \quad y(2)=0, \quad y''(2)=0.$$

We could integrate, but using the Laplace transform is even easier. We apply the transform in the x variable rather than the t variable. Let us again denote the transform of $y(x)$ as $Y(s)$.

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)=-e^{-s}.$$

We notice that $y(0)=0$ and $y''(0)=0$. Let us call $C_1=y'(0)$ and $C^2=y'''(0)$. We solve for $Y(s)$,

$$Y(s)=\frac{-e^{-s}}{s^4}+\frac{C_1}{s^2}+\frac{C_2}{s^4}.$$

We take the inverse Laplace transform utilizing the second shifting property Equation (6.2.14) to take the inverse of the first term.

$$y(x)=\frac{-(x-1)^3}{6}u(x-1)+C_1x+\frac{C_2}{6}x^3.$$

We still need to apply two of the endpoint conditions. As the conditions are at $x=2$ we can simply replace $u(x-1)=1$ when taking the derivatives. Therefore,

$$0=y(2)=\frac{-(2-1)^3}{6}+C_1(2)+\frac{C_2}{6}2^3=\frac{-1}{6}+2C_1+\frac{4}{3C_2},$$

and

$$0=y''(2)=\frac{-3\cdot 2\cdot(2-1)}{6}+\frac{C_2}{6}3\cdot 2\cdot 2=-1+2C_2$$

Hence $C_2=\frac{1}{2}$ and solving for C_1 using the first equation we obtain $C_1=\frac{-1}{4}$. Our solution for the beam deflection is

$$y(x)=\frac{-(x-1)^3}{6}u(x-1)-\frac{x}{4}+\frac{x^3}{12}.$$

8.4.5: Footnotes

[1] Named after the English physicist and mathematician Paul Adrien Maurice Dirac (1902–1984).

[2] You should really think of the integral going over $(-\infty,\infty)$ rather than over $[0,t]$ and simply assume that $f(t)$ and $x(t)$ are continuous and zero for negative t.

[3] Named for the Swiss mathematicians Jacob Bernoulli (1654–1705), Daniel Bernoulli —nephew of Jacob— (1700–1782), and Leonhard Paul Euler (1707–1783).

[4] E is the elastic modulus and I is the second moment of area. Let us not worry about the details and simply think of these as some given constants.

8.5: Solving PDEs with the Laplace Transform

The Laplace transform comes from the same family of transforms as does the Fourier series[1], which we used in Chapter 4 to solve partial differential equations (PDEs). It is therefore not surprising that we can also solve PDEs with the Laplace transform.

Given a PDE in two independent variables x and t, we use the Laplace transform on one of the variables (taking the transform of everything in sight), and derivatives in that variable become multiplications by the transformed variable s. The PDE becomes an ODE, which we solve. Afterwards we invert the transform to find a solution to the original problem. It is best to see the procedure on an example.

Example 8.5.1

Consider the first order PDE

$$y_t = -\alpha y_x, \qquad \text{for } x > 0, \;\; t > 0,$$

with side conditions

$$y(0,t) = C, \qquad y(x,0) = 0.$$

This equation is called the *convection equation* or sometimes the *transport equation*, and it already made an appearance in Section 1.9, with different conditions. See Figure 8.5.1 for a diagram of the setup.

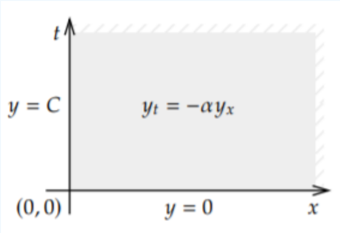


Figure 8.5.1: Transport equation on a half line.

A physical setup of this equation is a river of solid goo, as we do not want anything to diffuse. The function y is the concentration of some toxic substance.[2] The variable x denotes position where $x = 0$ is the location of a factory spewing the toxic substance into the river. The toxic substance flows into the river so that at $x = 0$ the concentration is always C. We wish to see what happens past the factory, that is at $x > 0$. Let t be the time, and assume the factory started operations at $t = 0$, so that at $t = 0$ the river is just pure goo.

Consider a function of two variables $y(x,t)$. Let us fix x and transform the t variable. For convenience, we treat the transformed s variable as a parameter, since there are no derivatives in s. That is, we write $Y(x)$ for the transformed function, and treat it as a function of x, leaving s as a parameter.

$$Y(x) = \mathcal{L}\{y(x,t)\} = \int_0^\infty y(x,t)e^{-st}\,ds.$$

The transform of a derivative with respect to x is just differentiating the transformed function:

$$\mathcal{L}\{y_x(x,t)\} = \int_0^\infty y_x(x,t)e^{-st}\,ds = \frac{d}{dx}\left[\int_0^\infty y(x,t)e^{-st}\,ds\right] = Y'(x).$$

To transform the derivative in t (the variable being transformed), we use the rules from Section 6.2:

$$\mathcal{L}\{y_t(x,t)\} = sY(x) - y(x,0).$$

In our specific case, $y(x,0) = 0$, and so $\mathcal{L}\{y_t(x,t)\} = sY(x)$. We transform the equation to find

$$sY(x) = -\alpha Y'(x).$$

This ODE needs an initial condition. The initial condition is the other side condition of the PDE, the one that depends on x. Everything is transformed, so we must also transform this condition

$$Y(0) = \mathcal{L}\{y(0,t)\} = \mathcal{L}\{C\} = \frac{C}{s}.$$

We solve the ODE problem $sY(x) = -\alpha Y'(x)$, $Y(0) = \frac{C}{s}$, to find

$$Y(x) = \frac{C}{s} e^{-\frac{s}{\alpha}x}.$$

We are not done, we have $Y(x)$, but we really want $y(x,t)$. We transform the s variable back to t. Let

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{otherwise} \end{cases}$$

be the Heaviside function. As

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)\, e^{-st}\, dt = \int_a^\infty e^{-st}\, dt = \frac{e^{-as}}{s},$$

then

$$y(x,t) = \mathcal{L}^{-1}\left\{\frac{C}{s} e^{-\frac{s}{\alpha}x}\right\} = Cu\left(t - \frac{x}{\alpha}\right).$$

In other words,

$$y(x,t) = \begin{cases} 0 & \text{if } t < \frac{x}{\alpha}, \\ C & \text{otherwise.} \end{cases}$$

See Figure 8.5.2 for a diagram of this solution. The line of slope $\frac{1}{\alpha}$ indicates the wavefront of the toxic substance in the picture as it is leaving the factory. What the equation does is simply move the initial condition to the right at speed α.

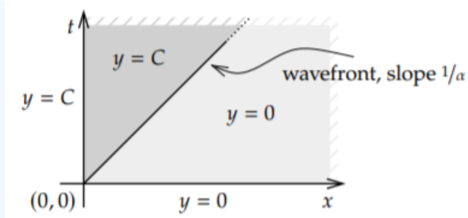


Figure $8.5.2$: Wavefront of toxic substance is a line of slope $\frac{1}{\alpha}$.

Shhh…y is not differentiable, it is not even continuous (nobody ever seems to notice). How could we plug something that's not differentiable into the equation? Well, just think of a differentiable function very very close to y. Or, if you recognize the derivative of the Heaviside function as the delta function, then all is well too:

$$y_t(x,t) = \frac{\partial}{\partial t}\left[Cu\left(t - \frac{x}{\alpha}\right)\right] = Cu'\left(t - \frac{x}{\alpha}\right) = C\delta\left(t - \frac{x}{\alpha}\right)$$

and

$$y_x(x,t) = \frac{\partial}{\partial x}\left[Cu\left(t - \frac{x}{\alpha}\right)\right] = -\frac{C}{\alpha}u'\left(t - \frac{x}{\alpha}\right) = -\frac{C}{\alpha}\delta\left(t - \frac{x}{\alpha}\right).$$

So $y_t = -\alpha y_x$.

Laplace equation is very good with constant coefficient equations. One advantage of Laplace is that it easily handles nonhomogeneous side conditions. Let us try a more complicated example.

Example $8.5.2$

Consider

$$\begin{aligned} y_t + y_x + y = 0, &\qquad \text{for } x > 0,\ \ t > 0, \\ y(0,t) = \sin(t), &\qquad y(x,0) = 0. \end{aligned} \tag{8.5.1}$$

Again, we transform t, and we write $Y(x)$ for the transformed function. As $y(x,0) = 0$, we find

$$sY(x) + Y'(x) + Y(x) = 0, \qquad Y(0) = \frac{1}{s^2+1}.$$

The solution of the transformed equation is

$$Y(x) = \frac{1}{s^2+1}e^{-(s+1)x} = \frac{1}{s^2+1}e^{-xs}e^{-x}.$$

Using the second shifting property (6.2.14) and linearity of the transform, we obtain the solution

$$y(x,t) = e^{-x}\sin(t-x)u(t-x).$$

We can also detect when the problem is in the sense that it has no solution. Let us change the equation to

$$\begin{aligned} -y_t + y_x = 0, &\qquad \text{for } x > 0,\ \ t > 0, \\ y(0,t) = \sin(t), &\qquad y(x,0) = 0. \end{aligned} \tag{8.5.2}$$

Then the problem has no solution. First, let us see this in the language of Section 1.9. The characteristic curves are $t = -x + C$. If τ is the the characteristic coordinate, then we find the equation $y_\tau = 0$ along the curve, meaning a solution is constant along characteristic curves. But these curves intersect both the x-axis and the t-axis. For example, the curve $t = -x + 1$ intersects at $(1,0)$ and $(0,1)$. The solution is constant along the curve so $y(1,0)$ should equal $y(0,1)$. But $y(1,0) = 0$ and $y(0,1) = \sin(1) \neq 0$. See Figure $8.5.3$.

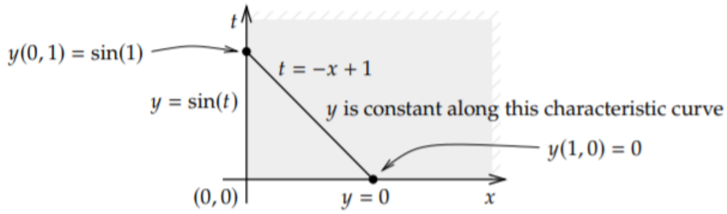


Figure 8.5.3: Ill-posed problem.

Now consider the transform. The transformed problem is

$$-sY(x) + Y'(x) = 0, \quad Y(0) = \frac{1}{s^2+1},$$

and the solution ought to be

$$Y(x) = \frac{1}{s^2+1} e^{sx}.$$

Importantly, this Laplace transform does not decay to zero at infinity! That is, since $x > 0$ in the region of interest, then

$$\lim_{s \to \infty} \frac{1}{s^2+1} e^{sx} = \infty \neq 0.$$

It almost looks as if we could use the shifting property, but notice that the shift is in the wrong direction. Of course, we need not restrict ourselves to first order equations, although the computations become more involved for higher order equations.

Example 8.5.3

Let us use Laplace for the following problem:

$$\begin{aligned} y_t &= y_{xx}, \qquad 0 < x < \infty, \quad t > 0, \\ y_x(0,t) &= f(t), \\ y(x,0) &= 0. \end{aligned}$$

Really we also impose other conditions on the solution so that for example the Laplace transform exists. For example, we might impose that y is bounded for each fixed time t. Transform the equation in the t variable to find

$$sY(x) = Y''(x).$$

The general solution to this ODE is

$$Y(x) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}.$$

First $A = 0$, since otherwise Y does not decay to zero as $s \to \infty$. Now consider the boundary condition. Transform $Y'(0) = F(s)$ and so $-\sqrt{s}B = F(s)$. In other words,

$$Y(x) = -F(s)\frac{1}{\sqrt{s}} e^{-\sqrt{s}x}.$$

If we look up the inverse transform in a table such as the one in Appendix B (or we spend the afternoon doing calculus), we find

$$\mathcal{L}^{-1}\left[e^{-\sqrt{s}x}\right] = \frac{x}{\sqrt{4\pi t^3}} e^{\frac{-x^2}{4t}},$$

or

$$\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}} e^{-\sqrt{s}x}\right] = \frac{1}{\sqrt{\pi t}} e^{\frac{-x^2}{4t}}.$$

So

$$y(x,t) = \mathcal{L}^{-1}\left[F(s)e^{-\sqrt{s}x}\right] = \int_0^t f(\tau)\frac{1}{\sqrt{\pi(t-\tau)}}e^{\frac{-x^2}{4(t-\tau)}}\,d\tau.$$

Laplace can solve problems where separation of variables fails. Laplace does not mind nonhomogeneity, but it is essentially only useful for constant coefficient equations.

8.5.1: Footnotes

[1] There is a corresponding Fourier transform on the real line as well that looks sort of like the Laplace transform.

[2] It’s a river of goo already, we’re not hurting the environment much more.

8.E: The Laplace Transform (Exercises)

These are homework exercises to accompany Libl's "Differential Equations for Engineering" Textmap. This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Prerequisite for the course is the basic calculus sequence.

8.E.1: 6.1: The Laplace transform

Exercise $8.E.6.1.1$

Find the Laplace transform of $3+t^5+\sin(\pi t)$.

Exercise $8.E.6.1.2$

Find the Laplace transform of $a+bt+ct^2$ for some constants a, b, and c.

Exercise $8.E.6.1.3$

Find the Laplace transform of $A\cos(\omega t)+B\sin(\omega t)$.

Exercise $8.E.6.1.4$

Find the Laplace transform of $\cos^2(\omega t)$.

Exercise $8.E.6.1.5$

Find the inverse Laplace transform of $\dfrac{4}{s^2-9}$.

Exercise $8.E.6.1.6$

Find the inverse Laplace transform of $\dfrac{2s}{s^2-1}$.

Exercise $8.E.6.1.7$

Find the inverse Laplace transform of $\dfrac{1}{(s-1)^2(s+1)}$.

Exercise $8.E.6.1.8$

Find the Laplace transform of $f(t)=\begin{cases} t & \text{if } t\geq 1, \\ 0 & \text{if } t<1. \end{cases}$.

Exercise $8.E.6.1.9$

Find the inverse Laplace transform of $\dfrac{s}{(s^2+s+2)(s+4)}$.

Exercise $8.E.6.1.10$

Find the Laplace transform of $\sin(\omega(t-a))$.

Exercise $8.E.6.1.11$

Find the Laplace transform of $t\sin(\omega t)$. Hint: Several integrations by parts.

Exercise $8.E.6.1.12$

Find the Laplace transform of $4(t+1)^2$.

Answer

$\frac{8}{s^3}+\frac{8}{s^2}+\frac{4}{s}$

Exercise $8.E.6.1.13$

Find the inverse Laplace transform of $\frac{8}{s^3(s+2)}$.

Answer

$2t^2-2t+1-e^{-2t}$

Exercise $8.E.6.1.14$

Find the Laplace transform of te^{-t} (Hint: integrate by parts).

Answer

$\frac{1}{(s+1)^2}$

Exercise $8.E.6.1.15$

Find the Laplace transform of $\sin(t)e^{-t}$ (Hint: integrate by parts).

Answer

$\frac{1}{s^2+2s+2}$

8.E.2: 6.2: Transforms of Derivatives and ODEs

Exercise $8.E.6.2.1$

Verify Table 6.2.1.

Exercise $8.E.6.2.2$

Using the Heaviside function write down the piecewise function that is 0 for $t<0$, t^2 for t in $[0,1]$ and t for $t>1$.

Exercise $8.E.6.2.3$

Using the Laplace transform solve

$$mx''+cx'+kx=0, \quad x(0)=a, \quad x'(0)=b. \tag{8.E.1}$$

where $m>0, c>0, k>0$, and $c^2-4km>0$ (system is overdamped).

Exercise $8.E.6.2.4$

Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b. \tag{8.E.2}$$

where $m > 0, c > 0, k > 0$, and $c^2 - 4km < 0$ (system is underdamped).

Exercise $8.E.6.2.5$

Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b. \tag{8.E.3}$$

where $m > 0, c > 0, k > 0$, and $c^2 = 4km$ (system is critically damped).

Exercise $8.E.6.2.6$

Solve $x'' + x = u(t-1)$ for initial conditions $x(0) = 0$ and $x'(0) = 0$.

Exercise $8.E.6.2.7$

Show the differentiation of the transform property. Suppose $\mathcal{L}\{f(t)\} = F(s)$, then show

$$\mathcal{L}\{-tf(t)\} = F'(s). \tag{8.E.4}$$

Hint: Differentiate under the integral sign.

Exercise $8.E.6.2.8$

Solve $x''' + x = t^3 u(t-1)$ for initial conditions $x(0) = 1$ and $x'(0) = 0$, $x''(0) = 0$.

Exercise $8.E.6.2.9$

Show the second shifting property: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$.

Exercise $8.E.6.2.10$

Let us think of the mass-spring system with a rocket from Example 6.2.2. We noticed that the solution kept oscillating after the rocket stopped running. The amplitude of the oscillation depends on the time that the rocket was fired (for 4 seconds in the example).

a. Find a formula for the amplitude of the resulting oscillation in terms of the amount of time the rocket is fired.
b. Is there a nonzero time (if so what is it?) for which the rocket fires and the resulting oscillation has amplitude 0 (the mass is not moving)?

Exercise $8.E.6.2.11$

Define

$$f(t) = \begin{cases} (t-1)^2 & if\ 1 \le t < 2, \\ 3-t & if\ 2 \le t < 3, \\ 0 & otherwise. \end{cases} \tag{8.E.5}$$

a. Sketch the graph of $f(t)$.
b. Write down $f(t)$ using the Heaviside function.
c. Solve $x'' + x = f(t), x(0) = 0, x'(0) = 0$ using Laplace transform.

Exercise $8.E.6.2.12$

Find the transfer function for $mx'' + cx' + kx = f(t)$ (assuming the initial conditions are zero).

Exercise $8.E.6.2.13$

Using the Heaviside function $u(t)$, write down the function

$$f(t) = \begin{cases} 0 & if \quad t < 1, \\ t-1 & if\, 1 \leq t < 2, \\ if \quad 2 \leq t. \end{cases} \tag{8.E.6}$$

Answer

$f(t) = (t-1)(u(t-1) - u(t-2)) + u(t-2)$

Exercise $8.E.6.2.14$

Solve $x'' - x = (t^2 - 1)u(t-1)$ for initial conditions $x(0) = 1, x'(0) = 2$ using the Laplace transform.

Answer

$x(t) = (2e^{t-1} - t^2 - 1)u(t-1) - \frac{1}{2}e^{-t} + \frac{3}{2}e^t$

Exercise $8.E.6.2.15$

Find the transfer function for $x' + x = f(t)$ (assuming the initial conditions are zero).

Answer

$H(s) = \frac{1}{s+1}$

8.E.3: 6.3: Convolution

Exercise $8.E.6.3.1$

Let $f(t) = t^2$ for $t \geq 0$, and $g(t) = u(t-1)$. Compute $f * g$.

Exercise $8.E.6.3.2$

Let $f(t) = t$ for $t \geq 0$, and $g(t) = \sin t$ for $t \geq 0$. Compute $f * g$.

Exercise $8.E.6.3.3$

Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0, c > 0, k > 0$, and $c^2 - 4km > 0$ (system is overdamped). Write the solution as a definite integral.

Exercise $8.E.6.3.4$

Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m>0, c>0, k>0$, and $c^2-4km<0$ (system is underdamped). Write the solution as a definite integral.

Exercise $8.E.6.3.5$

Find the solution to

$$mx''+cx'+kx=f(t), \quad x(0)=0, \quad x'(0)=0,$$

for an arbitrary function $f(t)$, where $m>0, c>0, k>0$, and $c^2=4km$ (system is critically damped). Write the solution as a definite integral.

Exercise $8.E.6.3.6$

Solve

$$x(t)=e^{-t}+\int_0^t \cos(t-\tau)x(\tau)\,d\tau.$$

Exercise $8.E.6.3.7$

Solve

$$x(t)=\cos t+\int_0^t \cos(t-\tau)x(\tau)\,d\tau.$$

Exercise $8.E.6.3.8$

Compute $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$ using convolution.

Exercise $8.E.6.3.9$

Write down the solution to $x''-2x=e^{-t^2}, x(0)=0, x'(0)=0$ as a definite integral. Hint: Do not try to compute the Laplace transform of e^{-t^2}.

Exercise $8.E.6.3.10$

Let $f(t)=\cos t$ for $t\geq 0$, and $g(t)=e^{-t}$. Compute $f*g$.

Answer

$\frac{1}{2}(\cos t+\sin t-e^{-t})$

Exercise $8.E.6.3.11$

Compute $\mathcal{L}^{-1}\left\{\frac{5}{s^4+s^2}\right\}$ using convolution.

Answer

$5t-5\sin t$

Exercise $8.E.6.3.12$

Solve $x''+x=\sin t, x(0)=0, x'(0)=0$ using convolution.

Answer

$\frac{1}{2}(\sin t-t\cos t)$

> **?** Exercise $8.E.6.3.13$
>
> Solve $x''' + x' = f(t), x(0) = 0, x'(0) = 0, x''(0) = 0$ using convolution. Write the result as a definite integral.
>
> **Answer**
>
> $\int_0^t f(\tau)(1 - \cos(t - \tau))d\tau$

8.E.4: 6.4: Dirac delta and impulse response

> **?** Exercise $8.E.6.4.1$
>
> Solve (find the impulse response) $x'' + x' + x = \delta(t), x(0) = 0, x'(0) = 0$.

> **?** Exercise $8.E.6.4.2$
>
> Solve (find the impulse response) $x'' + 2x' + x = \delta(t), x(0) = 0, x'(0) = 0$.

> **?** Exercise $8.E.6.4.3$
>
> A pulse can come later and can be bigger. Solve $x'' + 4x = 4\delta(t - 1), x(0) = 0, x'(0) = 0$.

> **?** Exercise $8.E.6.4.4$
>
> Suppose that $f(t)$ and $g(t)$ are differentiable functions and suppose that $f(t) = g(t) = 0$ for all $t \leq 0$. Show that
>
> $$(f * g)'(t) = (f' * g)(t) = (f * g')(t). \tag{8.E.7}$$

> **?** Exercise $8.E.6.4.5$
>
> Suppose that $Lx = \delta(t), x(0) = 0, x'(0) = 0,$ has the solution $x = e^{-t}$ for $t > 0$. Find the solution to $Lx = t^2, x(0) = 0, x'(0) = 0$ for $t > 0$.

> **?** Exercise $8.E.6.4.6$
>
> Compute $\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^2}\right\}$.

> **?** Exercise $8.E.6.4.7$: (challenging)
>
> Solve Example 6.4.3 via integrating 4 times in the x variable.

> **?** Exercise $8.E.6.4.8$
>
> Suppose we have a beam of length 1 simply supported at the ends and suppose that force $F = 1$ is applied at $x = \frac{3}{4}$ in the downward direction. Suppose that $EI = 1$ for simplicity. Find the beam deflection $y(x)$.

> **?** Exercise $8.E.6.4.9$
>
> Solve (find the impulse response) $x'' = \delta(t),\ x(0) = 0,\ x'(0) = 0$.
>
> **Answer**
>
> $x(t) = t$

Exercise $8.E.6.4.10$

Solve (find the impulse response) $x' + ax = \delta(t),\ x(0) = 0,\ x'(0) = 0$.

Answer

$x(t) = e^{-at}$

Exercise $8.E.6.4.11$

Suppose that $Lx = \delta(t), x(0) = 0, x'(0) = 0$, has the solution $x(t) = \cos(t)$ for $t > 0$. Find (in closed form) the solution to $Lx = \sin(t), x(0) = 0, x'(0) = 0 fort > 0$.

Answer

$x(t) = (\cos * \sin)(t) = \frac{1}{2} t \sin(t)$

Exercise $8.E.6.4.12$

Compute $\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\}$.

Answer

$\delta(t) - \sin(t)$

Exercise $8.E.6.4.13$

Compute $\mathcal{L}^{-1}\left\{\frac{3s^2e^{-s}+2}{s^2}\right\}$.

Answer

$3\delta(t-1) + 2t$

8.E.5: 6.5: Solving PDEs with the Laplace Transform

Exercise $8.E.6.5.1$

Solve

$$\begin{aligned} y_t + y_x = 1, \qquad 0 < x < \infty,\ \ t > 0, \\ y(0,t) = 1, \quad y(x,0) = 0. \end{aligned}$$

Exercise $8.E.6.5.2$

Solve

$$\begin{aligned} y_t + \alpha y_x = 0, \qquad 0 < x < \infty,\ \ t > 0, \\ y(0,t) = t, \quad y(x,0) = 0. \end{aligned}$$

Exercise $8.E.6.5.3$

Solve

$$\begin{aligned} y_t + 2y_x = x + t, \qquad 0 < x < \infty,\ \ t > 0, \\ y(0,t) = 0, \quad y(x,0) = 0. \end{aligned}$$

Exercise 8.E.6.5.4

For an $\alpha > 0$, solve

$$\begin{gathered} y_t + \alpha y_x + y = 0, \qquad 0 < x < \infty, \quad t > 0, \\ y(0,t) = \sin(t), \quad y(x,0) = 0. \end{gathered}$$

Exercise 8.E.6.5.5

Find the corresponding ODE problem for $Y(x)$, after transforming the t variable

$$\begin{gathered} y_{tt} + 3y_{xx} + y_{xt} + 3y_x + y = \sin(x) + t, \qquad 0 < x < 1, \quad t > 0, \\ y(0,t) = 1, \quad y(1,t) = t, \quad y(x,0) = 1 - x, \quad y_t(x,0) = 1. \end{gathered}$$

Do not solve the problem.

Exercise 8.E.6.5.6

Write down a solution to

$$\begin{gathered} y_t = y_{xx}, \qquad 0 < x < \infty, \quad t > 0, \\ y_x(0,t) = e^{-t}, \quad y(x,0) = 0, \end{gathered}$$

as an definite integral (convolution).

Exercise 8.E.6.5.7

Use the Laplace transform in t to solve

$$\begin{gathered} y_{tt} = y_{xx}, \qquad -\infty < x < \infty, \quad t > 0, \\ y_t(x,0) = \sin(x), \quad y(x,0) = 0. \end{gathered}$$

Hint: Note that e^{sx} does not go to zero as $s \to \infty$ for positive x, and e^{-sx} does not go to zero as $s \to \infty$ for negative x.

Exercise 8.E.6.5.8

Solve

$$\begin{gathered} y_t + y_x = 1, \qquad 0 < x < \infty, \quad t > 0, \\ y(0,t) = 0, \quad y(x,0) = 0. \end{gathered}$$

Answer

$y = (x - t)u(t - x) + t$

Exercise 8.E.6.5.9

For a $c > 0$, solve

$$\begin{gathered} y_t + y_x + cy = 0, \qquad 0 < x < \infty, \quad t > 0, \\ y(0,t) = \sin(t), \quad y(x,0) = 0. \end{gathered}$$

Answer

$y = e^{-cx} \sin(t - x) u(t - x)$

? Exercise $8.E.6.5.10$

Find the corresponding ODE problem for $Y(x)$, after transforming the t variable

$$\begin{aligned} &y_{tt} + 3y_{xx} + y = x + t, \qquad -1 < x < 1, \;\; t > 0, \\ &y(-1,t) = 0, \quad y(1,t) = 0, \quad y(x,0) = (1-x^2), \quad y_t(x,0) = 0. \end{aligned}$$

Do not solve the problem.

Answer

$s^2Y(x) - s(1+x^2) + 3Y''(x) + Y(x) = \frac{x}{s} + \frac{1}{s^2}, \quad Y(-1) = 0, \quad Y(1) = 0.$

? Exercise $8.E.6.5.11$

Use the Laplace transform in t to solve

$$\begin{aligned} &y_{tt} = y_{xx}, \qquad -\infty < x < \infty, \;\; t > 0, \\ &y_t(x,0) = x^2, \quad y(x,0) = 0. \end{aligned}$$

Hint: Note that e^{sx} does not go to zero as $s \to \infty$ for positive x, and e^{-sx} does not go to zero as $s \to \infty$ for negative x.

Answer

$y = tx^2 + \frac{t^3}{3}$

CHAPTER OVERVIEW

9: Power series methods

Thumbnail: The sine function and its Taylor approximations around $x_o = 0$ of 5th and 9th degree.

Contributors and Attributions

- Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

9.1: Power Series

Many functions can be written in terms of a power series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

If we assume that a solution of a differential equation is written as a power series, then perhaps we can use a method reminiscent of undetermined coefficients. That is, we will try to solve for the numbers a_k. Before we can carry out this process, let us review some results and concepts about power series.

9.1.1: Definition

As we said, a power series is an expression such as

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \cdots, \tag{9.1.1}$$

where $a_0, a_1, a_2, \ldots, a_k, \ldots$ and x_0 are constants. Let

$$S_n(x) = \sum_{k=0}^{n} a_k (x - x_0)^k = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \cdots + a_n (x - x_0)^n,$$

denote the so-called *partial sum*. If for some x, the limit

$$\lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \sum_{k=0}^{n} a_k (x - x_0)^k$$

exists, then we say that the series (9.1.1) converges at x. Note that for $x = x_0$, the series always converges to a_0. When (9.1.1) converges at any other point $x \neq x_0$, we say that (9.1.1) is a convergent power series. In this case we write

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k (x - x_0)^k.$$

If the series does not converge for any point $x \neq x_0$, we say that the series is divergent.

Example 9.1.1

The series

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

is convergent for any x. Recall that $k! = 1 \cdot 2 \cdot 3 \cdots k$ is the factorial. By convention we define $0! = 1$. In fact, you may recall that this series converges to e^x.

We say that (9.1.1) converges absolutely at x whenever the limit

$$\lim_{n \to \infty} \sum_{k=0}^{n} |a_k| \, |x - x_0|^k$$

exists. That is, the series $\sum_{k=0}^{\infty} |a_k| \, |x - x_0|^k$ is convergent. If (9.1.1) converges absolutely at x, then it converges at x. However, the opposite implication is not true.

Example 9.1.2

The series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

converges absolutely for all x in the interval $(-1, 1)$.

It converges at $x = -1$, as

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges (conditionally)

by the alternating series test.

But the power series does not converge absolutely at $x = -1$, because $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. The series diverges at $x = 1$.

9.1.2: Radius of Convergence

If a power series converges absolutely at some x_1, then for all x such that $|x - x_0| \leq |x_1 - x_0|$ (that is, x is closer than x_1 to x_0) we have $\left|a_k (x - x_0)^k\right| \leq \left|a_k (x_1 - x_0)^k\right|$ for all k. As the numbers $\left|a_k (x_1 - x_0)^k\right|$ sum to some finite limit, summing smaller positive numbers $\left|a_k (x - x_0)^k\right|$ must also have a finite limit. Hence, the series must converge absolutely at x.

Theorem 9.1.1

For a power series (9.1.1), there exists a number ρ (we allow $\rho = \infty$) called the radius of convergence such that the series converges absolutely on the interval $(x_0 - \rho,\ x_0 + \rho)$ and diverges for $x < x_0 - \rho$ and $x > x_0 + \rho$. We write $\rho = \infty$ if the series converges for all x.

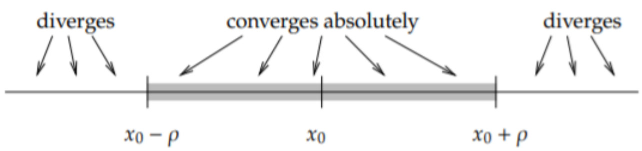


Figure 9.1.1: Convergence of a power series.

See Figure 9.1.1. In Example 9.1.1 the radius of convergence is $\rho = \infty$ as the series converges everywhere. In Example 9.1.2 the radius of convergence is $\rho = 1$. We note that $\rho = 0$ is another way of saying that the series is divergent. A useful test for convergence of a series is the *ratio test*. Suppose that

$$\sum_{k=0}^{\infty} c_k$$

is a series such that the limit

$$L = \lim_{n \to \infty} \left| \frac{c_{k+1}}{c_k} \right|$$

exists. Then the series converges absolutely if $L < 1$ and diverges if $L > 1$.

Let us apply this test to the series (9.1.1). That is we let $c_k = a_k (x - x_0)^k$ in the test. Compute

$$L = \lim_{n \to \infty} \left| \frac{c_{k+1}}{c_k} \right| = \lim_{n \to \infty} \left| \frac{a_{k+1} (x - x_0)^{k+1}}{a_k (x - x_0)^k} \right| = \lim_{n \to \infty} \left| \frac{a_{k+1}}{a_k} \right| |x - x_0|.$$

Define A by

$$A = \lim_{n \to \infty} \left| \frac{a_{k+1}}{a_k} \right|.$$

Then if $1 > L = A|x - x_0|$ the series (9.1.1) converges absolutely. If $A = 0$, then the series always converges. If $A > 0$, then the series converges absolutely if $|x - x_0| < \frac{1}{A}$, and diverges if $|x - x_0| > \frac{1}{A}$. That is, the radius of convergence is $\frac{1}{A}$.

A similar test is the *root test*. Suppose

$$L=\lim_{k\to\infty}\sqrt[3]{|c_k|}$$

exists. Then $\sum_{k=0}^{\infty} c_k$ converges absolutely if $L<1$ and diverges if $L>1$. We can use the same calculation as above to find A. Let us summarize.

Theorem 9.1.2: Ratio and Root Tests for Power Series

Let

$$\sum_{k=0}^{\infty} a_k\left(x-x_0\right)^k$$

be a power series such that

$$A=\lim_{n\to\infty}\left|\frac{a_{k+1}}{a_k}\right| \quad \text{or} \quad A=\lim_{k\to\infty}\sqrt[3]{|a_k|}$$

exists. If $A=0$, then the radius of convergence of the series is ∞. Otherwise the radius of convergence is $\frac{1}{A}$.

Below is a video on finding the interval of convergence of a power series.

Below is another video on finding the interval of convergence of a power series.

Example 9.1.3

Suppose we have the series

$$\sum_{k=0}^{\infty} 2^{-k}(x-1)^k.$$

First we compute,

$$A = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{2^{-k-1}}{2^{-k}}\right| = 2^{-1} = \frac{1}{2}.$$

Therefore the radius of convergence is 2, and the series converges absolutely on the interval $(-1,3)$. And we could just as well have used the root test:

$$A = \lim_{k\to\infty}\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{|2^{-k}|} = \lim_{k\to\infty} 2^{-1} = \frac{1}{2}.$$

Below is a video on the interval of convergence of a power series not centered at the origin.

Example 9.1.4

Consider

$$\sum_{k=0}^{\infty}\frac{1}{k^k}x^k.$$

Compute the limit for the root test,

$$A = \lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{\left|\frac{1}{k^k}\right|} = \lim_{k\to\infty}\sqrt[k]{\left|\frac{1}{k}\right|^k} = \lim_{k\to\infty}\frac{1}{k} = 0.$$

So the radius of convergence is ∞: the series converges everywhere. The ratio test would also work here.

The root or the ratio test does not always apply. That is the limit of $\left|\frac{a_{k+1}}{a_k}\right|$ or $\sqrt[k]{|a_k|}$ might not exist. There exist more sophisticated ways of finding the radius of convergence, but those would be beyond the scope of this chapter. The two methods above cover many of the series that arise in practice. Often if the root test applies, so does the ratio test, and vice versa, though the limit might be easier to compute in one way than the other.

9.1.3: Analytic Functions

Functions represented by power series are called analytic functions. Not every function is analytic, although the majority of the functions you have seen in calculus are. An analytic function $f(x)$ is equal to its Taylor series[1] near a point x_0. That is, for x near x_0 we have

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \tag{9.1.2}$$

where $f^{(k)}(x_0)$ denotes the k^{th} derivative of $f(x)$ at the point x_0.

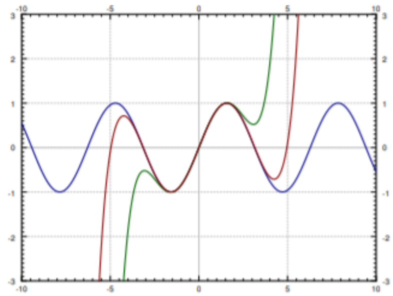


Figure 9.1.2: The sine function and its Taylor approximations around $x_o=0$ of 5th and 9th degree.

For example, sine is an analytic function and its Taylor series around $x_0=0$ is given by

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

In Figure $9.1.2$ we plot $\sin(x)$ and the truncations of the series up to degree 5 and 9. You can see that the approximation is very good for x near 0, but gets worse for x further away from 0. This is what happens in general. To get a good approximation far away from x_0 you need to take more and more terms of the Taylor series.

9.1.4: Manipulating Power Series

One of the main properties of power series that we will use is that we can differentiate them term by term. That is, suppose that $\sum a_k(x-x_0)^k$ is a convergent power series. Then for x in the radius of convergence we have

$$\frac{d}{dx}\left[\sum_{k=0}^{\infty}a_k(x-x_0)^k\right]=\sum_{k=1}^{\infty}ka_k(x-x_0)^{k-1}.$$

Notice that the term corresponding to $k=0$ disappeared as it was constant. The radius of convergence of the differentiated series is the same as that of the original.

Example $9.1.5$

Let us show that the exponential $y=e^x$ solves $y'=y$. First write

$$y=e^x=\sum_{k=0}^{\infty}\frac{1}{k!}x^k.$$

Now differentiate

$$y'=\sum_{k=1}^{\infty}k\frac{1}{k!}x^{k-1}=\sum_{k=1}^{\infty}\frac{1}{(k-1)!}x^{k-1}.$$

We *reindex* the series by simply replacing k with $k+1$. The series does not change, what changes is simply how we write it. After reindexing the series starts at $k=0$ again.

$$\sum_{k=1}^{\infty}\frac{1}{(k-1)!}x^{k-1}=\sum_{k+1=1}^{\infty}\frac{1}{((k+1)-1)!}x^{(k+1)-1}=\sum_{k=0}^{\infty}\frac{1}{k!}x^k.$$

That was precisely the power series for e^x that we started with, so we showed that $\frac{d}{dx}[e^x]=e^x$.

Convergent power series can be added and multiplied together, and multiplied by constants using the following rules. First, we can add series by adding term by term,

$$\left(\sum_{k=0}^{\infty} a_k (x-x_0)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-x_0)^k\right) = \sum_{k=0}^{\infty} (a_k + b_k)(x-x_0)^k.$$

We can multiply by constants,

$$\alpha \left(\sum_{k=0}^{\infty} a_k (x-x_0)^k\right) = \sum_{k=0}^{\infty} \alpha a_k (x-x_0)^k.$$

We can also multiply series together,

$$\left(\sum_{k=0}^{\infty} a_k (x-x_0)^k\right) \left(\sum_{k=0}^{\infty} b_k (x-x_0)^k\right) = \sum_{k=0}^{\infty} c_k (x-x_0)^k,$$

where $c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$. The radius of convergence of the sum or the product is at least the minimum of the radii of convergence of the two series involved.

Below is a video on differentiation and integration using power series.

9.1.5: Power Series for Rational Functions

Polynomials are simply finite power series. That is, a polynomial is a power series where the a_k are zero for all k large enough. We can always expand a polynomial as a power series about any point x_0 by writing the polynomial as a polynomial in $(x-x_0)$. For example, let us write $2x^2 - 3x + 4$ as a power series around $x_0 = 1$:

$$2x^2 - 3x + 4 = 3 + (x-1) + 2(x-1)^2.$$

In other words $a_0 = 3$, $a_1 = 1$, $a_2 = 2$, and all other $a_k = 0$. To do this, we know that $a_k = 0$ for all $k \geq 3$ as the polynomial is of degree 2.

We write $a_0 + a_1(x-1) + a_2(x-1)^2$, we expand, and we solve for a_0, a_1, and a_2. We could have also differentiated at $x = 1$and used the Taylor series formula $(9.1.2)$.

Let us look at rational functions, that is, ratios of polynomials. An important fact is that a series for a function only defines the function on an interval even if the function is defined elsewhere. For example, for $-1 < x < 1$ we have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

This series is called the *geometric series*. The ratio test tells us that the radius of convergence is 1. The series diverges for $x \leq -1$ and $x \geq 1$, even though $\frac{1}{1-x}$ is defined for all $x \neq 1$.

We can use the geometric series together with rules for addition and multiplication of power series to expand rational functions around a point, as long as the denominator is not zero at x_0. Note that as for polynomials, we could equivalently use the Taylor series expansion $(9.1.2)$.

Below is a video on finding a power series to represent a rational function.

Example 9.1.5

Expand $\frac{x}{1+2x+x^2}$ as a power series around the origin $(x_0 = 0)$ and find the radius of convergence. First, write $1+2x+x^2 = (1+x)^2 = \big(1-(-x)\big)^2$. Now we compute

$$\begin{aligned} \frac{x}{1+2x+x^2} &= x\left(\frac{1}{1-(-x)}\right)^2 \\ &= x\left(\sum_{k=0}^{\infty}(-1)^k x^k\right)^2 \\ &= x\left(\sum_{k=0}^{\infty} c_k x^k\right) \\ &= \sum_{k=0}^{\infty} c_k x^{k+1}, \end{aligned} \tag{9.1.3}$$

where using the formula for the product of series we obtain, $c_0 = 1$, $c_1 = -1-1 = -2$, $c_2 = 1+1+1 = 3$, etc . . ..

Therefore

$$\frac{x}{1+2x+x^2} = \sum_{k=1}^{\infty}(-1)^{k+1}kx^k = x - 2x^2 + 3x^3 - 4x^4 + \cdots$$

The radius of convergence is at least 1. We use the ratio test

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+2}(k+1)}{(-1)^{k+1}k}\right| = \lim_{k\to\infty}\frac{k+1}{k} = 1.$$

So the radius of convergence is actually equal to 1.

When the rational function is more complicated, it is also possible to use method of partial fractions. For example, to find the Taylor series for $\frac{x^3+x}{x^2-1}$, we write

$$\frac{x^3+x}{x^2-1} = x + \frac{1}{1+x} - \frac{1}{1-x} = x + \sum_{k=0}^{\infty}(-1)^k x^k - \sum_{k=0}^{\infty} x^k = -x + \sum_{\substack{k=3 \\ k \text{ odd}}}^{\infty}(-2)x^k.$$

9.1.6: Footnotes

[1] Named after the English mathematician Sir Brook Taylor (1685–1731).

9.2: Series Solutions of Linear Second Order ODEs

Suppose we have a linear second order homogeneous ODE of the form

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{9.2.1}$$

Suppose that $p(x)$, $q(x)$, and $r(x)$ are polynomials. We will try a solution of the form

$$y = \sum_{k=0}^{\infty} a_k (x - x_o)^2$$

and solve for the a_k to try to obtain a solution defined in some interval around x_o.

Definition: Ordinary and Singular Points

The point x_o is called an *ordinary point* if $p(x_o) \neq 0$ in linear second order homogeneous ODE of the form in Equation 9.2.1. That is, the functions

$$\frac{q(x)}{p(x)} \quad \text{and} \quad \frac{r(x)}{p(x)}$$

are defined for x near x_o.

If $p(x_0) = 0$, then we say x_o is a *singular point*.

Handling singular points is harder than ordinary points and so we now focus only on ordinary points.

Example 9.2.1: Expansion around an Ordinary Point

Let us start with a very simple example

$$y'' - y = 0$$

Let us try a power series solution near $x_o = 0$, which is an ordinary point.

Solution

Every point is an ordinary point in fact, as the equation is constant coefficient. We already know we should obtain exponentials or the hyperbolic sine and cosine, but let us pretend we do not know this.

We try

$$y = \sum_{k=0}^{\infty} a_k x^k$$

If we differentiate, the $k = 0$ term is a constant and hence disappears. We therefore get

$$y' = \sum_{k=1}^{\infty} k\, a_k\, x^{k-1}$$

We differentiate yet again to obtain (now the $k = 1$ term disappears)

$$y'' = \sum_{k=2}^{\infty} k\,(k-1)\, a_k\, x^{k-2}$$

We reindex the series (replace k with $k + 2$) to obtain

$$y'' = \sum_{k=0}^{\infty} (k+2)\,(k+1)\, a_{k+2}\, x^k$$

Now we plug y and y'' into the differential equation.

$$\begin{aligned} 0 = y'' - y &= \left(\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k\right) - \left(\sum_{k=0}^{\infty}a_kx^k\right) \\ &= \sum_{k=0}^{\infty}\left((k+2)(k+1)a_{k+2}x^k - a_kx^k\right) \\ &= \sum_{k=0}^{\infty}\left((k+2)(k+1)a_{k+2} - a_k\right)x^k. \end{aligned} \tag{9.2.2}$$

As $y'' - y$ is supposed to be equal to 0, we know that the coefficients of the resulting series must be equal to 0. Therefore,

$$(k+2)(k+1)a_{k+2} - a_k = 0, \qquad \text{or} \qquad a_{k+2} = \frac{a_k}{(k+2)(k+1)}.$$

The above equation is called a *recurrence relation* for the coefficients of the power series. It did not matter what a_0 or a_1 was. They can be arbitrary. But once we pick a_0 and a_1, then all other coefficients are determined by the recurrence relation.

Let us see what the coefficients must be. First, a_0 and a_1 are arbitrary

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{(3)(2)}, \quad a_4 = \frac{a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)}, \quad a_5 = \frac{a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)}, \quad \ldots$$

So we note that for even k, that is $k = 2n$ we get

$$a_k = a_{2n} = \frac{a_o}{(2n)!}$$

and for odd k that is $k = 2n+1$ we have

$$a_k = a_{2n+1} = \frac{a_1}{(2n+1)!}$$

Let us write down the series

$$y = \sum_{k=0}^{\infty}a_kx^k = \sum_{n=0}^{\infty}\left(\frac{a_0}{(2n)!}x^{2n} + \frac{a_1}{(2n+1)!}x^{2n+1}\right) = a_0\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n} + a_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}.$$

We recognize the two series as the hyperbolic sine and cosine. Therefore,

$$y = a_o\cosh x + a_1\sinh x$$

Of course, in general we will not be able to recognize the series that appears, since usually there will not be any elementary function that matches it. In that case we will be content with the series.

Example 9.2.2

Let us do a more complex example. Suppose we wish to solve *Airy's equation*[1], that is

$$y'' - xy = 0$$

near the point $x_0 = 0$, which is an ordinary point.

We try

$$y = \sum_{k=0}^{\infty}a_kx^k$$

We differentiate twice (as above) to obtain

$$y'' = \sum_{k=2}^{\infty}k(k-1)a_kx^{k-2}$$

We plug y into the equation

$$\begin{aligned} 0=y''-xy &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - x\left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right). \end{aligned} \tag{9.2.3}$$

We reindex to make things easier to sum

$$\begin{aligned} 0=y''-xy &= \left(2a_2+\sum_{k=1}^{\infty}(k+2)\,(k+1)\,a_{k+2}x^k\right) - \left(\sum_{k=1}^{\infty} a_{k-1}x^k\right). \\ &= 2a_2+\sum_{k=1}^{\infty}\Big((k+2)\,(k+1)\,a_{k+2}-a_{k-1}\Big)\,x^k. \end{aligned} \tag{9.2.4}$$

Again $y''-xy$ is supposed to be 0 so first we notice that $a_2=0$ and also

$$(k+2)\,(k+1)\,a_{k+2}-a_{k-1}=0, \qquad \text{or} \qquad a_{k+2}=\frac{a_{k-1}}{(k+2)(k+1)}.$$

Now we jump in steps of three. First we notice that since $a_2=0$ we must have that, $a_5=0$, $a_8=0$, $a_{11}=0$, etc …. In general $a_{3n+2}=0$. The constants a_0 and a_1 are arbitrary and we obtain

$$a_3=\frac{a_0}{(3)(2)}, \quad a_4=\frac{a_1}{(4)(3)}, \quad a_6=\frac{a_3}{(6)(5)}=\frac{a_0}{(6)(5)(3)(2)}, \quad a_7=\frac{a_4}{(7)(6)}=\frac{a_1}{(7)(6)(4)(3)}, \quad \ldots$$

For a_k where k is a multiple of 3, that is $k=3n$ we notice that

$$a_{3n}=\frac{a_0}{(2)(3)(5)(6)\cdots(3n-1)(3n)}.$$

For a_k where $k=3n+1$, we notice

$$a_{3n+1}=\frac{a_1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}.$$

In other words, if we write down the series for y we notice that it has two parts

$$\begin{aligned} y &= \left(a_0+\frac{a_0}{6}x^3+\frac{a_0}{180}x^6+\cdots+\frac{a_0}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n}+\cdots\right) \\ &\quad+\left(a_1x+\frac{a_1}{12}x^4+\frac{a_1}{504}x^7+\cdots+\frac{a_1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1}+\cdots\right) \\ &= a_0\left(1+\frac{1}{6}x^3+\frac{1}{180}x^6+\cdots+\frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n}+\cdots\right) \\ &\quad+a_1\left(x+\frac{1}{12}x^4+\frac{1}{504}x^7+\cdots+\frac{1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1}+\cdots\right). \end{aligned} \tag{9.2.5}$$

We define

$$\begin{aligned} y_1(x) &= 1+\frac{1}{6}x^3+\frac{1}{180}x^6+\cdots+\frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n}+\cdots, \\ y_2(x) &= x+\frac{1}{12}x^4+\frac{1}{504}x^7+\cdots+\frac{1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1}+\cdots, \end{aligned} \tag{9.2.6}$$

and write the general solution to the equation as $y(x)=a_0y_1(x)+a_1y_2(x)$. Notice from the power series that $y_1(0)=1$ and $y_2(0)=0$. Also, $y_1'(0)=0$ and $y_2'(0)=1$. Therefore $y(x)$ is a solution that satisfies the initial conditions $y(0)=a_0$ and $y'(0)=a_1$.

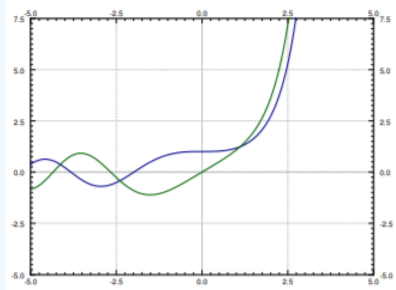


Figure 9.2.1: The two solutions y_1 and y_2 to Airy's equation.

The functions y_1 and y_2 cannot be written in terms of the elementary functions that you know. See Figure 9.2.1 for the plot of the solutions y_1 and y_2. These functions have many interesting properties. For example, they are oscillatory for negative x (like solutions to $y'' + y = 0$) and for positive x they grow without bound (like solutions to $y'' - y = 0$).

Sometimes a solution may turn out to be a polynomial.

✓ Example 9.2.3: Hermite Equation

Let us find a solution to the so-called Hermite's equation of order n[2] is the equation

$$y'' - 2xy' + 2ny = 0.$$

Find a solution around the point $x_0 = 0$.

Solution

We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We differentiate (as above) to obtain

$$\begin{aligned} y' &= \sum_{k=1}^{\infty} k a_k x^{k-1}, \\ y'' &= \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}. \end{aligned} \tag{9.2.7}$$

Now we plug into the equation

$$\begin{aligned} 0 &= y'' - 2xy' + 2ny \\ &= \left(\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}\right) - 2x\left(\sum_{k=1}^{\infty} k a_k x^{k-1}\right) + 2n\left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= \left(\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}\right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k\right) + \left(\sum_{k=0}^{\infty} 2n a_k x^k\right) \\ &= \left(2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} x^k\right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k\right) + \left(2na_0 + \sum_{k=1}^{\infty} 2n a_k x^k\right) \\ &= 2a_2 + 2na_0 + \sum_{k=1}^{\infty} \left((k+2)(k+1) a_{k+2} - 2k a_k + 2n a_k\right) x^k. \end{aligned} \tag{9.2.8}$$

As $y'' - 2xy' + 2ny = 0$ we have

$$(k+2)(k+1)a_{k+2}+(-2k+2n)a_k=0, \qquad \text{or} \qquad a_{k+2}=\frac{(2k-2n)}{(k+2)(k+1)}a_k.$$

This recurrence relation actually includes $a_2=-na_0$ (which comes about from $2a_2+2na_0=0$). Again a_0 and a_1 are arbitrary.

$$\begin{aligned} a_2 &= \frac{-2n}{(2)(1)}a_0, \qquad a_3=\frac{2(1-n)}{(3)(2)}a_1, \\ a_4 &= \frac{2(2-n)}{(4)(3)}a_2=\frac{2^2(2-n)(-n)}{(4)(3)(2)(1)}a_0, \\ a_5 &= \frac{2(3-n)}{(5)(4)}a_3=\frac{2^2(3-n)(1-n)}{(5)(4)(3)(2)}a_1, \quad \ldots \end{aligned} \tag{9.2.9}$$

Let us separate the even and odd coefficients. We find that

$$\begin{aligned} a_{2m} &= \frac{2^m(-n)(2-n)\cdots(2m-2-n)}{(2m)!}, \\ a_{2m+1} &= \frac{2^m(1-n)(3-n)\cdots(2m-1-n)}{(2m+1)!}. \end{aligned} \tag{9.2.10}$$

Let us write down the two series, one with the even powers and one with the odd.

$$\begin{aligned} y_1(x) &= 1+\frac{2(-n)}{2!}x^2+\frac{2^2(-n)(2-n)}{4!}x^4+\frac{2^3(-n)(2-n)(4-n)}{6!}x^6+\cdots, \\ y_2(x) &= x+\frac{2(1-n)}{3!}x^3+\frac{2^2(1-n)(3-n)}{5!}x^5+\frac{2^3(1-n)(3-n)(5-n)}{7!}x^7+\cdots. \end{aligned} \tag{9.2.11}$$

We then write

$$y(x)=a_0y_1(x)+a_1y_2(x).$$

We also notice that if n is a positive even integer, then $y_1(x)$ is a polynomial as all the coefficients in the series beyond a certain degree are zero. If n is a positive odd integer, then $y_2(x)$ is a polynomial. For example, if $n=4$, then

$$y_1(x)=1+\frac{2(-4)}{2!}x^2+\frac{2^2(-4)(2-4)}{4!}x^4=1-4x^2+\frac{4}{3}x^4.$$

9.2.1: Footnotes

[1] Named after the English mathematician Sir George Biddell Airy (1801 – 1892).

[2] Named after the French mathematician Charles Hermite (1822–1901).

9.3: Singular Points and the Method of Frobenius

9.3.1: Examples

While behavior of ODEs at singular points is more complicated, certain singular points are not especially difficult to solve. Let us look at some examples before giving a general method. We may be lucky and obtain a power series solution using the method of the previous section, but in general we may have to try other things.

Example 9.3.1

Let us first look at a simple first order equation

$$2xy' - y = 0. \tag{9.3.1}$$

Note that $x = 0$ is a singular point. If we only try to plug in

$$y = \sum_{k=0}^{\infty} a_k x^k, \tag{9.3.2}$$

we obtain

$$\begin{aligned} 0 = 2xy' - y &= 2x \left(\sum_{k=1}^{\infty} k a_k x^{k-1} \right) - \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= a_0 + \sum_{k=1}^{\infty} (2k a_k - a_k)\, x^k. \end{aligned} \tag{9.3.3}$$

First, $a_0 = 0$. Next, the only way to solve $0 = 2ka_k - a_k = (2k-1)\, a_k$ for $k = 1, 2, 3, \ldots$ is for $a_k = 0$ for all k. Therefore we only get the trivial solution $y = 0$. We need a nonzero solution to get the general solution.

Let us try $y = x^r$ for some real number r. Consequently our solution---if we can find one---may only make sense for positive x. Then $y' = rx^{r-1}$. So

$$0 = 2xy' - y = 2xrx^{r-1} - x^r = (2r-1)x^r.$$

Therefore $r = \frac{1}{2}$, or in other words $y = x^{1/2}$. Multiplying by a constant, the general solution for positive x is

$$y = Cx^{1/2}.$$

If $C \neq 0$ then the derivative of the solution "blows up" at $x = 0$ (the singular point). There is only one solution that is differentiable at $x = 0$ and that's the trivial solution $y = 0$.

Not every problem with a singular point has a solution of the form $y = x^r$, of course. But perhaps we can combine the methods. What we will do is to try a solution of the form

$$y = x^r f(x)$$

where $f(x)$ is an analytic function.

Example 9.3.2

Suppose that we have the equation

$$4x^2 y'' - 4x^2 y' + (1-2x)y = 0, \tag{9.3.4}$$

and again note that $x = 0$ is a singular point. Let us try

$$y = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}, \tag{9.3.5}$$

where r is a real number, not necessarily an integer. Again if such a solution exists, it may only exist for positive x. First let us find the derivatives

$$
\begin{aligned}
y' &= \sum_{k=0}^{\infty}(k+r)\, a_k x^{k+r-1}, \\
y'' &= \sum_{k=0}^{\infty}(k+r)\,(k+r-1)\, a_k x^{k+r-2}.
\end{aligned} \tag{9.3.6}
$$

Plugging Equations 9.3.5 - 9.3.6 into our original differential equation (Equation 9.3.4) we obtain

$$
\begin{aligned}
0 &= 4x^2y'' - 4x^2y' + (1-2x)y \\
&= 4x^2 \left(\sum_{k=0}^{\infty}(k+r)\,(k+r-1)\, a_k x^{k+r-2}\right) - 4x^2 \left(\sum_{k=0}^{\infty}(k+r)\, a_k x^{k+r-1}\right) + (1-2x) \left(\sum_{k=0}^{\infty} a_k x^{k+r}\right) \\
&= \left(\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r}\right) - \left(\sum_{k=0}^{\infty} 4(k+r)\, a_k x^{k+r+1}\right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r}\right) - \left(\sum_{k=0}^{\infty} 2a_k x^{k+r+1}\right) \\
&= \left(\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r}\right) - \left(\sum_{k=1}^{\infty} 4(k+r-1)\, a_{k-1} x^{k+r}\right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r}\right) - \left(\sum_{k=1}^{\infty} 2a_{k-1} x^{k+r}\right) \\
&= 4r(r-1)\, a_0 x^r + a_0 x^r + \sum_{k=1}^{\infty} \left(4(k+r)\,(k+r-1)\, a_k - 4(k+r-1)\, a_{k-1} + a_k - 2a_{k-1}\right)\, x^{k+r} \\
&= (4r(r-1)+1)\; a_0 x^r + \sum_{k=1}^{\infty} \left((4(k+r)\,(k+r-1)+1)\; a_k - (4(k+r-1)+2)\; a_{k-1}\right)\, x^{k+r}.
\end{aligned} \tag{9.3.7}
$$

To have a solution we must first have $(4r(r-1)+1)\; a_0 = 0$. Supposing that $a_0 \neq 0$ we obtain

$$
4r(r-1)+1 = 0.
$$

This equation is called the *indicial equation*. This particular indicial equation has a double root at $r = \frac{1}{2}$.

OK, so we know what r has to be. That knowledge we obtained simply by looking at the coefficient of x^r. All other coefficients of x^{k+r} also have to be zero so

$$
(4(k+r)\,(k+r-1)+1)\; a_k - (4(k+r-1)+2)\; a_{k-1} = 0.
$$

If we plug in $r = \frac{1}{2}$ and solve for a_k we get

$$
a_k = \frac{4(k+\frac{1}{2}-1)+2}{4(k+\frac{1}{2})\,(k+\frac{1}{2}-1)+1}\; a_{k-1} = \frac{1}{k}\, a_{k-1}.
$$

Let us set $a_0 = 1$. Then

$$
a_1 = \frac{1}{1} a_0 = 1, \qquad a_2 = \frac{1}{2} a_1 = \frac{1}{2}, \qquad a_3 = \frac{1}{3} a_2 = \frac{1}{3\cdot 2}, \qquad a_4 = \frac{1}{4} a_3 = \frac{1}{4\cdot 3\cdot 2}, \qquad \ldots
$$

Extrapolating, we notice that

$$
a_k = \frac{1}{k(k-1)(k-2)\cdots 3\cdot 2} = \frac{1}{k!}.
$$

In other words,

$$
y = \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+1/2} = x^{1/2} \sum_{k=0}^{\infty} \frac{1}{k!} x^k = x^{1/2} e^x.
$$

That was lucky! In general, we will not be able to write the series in terms of elementary functions. We have one solution, let us call it $y_1 = x^{1/2}e^x$. But what about a second solution? If we want a general solution, we need two linearly independent solutions. Picking a_0 to be a different constant only gets us a constant multiple of y_1, and we do not have any other r to try; we only have one solution to the indicial equation. Well, there are powers of x floating around and we are taking derivatives, perhaps the logarithm (the antiderivative of x^{-1}) is around as well. It turns out we want to try for another solution of the form

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+r} + (\ln x) y_1,$$

which in our case is

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+1/2} + (\ln x) x^{1/2} e^x.$$

We now differentiate this equation, substitute into the differential equation and solve for b_k. A long computation ensues and we obtain some recursion relation for b_k. The reader can (and should) try this to obtain for example the first three terms

$$b_1 = b_0 - 1, \qquad b_2 = \frac{2b_1 - 1}{4}, \qquad b_3 = \frac{6b_2 - 1}{18}, \qquad \dots$$

We then fix b_0 and obtain a solution y_2. Then we write the general solution as $y = Ay_1 + By_2$.

9.3.2: Method of Frobenius

Before giving the general method, let us clarify when the method applies. Let

$$p(x)y'' + q(x)y' + r(x)y = 0$$

be an ODE. As before, if $p(x_0) = 0$, then x_0 is a singular point. If, furthermore, the limits

$$\lim_{x \to x_0} (x - x_0)\frac{q(x)}{p(x)} \qquad \text{and} \qquad \lim_{x \to x_0} (x - x_0)^2 \frac{r(x)}{p(x)}$$

both exist and are finite, then we say that x_0 is a *regular singular point*.

Example 9.3.3: Expansion around a regular singular point

Often, and for the rest of this section, $x_0 = 0$. Consider

$$x^2 y'' + x(1+x)y' + (\pi + x^2)y = 0.$$

Write

$$\begin{aligned} \lim_{x \to 0} x\frac{q(x)}{p(x)} &= \lim_{x \to 0} x\frac{x(1+x)}{x^2} = \lim_{x \to 0} (1+x) = 1, \\ \lim_{x \to 0} x^2\frac{r(x)}{p(x)} &= \lim_{x \to 0} x^2 frac(\pi + x^2)x^2 = \lim_{x \to 0} (\pi + x^2) = \pi \end{aligned} \tag{9.3.8}$$

So $x = 0$ is a regular singular point.

On the other hand if we make the slight change

$$x^2 y'' + (1+x)y' + (\pi + x^2)y = 0,$$

then

$$\lim_{x \to 0} x\frac{q(x)}{p(x)} = \lim_{x \to 0} x\frac{(1+x)}{x^2} = \lim_{x \to 0} \frac{1+x}{x} = \text{DNE}.$$

Here DNE stands for *does not exist*. The point 0 is a singular point, but not a regular singular point.

Below is part 1 of a video on the method of Frobenius.

Below is part 2 of a video on the method of Frobenius.

Let us now discuss the general *Method of Frobenius*[1]. Let us only consider the method at the point $x = 0$ for simplicity. The main idea is the following theorem.

Theorem 9.3.1

Method of Frobenius

Suppose that

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{9.3.9}$$

has a regular singular point at $x = 0$, then there exists at least one solution of the form

$$y = x^r \sum_{k=0}^{\infty} a_k x^k.$$

A solution of this form is called a *Frobenius-type solution*.

The method usually breaks down like this.

i. We seek a Frobenius-type solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}.$$

We plug this y into equation (9.3.9). We collect terms and write everything as a single series.

ii. The obtained series must be zero. Setting the first coefficient (usually the coefficient of x^r) in the series to zero we obtain the *indicial equation*, which is a quadratic polynomial in r.

iii. If the indicial equation has two real roots r_1 and r_2 such that $r_1 - r_2$ is not an integer, then we have two linearly independent Frobenius-type solutions. Using the first root, we plug in

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and we solve for all a_k to obtain the first solution. Then using the second root, we plug in

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k,$$

and solve for all b_k to obtain the second solution.

iv. If the indicial equation has a doubled root r, then there we find one solution

$$y_1 = x^{r} \sum_{k=0}^{\infty} a_k x^k,$$

and then we obtain a new solution by plugging

$$y_2 = x^{r} \sum_{k=0}^{\infty} b_k x^k + (\ln x) y_1,$$

into Equation (9.3.9) and solving for the constants b_k.

v. If the indicial equation has two real roots such that $r_1 - r_2$ is an integer, then one solution is

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and the second linearly independent solution is of the form

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k + C(\ln x) y_1,$$

where we plug y_2 into (9.3.9) and solve for the constants b_k and C.

vi. Finally, if the indicial equation has complex roots, then solving for a_k in the solution

$$y = x^{r_1} \sum_{k=0}^{\infty} a_k x^k$$

results in a complex-valued function---all the a_k are complex numbers. We obtain our two linearly independent solutions[2] by taking the real and imaginary parts of y.

The main idea is to find at least one Frobenius-type solution. If we are lucky and find two, we are done. If we only get one, we either use the ideas above or even a different method such as reduction of order (Exercise 2.1.8) to obtain a second solution.

Below is a video on using the method of Frobenious to solve a differential equation.

Below is another video on using the method of Frobenious to solve a differential equation.

9.3.3: Bessel Functions

An important class of functions that arises commonly in physics are the *Bessel functions*[3]. For example, these functions appear when solving the wave equation in two and three dimensions. First we have *Bessel's equation* of order p:

$$x^2y'' + xy' + \left(x^2 - p^2\right) y = 0.$$

We allow p to be any number, not just an integer, although integers and multiples of $\frac{1}{2}$ are most important in applications. When we plug

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

into Bessel's equation of order p we obtain the indicial equation

$$r(r-1) + r - p^2 = (r-p)(r+p) = 0.$$

Therefore we obtain two roots $r_1 = p$ and $r_2 = -p$. If p is not an integer following the method of Frobenius and setting $a_0 = 1$, we obtain linearly independent solutions of the form

$$\begin{aligned} y_1 &= x^p \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!(k+p)(k-1+p)\cdots(2+p)(1+p)}, \\ y_2 &= x^{-p} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!(k-p)(k-1-p)\cdots(2-p)(1-p)}. \end{aligned} \tag{9.3.10}$$

? Exercise 9.3.1

a. Verify that the indicial equation of Bessel's equation of order p is $(r-p)(r+p) = 0$.
b. Suppose that p is not an integer. Carry out the computation to obtain the solutions y_1 and y_2 above.

Bessel functions will be convenient constant multiples of y_1 and y_2. First we must define the *gamma function*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Notice that $\Gamma(1) = 1$. The gamma function also has a wonderful property

$$\Gamma(x+1) = x\Gamma(x).$$

From this property, one can show that $\Gamma(n) = (n-1)!$ when n is an integer, so the gamma function is a continuous version of the factorial. We compute:

$$\begin{aligned} \Gamma(k+p+1) &= (k+p)(k-1+p)\cdots(2+p)(1+p)\Gamma(1+p), \\ \Gamma(k-p+1) &= (k-p)(k-1-p)\cdots(2-p)(1-p)\Gamma(1-p). \end{aligned} \tag{9.3.11}$$

Exercise 9.3.2

Verify the above identities using $\Gamma(x+1) = x\Gamma(x)$.

We define the *Bessel functions of the first kind* of order p and $-p$ as

$$\begin{aligned} J_p(x) &= \frac{1}{2^p\Gamma(1+p)}y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k+p}, \\ J_{-p}(x) &= \frac{1}{2^{-}\Gamma(1-p)}y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-p+1)}\left(\frac{x}{2}\right)^{2k-p}. \end{aligned} \tag{9.3.12}$$

As these are constant multiples of the solutions we found above, these are both solutions to Bessel's equation of order p. The constants are picked for convenience.

When p is not an integer, J_p and J_{-p} are linearly independent. When n is an integer we obtain

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}.$$

In this case it turns out that

$$J_n(x) = (-1)^n J_{-n}(x),$$

and so we do not obtain a second linearly independent solution. The other solution is the so-called *Bessel function of second kind*. These make sense only for integer orders n and are defined as limits of linear combinations of $J_p(x)$ and $J_{-p}(x)$ as p approaches n in the following way:

$$Y_n(x) = \lim_{p\to n} \frac{\cos(p\pi)J_p(x) - J_{-p}(x)}{\sin(p\pi)}.$$

As each linear combination of $J_p(x)$ and $J_{-p}(x)$ is a solution to Bessel's equation of order p, then as we take the limit as p goes to n, $Y_n(x)$ is a solution to Bessel's equation of order n. It also turns out that $Y_n(x)$ and $J_n(x)$ are linearly independent. Therefore when n is an integer, we have the general solution to Bessel's equation of order n

$$y = AJ_n(x) + BY_n(x),$$

for arbitrary constants A and B. Note that $Y_n(x)$ goes to negative infinity at $x=0$. Many mathematical software packages have these functions $J_n(x)$ and $Y_n(x)$ defined, so they can be used just like say $\sin(x)$ and $\cos(x)$. In fact, they have some similar properties. For example, $-J_1(x)$ is a derivative of $J_0(x)$, and in general the derivative of $J_n(x)$ can be written as a linear combination of $J_{n-1}(x)$ and $J_{n+1}(x)$. Furthermore, these functions oscillate, although they are not periodic. See Figure 9.3.1 for graphs of Bessel functions.

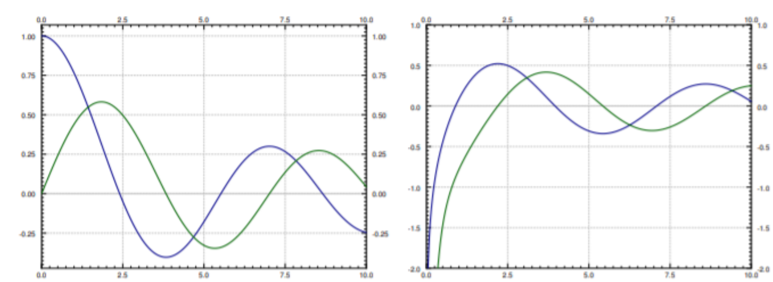


Figure 9.3.1: Plot of the $J_0(x)$ and $J_1(x)$ in the first graph and $Y_0(x)$ and $Y_1(x)$ in the second graph.

Example 9.3.4: Using Bessel functions to Solve a ODE

Other equations can sometimes be solved in terms of the Bessel functions. For example, given a positive constant λ,

$$xy'' + y' + \lambda^2 xy = 0,$$

can be changed to $x^2y'' + xy' + \lambda^2x^2y = 0$. Then changing variables $t = \lambda x$ we obtain via chain rule the equation in y and t:

$$t^2y'' + ty' + t^2y = 0,$$

which can be recognized as Bessel's equation of order 0. Therefore the general solution is $y(t) = AJ_0(t) + BY_0(t)$, or in terms of x:

$$y = AJ_0(\lambda x) + BY_0(\lambda x).$$

This equation comes up for example when finding fundamental modes of vibration of a circular drum, but we digress.

9.3.4: Footnotes

[1] Named after the German mathematician Ferdinand Georg Frobenius (1849 – 1917).

[2] See Joseph L. Neuringera, The Frobenius method for complex roots of the indicial equation, International Journal of Mathematical Education in Science and Technology, Volume 9, Issue 1, 1978, 71–77.

[3] Named after the German astronomer and mathematician Friedrich Wilhelm Bessel (1784 – 1846).

9.E: Power series methods (Exercises)

9.E.1: 7.1: Power Series

Exercise $9.E.7.1.1$

Is the power series $\sum_{k=0}^{\infty} e^k x^k$ convergent? If so, what is the radius of convergence?

Exercise $9.E.7.1.2$

Is the power series $\sum_{k=0}^{\infty} k x^k$ convergent? If so, what is the radius of convergence?

Exercise $9.E.7.1.3$

Is the power series $\sum_{k=0}^{\infty} k! x^k$ convergent? If so, what is the radius of convergence?

Exercise $9.E.7.1.4$

Is the power series $\sum_{k=0}^{\infty} \frac{1}{(2k)!}(x-10)^k$ convergent? If so, what is the radius of convergence?

Exercise $9.E.7.1.5$

Determine the Taylor series for $\sin x$ around the point $x_0 = \pi$.

Exercise $9.E.7.1.6$

Determine the Taylor series for $\ln x$ around the point $x_0 = 1$, and find the radius of convergence.

Exercise $9.E.7.1.7$

Determine the Taylor series and its radius of convergence of $\dfrac{1}{1+x}$ around $x_0 = 0$.

Exercise $9.E.7.1.8$

Determine the Taylor series and its radius of convergence of $\dfrac{x}{4-x^2}$ around $x_0 = 0$. Hint: You will not be able to use the ratio test.

Exercise $9.E.7.1.9$

Expand $x^5 + 5x + 1$ as a power series around $x_0 = 5$.

Exercise $9.E.7.1.10$

Suppose that the ratio test applies to a series $\sum_{k=0}^{\infty} a_k x^k$. Show, using the ratio test, that the radius of convergence of the differentiated series is the same as that of the original series.

Exercise $9.E.7.1.11$

Suppose that f is an analytic function such that $f^{(n)}(0) = n$. Find $f(1)$.

Exercise $9.E.7.1.12$

Is the power series $\sum_{n=1}^{\infty}(0.1)^n x^n$ convergent? If so, what is the radius of convergence?

Answer

Yes. Radius of convergence is 10.

Exercise $9.E.7.1.13$: (challenging)

Is the power series $\sum_{n=1}^{\infty}\frac{n!}{n^n}x^n$ convergent? If so, what is the radius of convergence?

Answer

Yes. Radius of convergence is e.

Exercise $9.E.7.1.14$

Using the geometric series, expand $\frac{1}{1-x}$ around $x_0=2$. For what x does the series converge?

Answer

$\frac{1}{1-x}=-\frac{1}{1-(2-x)}$ so $\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$, which converges for $1<x<3$.

Exercise $9.E.7.1.15$: (challenging)

Find the Taylor series for x^7e^x around $x_0=0$.

Answer

$\sum_{n=7}^{\infty}\frac{1}{(n-7)!}x^n$

Exercise $9.E.7.1.16$: (challenging)

Imagine f and g are analytic functions such that $f^{(k)}(0)=g^{(k)}(0)$ for all large enough k. What can you say about $f(x)-g(x)$?

Answer

$f(x)-g(x)$ is a polynomial. Hint: Use Taylor series.

9.E.2: 7.2: Series solutions of linear second order ODEs

In the following exercises, when asked to solve an equation using power series methods, you should find the first few terms of the series, and if possible find a general formula for the k^{th} coefficient.

Exercise $9.E.7.2.1$

Use power series methods to solve $y''+y=0$ at the point $x_0=1$.

Exercise $9.E.7.2.2$

Use power series methods to solve $y''+4xy=0$ at the point $x_0=0$.

Exercise $9.E.7.2.3$

Use power series methods to solve $y'' - xy = 0$ at the point $x_0 = 1$.

Exercise $9.E.7.2.4$

Use power series methods to solve $y'' + x^2 y = 0$ at the point $x_0 = 0$.

Exercise $9.E.7.2.5$

The methods work for other orders than second order. Try the methods of this section to solve the first order system $y' - xy = 0$ at the point $x_0 = 0$.

Exercise $9.E.7.2.6$

Chebyshev's equation of order p:

a. Solve $(1-x^2)y'' - xy' + p^2 y = 0$ using power series methods at $x_0 = 0$.
b. For what p is there a polynomial solution?

Exercise $9.E.7.2.7$

Find a polynomial solution to $(x^2+1)y'' - 2xy' + 2y = 0$ using power series methods.

Exercise $9.E.7.2.8$

a. Use power series methods to solve $(1-x)y'' + y = 0$ at the point $x_0 = 0$.
b. Use the solution to part a) to find a solution for $xy'' + y = 0$ around the point $x_0 = 1$.

Exercise $9.E.7.2.9$

Use power series methods to solve $y'' + 2x^3 y = 0$ at the point $x_0 = 0$.

Answer

$a_2 = 0$, $a_3 = 0$, $a_4 = 0$, recurrence relation (for $k \geq 5$): $a_k = \frac{-2a_{k-5}}{k(k-1)}$, so $y(x) = a_0 + a_1 x - \frac{a_0}{10}x^5 - \frac{a_1}{15}x^6 + \frac{a_0}{450}x^{10} + \frac{a_1}{825}x^{11} - \frac{a_0}{47250}x^{15} - \frac{a_1}{99000}x^{16} + \cdots$

Exercise $9.E.7.2.10$: (challenging)

We can also use power series methods in nonhomogeneous equations.

a. Use power series methods to solve $y'' - xy = \frac{1}{1-x}$ at the point $x_0 = 0$. Hint: Recall the geometric series.
b. Now solve for the initial condition $y(0) = 0$, $y'(0) = 0$.

Answer

a. $a_2 = \frac{1}{2}$, and for $k \geq 1$ we have $a_k = \frac{a_{k-3}+1}{k(k-1)}$, so
$y(x) = a_0 + a_1 x + \frac{1}{2}x^2 + \frac{a_0+1}{6}x^3 + \frac{a_1+1}{12}x^4 + \frac{3}{40}x^5 + \frac{a_0+2}{30}x^6 + \frac{a_1+2}{42}x^7 + \frac{5}{112}x^8 + \frac{a_0+3}{72}x^9 + \frac{a_1+3}{90}x^{10} + \cdots$
b. $y(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 + \frac{1}{15}x^6 + \frac{1}{21}x^7 + \frac{5}{112}x^8 + \frac{1}{24}x^9 + \frac{1}{30}x^{10} + \cdots$

Exercise $9.E.7.2.11$

Attempt to solve $x^2y'' - y = 0$ at $x_0 = 0$ using the power series method of this section (x_0 is a singular point). Can you find at least one solution? Can you find more than one solution?

Answer

Applying the method of this section directly we obtain $a_k = 0$ for all k and so $y(x) = 0$ is the only solution we find.

9.E.3: 7.3: Singular points and the method of Frobenius

Exercise $9.E.7.3.1$

Find a particular (Frobenius-type) solution of $x^2y'' + xy' + (1+x)y = 0$.

Exercise $9.E.7.3.2$

Find a particular (Frobenius-type) solution of $xy'' - y = 0$.

Exercise $9.E.7.3.3$

Find a particular (Frobenius-type) solution of $y'' + \frac{1}{x}y' - xy = 0$.

Exercise $9.E.7.3.4$

Find the general solution of $2xy'' + y' - x^2y = 0$.

Exercise $9.E.7.3.5$

Find the general solution of $x^2y'' - xy' - y = 0$.

Exercise $9.E.7.3.6$

In the following equations classify the point $x = 0$ as ordinary, regular singular, or singular but not regular singular.

a. $x^2(1+x^2)y'' + xy = 0$
b. $x^2y'' + y' + y = 0$
c. $xy'' + x^3y' + y = 0$
d. $xy'' + xy' - e^xy = 0$
e. $x^2y'' + x^2y' + x^2y = 0$

Exercise $9.E.7.3.7$

In the following equations classify the point $x = 0$ as ordinary, regular singular, or singular but not regular singular.

a. $y'' + y = 0$
b. $x^3y'' + (1+x)y = 0$
c. $xy'' + x^5y' + y = 0$
d. $\sin(x)y'' - y = 0$
e. $\cos(x)y'' - \sin(x)y = 0$

Answer

a. ordinary,
b. singular but not regular singular,
c. regular singular,

d. regular singular,
e. ordinary.

Exercise $9.E.7.3.8$

Find the general solution of $x^2y'' - y = 0$.

Answer

$$y = Ax^{\frac{1+\sqrt{5}}{2}} + Bx^{\frac{1-\sqrt{5}}{2}}$$

Exercise $9.E.7.3.9$

Find a particular solution of $x^2y'' + (x - \frac{3}{4})y = 0$.

Answer

$y = x^{3/2}\sum_{k=0}^{\infty} \frac{(-1)^{-1}}{k!(k+2)!}x^k$ (Note that for convenience we did not pick $a_0 = 1$.)

Exercise $9.E.7.3.10$: (tricky)

Find the general solution of $x^2y'' - xy' + y = 0$.

Answer

$$y = Ax + Bx\ln(x)$$

CHAPTER OVERVIEW

10: Nonlinear Systems

Linear equations suffice in many applications, but in reality most phenomena require nonlinear equations. Nonlinear equations, however, are notoriously more difficult to understand than linear ones, and many strange new phenomena appear when we allow our equations to be nonlinear.

Thumbnail: A double rod pendulum animation showing chaotic behavior. Starting the pendulum from a slightly different initial condition would result in a completely different trajectory. The double rod pendulum is one of the simplest dynamical systems that has chaotic solutions. (Public Domain; Catslash).

Contributors and Attributions

- Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

10.1: Linearization, Critical Points, and Equilibria

Except for a few brief detours in Chapter 1, we considered mostly linear equations. Linear equations suffice in many applications, but in reality most phenomena require nonlinear equations. Nonlinear equations, however, are notoriously more difficult to understand than linear ones, and many strange new phenomena appear when we allow our equations to be nonlinear.

Not to worry, we did not waste all this time studying linear equations. Nonlinear equations can often be approximated by linear ones if we only need a solution "locally," for example, only for a short period of time, or only for certain parameters. Understanding linear equations can also give us qualitative understanding about a more general nonlinear problem. The idea is similar to what you did in calculus in trying to approximate a function by a line with the right slope.

In Section 2.4 we looked at the pendulum of mass m and length L. The goal was to solve for the angle $\theta(t)$ as a function of the time t. The equation for the setup is the nonlinear equation

$$\theta'' + \frac{g}{L}\sin\theta = 0.$$

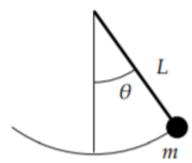


Figure 10.1.1

Instead of solving this equation, we solved the rather easier linear equation

$$\theta'' + \frac{g}{L}\theta = 0.$$

While the solution to the linear equation is not exactly what we were looking for, it is rather close to the original, as long as the angle θ is small and the time period involved is short.

You might ask: Why don't we just solve the nonlinear problem? Well, it might be very difficult, impractical, or impossible to solve analytically,depending on the equation in question. We may not even be interested in the actual solution, we might only be interested in some qualitative idea of what the solution is doing. For example, what happens as time goes to infinity?

10.1.1: Autonomous Systems and Phase Plane Analysis

We restrict our attention to a two dimensional autonomous system

$$x' = f(x,y), \qquad y' = g(x,y),$$

where $f(x,y)$ and $g(x,y)$ are functions of two variables, and the derivatives are taken with respect to time t. Solutions are functions $x(t)$ and $y(t)$ such that

$$x'(t) = f\big(x(t),y(t)\big), \qquad y'(t) = g\big(x(t),y(t)\big).$$

The way we will analyze the system is very similar to Section 1.6, where we studied a single autonomous equation. The ideas in two dimensions are the same, but the behavior can be far more complicated.

It may be best to think of the system of equations as the single vector equation

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}. \tag{10.1.1}$$

As in Section 3.1 we draw the *phase portrait* (or *phase diagram*), where each point (x,y) corresponds to a specific state of the system. We draw the *vector field* given at each point (x,y) by the vector $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$. And as before if we find solutions, we draw the trajectories by plotting all points $\big(x(t),y(t)\big)$ for a certain range of t.

Example 10.1.1

Consider the second order equation $x'' = -x + x^2$. Write this equation as a first order nonlinear system

$$x' = y, \qquad y' = -x + x^2.$$

The phase portrait with some trajectories is drawn in Figure 10.1.2.

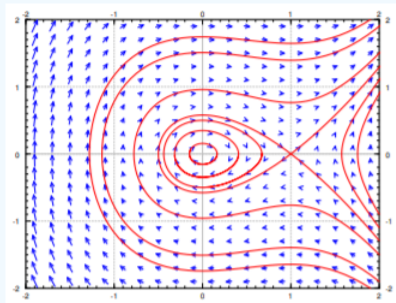


Figure 10.1.2: Phase portrait with some trajectories of $x' = y, y' = -x + x^2$.

From the phase portrait it should be clear that even this simple system has fairly complicated behavior. Some trajectories keep oscillating around the origin, and some go off towards infinity. We will return to this example often, and analyze it completely in this (and the next) section.

If we zoom into the diagram near a point where $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ is not zero, then nearby the arrows point generally in essentially that same direction and have essentially the same magnitude. In other words the behavior is not that interesting near such a point. We are of course assuming that $f(x, y)$ and $g(x, y)$ are continuous.

Let us concentrate on those points in the phase diagram above where the trajectories seem to start, end, or go around. We see two such points: $(0, 0)$ and $(1, 0)$. The trajectories seem to go around the point $(0, 0)$, and they seem to either go in or out of the point $(1, 0)$. These points are precisely those points where the derivatives of both x and y are zero. Let us define the *critical points* as the points (x, y) such that

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \vec{0}.$$

In other words, the points where both $f(x, y) = 0$ and $g(x, y) = 0$.

The critical points are where the behavior of the system is in some sense the most complicated. If $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ is zero, then nearby, the vector can point in any direction whatsoever. Also, the trajectories are either going towards, away from, or around these points, so if we are looking for long term behavior of the system, we should look at what happens there.

Critical points are also sometimes called *equilibria*, since we have so-called *equilibrium solutions* at critical points. If (x_0, y_0) is a critical point, then we have the solutions

$$x(t) = x_0, \quad y(t) = y_0.$$

In Example 10.1.1, there are two equilibrium solutions:

$$x(t) = 0, \quad y(t) = 0, \qquad \text{and} \qquad x(t) = 1, \quad y(t) = 0.$$

Compare this discussion on equilibria to the discussion in Section 1.6. The underlying concept is exactly the same.

10.1.2: Linearization

In Section 3.5 we studied the behavior of a homogeneous linear system of two equations near a critical point. For a linear system of two variables the only critical point is generally the origin $(0, 0)$. Let us put the understanding we gained in that section to good use understanding what happens near critical points of nonlinear systems.

In calculus we learned to estimate a function by taking its derivative and linearizing. We work similarly with nonlinear systems of ODE. Suppose (x_0, y_0) is a critical point. First change variables to (u, v), so that $(u, v) = (0, 0)$ corresponds to (x_0, y_0). That is,

$$u = x - x_0, \qquad v = y - y_0.$$

Next we need to find the derivative. In multivariable calculus you may have seen that the several variables version of the derivative is the *Jacobian matrix*[1]. The Jacobian matrix of the vector-valued function $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ at (x_0, y_0) is

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix}.$$

This matrix gives the best linear approximation as u and v (and therefore x and y) vary. We define the *linearization* of the equation (10.1.1) as the linear system

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Example 10.1.2

Let us keep with the same equations as Example 10.1.1: $x' = y$, $y' = -x + x^2$. There are two critical points, $(0, 0)$ and $(1, 0)$. The Jacobian matrix at any point is

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\ \frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + 2x & 0 \end{bmatrix}.$$

Therefore at $(0, 0)$, we have $u = x$ and $v = y$, and the linearization is

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

where $u = x$ and $v = y$.

At the point $(1, 0)$, we have $u = x - 1$ and $v = y$, and the linearization is

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The phase diagrams of the two linearizations at the point $(0, 0)$ and $(1, 0)$ are given in Figure 10.1.3. Note that the variables are now u and v. Compare Figure 10.1.3 with Figure 10.1.2, and look especially at the behavior near the critical points.

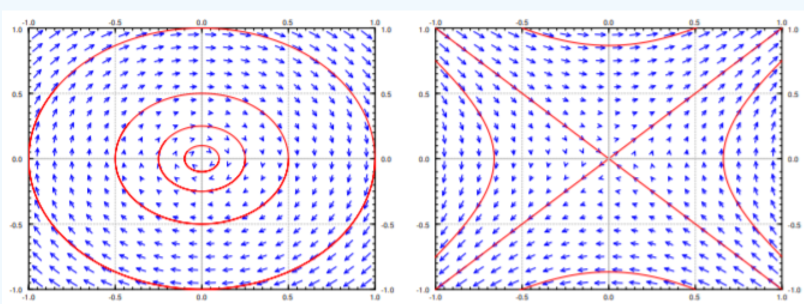


Figure 10.1.3: Phase diagram with some trajectories of linearizations at the critical points $(0, 0)$ (left) and $(1, 0)$ (right) of $x' = y$, $y' = -x + x^2$.

10.1.3: Footnotes

[1] Named for the German mathematician Carl Gustav Jacob Jacobi (1804–1851).

10.2: Stability and Classification of Isolated Critical Points

10.2.1: Isolated Critical Points and Almost Linear Systems

A *critical point* is *isolated* if it is the only critical point in some small "neighborhood" of the point. That is, if we zoom in far enough it is the only critical point we see. In the above example, the critical point was isolated. If on the other hand there would be a whole curve of critical points, then it would not be isolated.

A system is called *almost linear* (at a critical point (x_0, y_0)) if the critical point is isolated and the Jacobian at the point is invertible, or equivalently if the linearized system has an isolated critical point. In such a case, the nonlinear terms will be very small and the system will behave like its linearization, at least if we are close to the critical point.

In particular the system we have just seen in Examples 8.1.1 and 8.1.2 has two isolated critical points $(0,0)$ and $(0,1)$, and is almost linear at both critical points as both of the Jacobian matrices $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are *invertible*.

On the other hand a system such as $x' = x^2$,$y' = y^2$ has an isolated critical point at $(0,0)$, however the Jacobian matrix

$$\begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$$

is zero when $(x,y) = (0,0)$. Therefore the system is not almost linear. Even a worse example is the system $x' = x$,$y' = x^2$,which does not have an isolated critical point, as x' and y' are both zero whenever $x = 0$, that is, the entire y axis.

Fortunately, most often critical points are isolated, and the system is almost linear at the critical points. So if we learn what happens here, we have figured out the majority of situations that arise in applications.

10.2.2: Stability and Classification of Isolated Critical Points

Once we have an isolated critical point, the system is almost linear at that critical point, and we computed the associated linearized system, we can classify what happens to the solutions. We more or less use the classification for linear two-variable systems from Section 3.5, with one minor caveat. Let us list the behaviors depending on the eigenvalues of the Jacobian matrix at the critical point in Table $10.2.1$. This table is very similar to Table 3.5.1, with the exception of missing “center” points. We will discuss centers later, as they are more complicated.

Table 10.2.1: Behavior of an almost linear system near an isolated critical point.

Eigenvalues of the Jacobian matrix	Behavior	Stability
real and both positive	source / unstable node	unstable
real and both negative	sink / stable node	asymptotically stable
real and opposite signs	saddle	unstable
complex with positive real part	spiral source	unstable
complex with negative real part	spiral sink	asymptotically stable

In the new third column, we have marked points as *asymptotically stable* or *unstable*. Formally, a *stable critical point* (x_0, y_0) is one where given any small distance ϵ to (x_0, y_0),and any initial condition within a perhaps smaller radius around (x_0, y_0),the trajectory of the system will never go further away from (x_0, y_0) than ϵ. An *unstable critical point* is one that is not stable. Informally, a point is stable if we start close to a critical point and follow a trajectory we will either go towards, or at least not get away from, this critical point.

A stable critical point (x_0, y_0) is called *asymptotically stable* if given any initial condition sufficiently close to (x_0, y_0) and any solution $\big(x(t), y(t)\big)$ given that condition, then

$$\lim_{t \to \infty} \big(x(t), y(t)\big) = (x_0, y_0).$$

That is, the critical point is asymptotically stable if any trajectory for a sufficiently close initial condition goes towards the critical point (x_0, y_0).

Example 10.2.1

Consider $x' = -y - x^2$, $y' = -x + y^2$. See Figure 10.2.1 for the phase diagram. Let us find the critical points. These are the points where $-y - x^2 = 0$ and $-x + y^2 = 0$. The first equation means $y = -x^2$, and so $y^2 = x^4$. Plugging into the second equation we obtain $-x + x^4 = 0$. Factoring we obtain $x(1 - x^3) = 0$. Since we are looking only for real solutions we get either $x = 0$ or $x = 1$. Solving for the corresponding y using $y = -x^2$,we get two critical points, one being $(0, 0)$ and the other being $(1, -1)$. Clearly the critical points are isolated. Let us compute the Jacobian matrix:

$$\begin{bmatrix} -2x & -1 \\ -1 & 2y \end{bmatrix}.$$

At the point $(0, 0)$ we get the matrix $\left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right]$ and so the two eigenvalues are 1 and -1. As the matrix is invertible, the system is almost linear at $(0, 0)$. As the eigenvalues are real and of opposite signs, we get a saddle point, which is an unstable equilibrium point.

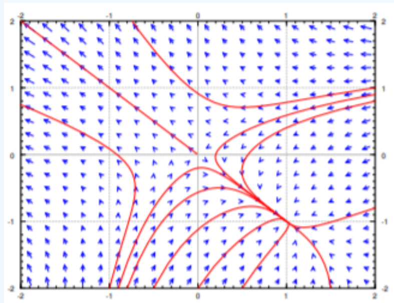


Figure 10.2.1: The phase portrait with few sample trajectories of $x' = -y - x^2$, $y' = -x + y^2$.

At the point $(1, -1)$ we get the matrix $\left[\begin{smallmatrix} -2 & -1 \\ -1 & -2 \end{smallmatrix}\right]$ and computing the eigenvalues we get -1, -3.The matrix is invertible, and so the system is almost linear at $(1, -1)$. As we have real eigenvalues both negative, the critical point is a sink, and therefore an asymptotically stable equilibrium point. That is, if we start with any point (x_i, y_i) close to $(1, -1)$ as an initial condition and plot a trajectory, it will approach $(1, -1)$. In other words,

$$\lim_{t\to\infty} \big(x(t), y(t)\big) = (1, -1).$$

As you can see from the diagram, this behavior is true even for some initial points quite far from $(1, -1)$,but it is definitely not true for all initial points.

Example 10.2.2

Let us look at $x' = y + y^2 e^x$, $y' = x$. First let us find the critical points. These are the points where $y + y^2 e^x = 0$ and $x = 0$. Simplifying we get $0 = y + y^2 = y(y + 1)$. So the critical points are $(0, 0)$ and $(0, -1)$,and hence are isolated. Let us compute the Jacobian matrix:

$$\begin{bmatrix} y^2 e^x & 1 + 2ye^x \\ 1 & 0 \end{bmatrix}.$$

At the point $(0, 0)$ we get the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and so the two eigenvalues are 1 and -1. As the matrix is invertible, the system is almost linear at $(0, 0)$. And, as the eigenvalues are real and of opposite signs, we get a saddle point, which is an unstable equilibrium point.

At the point $(0, -1)$ we get the matrix $\left[\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ whose eigenvalues are $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The matrix is invertible, and so the system is almost linear at $(0, -1)$. As we have complex eigenvalues with positive real part, the critical point is a spiral source, and therefore an unstable equilibrium point.

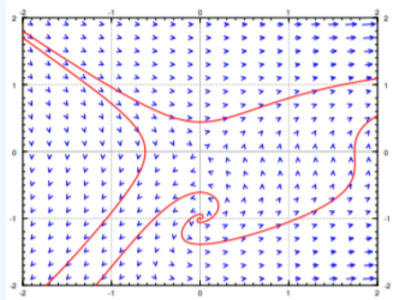


Figure 10.2.2: The phase portrait with few sample trajectories of $x' = y + y^2 e^x$, $y' = x$.

See Figure 10.2.2 for the phase diagram. Notice the two critical points, and the behavior of the arrows in the vector field around these points.

10.2.3: Trouble with Centers

Recall, a linear system with a center meant that trajectories traveled in closed elliptical orbits in some direction around the critical point. Such a critical point we would call a *center* or a *stable center*. It would not be an asymptotically stable critical point, as the trajectories would never approach the critical point, but at least if you start sufficiently close to the critical point, you will stay close to the critical point. The simplest example of such behavior is the linear system with a center. Another example is the critical point $(0, 0)$ in Example 8.1.1.

The trouble with a center in a nonlinear system is that whether the trajectory goes towards or away from the critical point is governed by the sign of the real part of the eigenvalues of the Jacobian. Since this real part is zero at the critical point itself, it can have either sign nearby, meaning the trajectory could be pulled towards or away from the critical point.

Example 10.2.3

An easy example where such a problematic behavior is exhibited is the system $x' = y, y' = -x + y^3$. The only critical point is the origin $(0, 0)$. The Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ -1 & 3y^2 \end{bmatrix}.$$

At $(0, 0)$ the Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,which has eigenvalues $\pm i$. Therefore, the linearization has a center.

Using the quadratic equation, the eigenvalues of the Jacobian matrix at any point (x, y) are

$$\lambda = \frac{3}{2}y^2 \pm i\frac{\sqrt{4 - 9y^4}}{2}.$$

At any point where $y \neq 0$ (so at most points near the origin), the eigenvalues have a positive real part (y^2 can never be negative). This positive real part will pull the trajectory away from the origin. A sample trajectory for an initial condition near the origin is given in Figure 10.2.3.

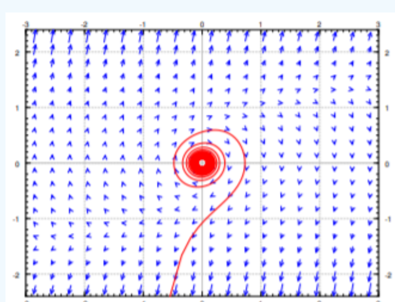


Figure 10.2.3: An unstable critical point (spiral source) at the origin for $x' = y, y' = -x + y^3$, even if the linearization has a center.

The moral of the example is that further analysis is needed when the linearization has a center. The analysis will in general be more complicated than in the above example, and is more likely to involve case-by-case consideration. Such a complication should not be surprising to you. By now in your mathematical career, you have seen many places where a simple test is inconclusive, perhaps starting with the second derivative test for maxima or minima, and requires more careful, and perhaps ad hoc analysis of the situation.

10.2.4: Conservative Equations

An equation of the form

$$x'' + f(x) = 0$$

for an arbitrary function $f(x)$ is called a *conservative equation*. For example the pendulum equation is a conservative equation. The equations are conservative as there is no friction in the system so the energy in the system is "conserved." Let us write this equation as a system of nonlinear ODE.

$$x' = y, \qquad y' = -f(x).$$

These types of equations have the advantage that we can solve for their trajectories easily. The trick is to first think of y as a function of x for a moment. Then use the chain rule

$$x'' = y' = y\frac{dy}{dx},$$

where the prime indicates a derivative with respect to t. We obtain $y\frac{dy}{dx} + f(x) = 0$. We integrate with respect to x to get $\int y\frac{dy}{dx}\,dx + \int f(x)\,dx = C$. In other words

$$\frac{1}{2}y^2 + \int f(x)\,dx = C.$$

We obtained an implicit equation for the trajectories, with different C giving different trajectories. The value of C is conserved on any trajectory. This expression is sometimes called the *Hamiltonian* or the energy of the system. If you look back to Section 1.8, you will notice that $y\frac{dy}{dx} + f(x) = 0$ is an exact equation, and we just found a potential function.

Example 10.2.4

Let us find the trajectories for the equation $x'' + x - x^2 = 0$, which is the equation from Example 8.1.1. The corresponding first order system is

$$x' = y, \qquad y' = -x + x^2.$$

Trajectories satisfy

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3 = C.$$

We solve for y

$$y = \pm\sqrt{-x^2 + \frac{2}{3}x^3 + 2C}.$$

Plotting these graphs we get exactly the trajectories in Figure 8.1.2. In particular we notice that near the origin the trajectories are *closed curves*: they keep going around the origin, never spiraling in or out. Therefore we discovered a way to verify that the critical point at $(0, 0)$ is a stable center. The critical point at $(0, 1)$ is a saddle as we already noticed. This example is typical for conservative equations.

Consider an arbitrary conservative equation. The trajectories are given by

$$y = \pm\sqrt{-2\int f(x)\,dx + 2C}.$$

So all trajectories are mirrored across the x-axis. In particular, there can be no spiral sources nor sinks. All critical points occur when $y = 0$ (the x-axis), that is when $x' = 0$. The critical points are simply those points on the x-axis where $f(x) = 0$. The Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ -f'(x) & 0 \end{bmatrix}.$$

So the critical point is almost linear if $f'(x) \neq 0$ at the critical point. Let J denote the Jacobian matrix, then the eigenvalues of J are solutions to

$$0 = \det(J - \lambda I) = \lambda^2 + f'(x).$$

Therefore $\lambda = \pm\sqrt{-f'(x)}$. In other words, either we get real eigenvalues of opposite signs, or we get purely imaginary eigenvalues. There are only two possibilities for critical points, either an *unstable saddle point*, or a *stable center*. There are never any asymptotically stable points, sinks, or sources.

10.3: Applications of Nonlinear Systems

In this section we study two very standard examples of nonlinear systems. First, we look at the nonlinear pendulum equation. We saw the pendulum equation's linearization before, but we noted it was only valid for small angles and short times. Now we find out what happens for large angles. Next, we look at the predator-prey equation, which finds various applications in modeling problems in biology, chemistry, economics, and elsewhere.

10.3.1: Pendulum

The first example we study is the pendulum equation $\theta'' + \frac{g}{L}\sin\theta = 0$. Here, θ is the angular displacement, g is the gravitational acceleration, and L is the length of the pendulum. In this equation we disregard friction, so we are talking about an idealized pendulum.

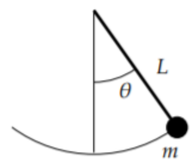


Figure 10.3.1

This equation is a conservative equation, so we can use our analysis of conservative equations from the previous section. Let us change the equation to a two-dimensional system in variables (θ, ω) by introducing the new variable ω:

$$\begin{bmatrix} \theta \\ \omega \end{bmatrix}' = \begin{bmatrix} \omega \\ -\frac{g}{L}\sin\theta \end{bmatrix}.$$

The critical points of this system are when $\omega = 0$ and $-\frac{g}{L}\sin\theta = 0$, or in other words if $\sin\theta = 0$. So the critical points are when $\omega = 0$ and θ is a multiple of π. That is, the points are $\dots(-2\pi,0),(-\pi,0),(0,0),(\pi,0),(2\pi,0)\dots$. While there are infinitely many critical points, they are all isolated. Let us compute the Jacobian matrix:

$$\begin{bmatrix} \frac{\partial}{\partial\theta}\left(\omega\right) & \frac{\partial}{\partial\omega}\left(\omega\right) \\ \frac{\partial}{\partial\theta}\left(-\frac{g}{L}\sin\theta\right) & \frac{\partial}{\partial\omega}\left(-\frac{g}{L}\sin\theta\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos\theta & 0 \end{bmatrix}.$$

For conservative equations, there are two types of critical points. Either stable centers, or saddle points. The eigenvalues of the Jacobian matrix are $\lambda = \pm\sqrt{-\frac{g}{L}\cos\theta}$.

The eigenvalues are going to be real when $\cos\theta < 0$. This happens at the odd multiples of π. The eigenvalues are going to be purely imaginary when $\cos\theta > 0$. This happens at the even multiples of π. Therefore the system has a stable center at the points $\dots(-2\pi,0),(0,0),(2\pi,0)\dots$, and it has an unstable saddle at the points $\dots(-3\pi,0),(-\pi,0),(\pi,0),(3\pi,0)\dots$. Look at the phase diagram in Figure $10.3.2$, where for simplicity we let $\frac{g}{L} = 1$.

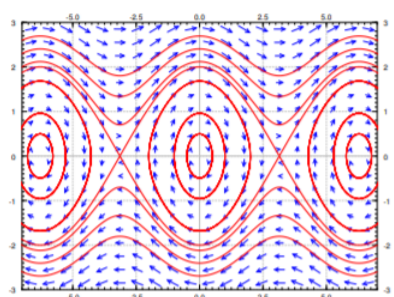


Figure 10.3.2: Phase plane diagram and some trajectories of the nonlinear pendulum equation.

In the linearized equation we have only a single critical point, the center at $(0,0)$. Now we see more clearly what we meant when we said the linearization is good for small angles. The horizontal axis is the deflection angle. The vertical axis is the angular velocity of the pendulum. Suppose we start at $\theta = 0$ (no deflection), and we start with a small angular velocity ω. Then the trajectory keeps going around the critical point $(0,0)$ in an approximate circle. This corresponds to short swings of the pendulum back and forth. When θ stays small, the trajectories really look like circles and hence are very close to our linearization.

When we give the pendulum a big enough push, it goes across the top and keeps spinning about its axis. This behavior corresponds to the wavy curves that do not cross the horizontal axis in the phase diagram. Let us suppose we look at the top curves, when the angular velocity ω is large and positive. Then the pendulum is going around and around its axis. The velocity is going to be large when the pendulum is near the bottom, and the velocity is the smallest when the pendulum is close to the top of its loop.

At each critical point, there is an equilibrium solution. Consider the solution $\theta=0$; the pendulum is not moving and is hanging straight down. This is a stable place for the pendulum to be, hence this is a *stable* equilibrium.

The other type of equilibrium solution is at the unstable point, for example $\theta=\pi$. Here the pendulum is upside down. Sure you can balance the pendulum this way and it will stay, but this is an *unstable* equilibrium. Even the tiniest push will make the pendulum start swinging wildly.

See Figure $10.3.3$ for a diagram. The first picture is the stable equilibrium $\theta=0$. The second picture corresponds to those in the phase diagram around $\theta=0$ when the angular velocity is small. The next picture is the unstable equilibrium $\theta=\pi$. The last picture corresponds to the wavy lines for large angular velocities.

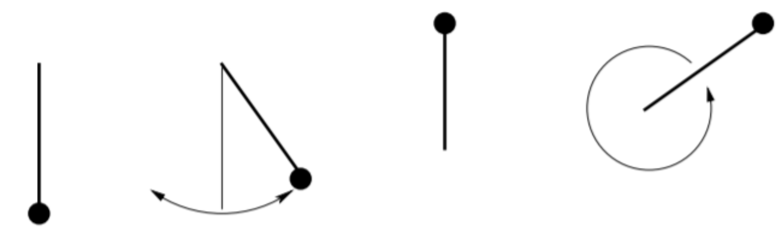


Figure 10.3.3: Various possibilities for the motion of the pendulum.

The quantity

$$\frac{1}{2}\omega^2-\frac{g}{L}\cos\theta$$

is conserved by any solution. This is the energy or the Hamiltonian of the system.

We have a conservative equation and so (exercise) the trajectories are given by

$$\omega=\pm\sqrt{\frac{2g}{L}\cos\theta+C},$$

for various values of C. Let us look at the initial condition of $(\theta_0,0)$, that is, we take the pendulum to angle θ_0, and just let it go (initial angular velocity 0). We plug the initial conditions into the above and solve for C to obtain

$$C=-\frac{2g}{L}\cos\theta_0.$$

Thus the expression for the trajectory is

$$\omega=\pm\sqrt{\frac{2g}{L}}\sqrt{\cos\theta-\cos\theta_0}.$$

Let us figure out the period. That is, the time it takes for the pendulum to swing back and forth. We notice that the oscillation about the origin in the phase plane is symmetric about both the θ and the ω axis. That is, in terms of θ, the time it takes from θ_0 to $-\theta_0$ is the same as it takes from $-\theta_0$ back to θ_0. Furthermore, the time it takes from $-\theta_0$ to 0 is the same as to go from 0 to θ_0. Therefore, let us find how long it takes for the pendulum to go from angle 0 to angle θ_0, which is a quarter of the full oscillation and then multiply by 4.

We figure out this time by finding $\frac{dt}{d\theta}$ and integrating from 0 to ω_0. The period is four times this integral. Let us stay in the region where ω is positive. Since $\omega=\frac{d\theta}{dt}$, inverting we get

$$\frac{dt}{d\theta} = \sqrt{\frac{L}{2g}} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}.$$

Therefore the period T is given by

$$T = 4\sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\, d\theta.$$

The integral is an improper integral, and we cannot in general evaluate it symbolically. We must resort to numerical approximation if we want to compute a particular T.

Recall from Section 2.4, the linearized equation $\theta'' + \frac{g}{L}\theta = 0$ has period

$$T_{\text{linear}} = 2\pi\sqrt{\frac{L}{g}}.$$

We plot T, T_{linear}, and the relative error $\frac{T - T_{\text{linear}}}{T}$ in Figure 10.3.4 The relative error says how far is our approximation from the real period percentage-wise. Note that T_{linear} is simply a constant, it does not change with the initial angle θ_0. The actual period T gets larger and larger as θ_0 gets larger. Notice how the relative error is small when θ_0 is small. It is still only 15% when $\theta_0 = \frac{\pi}{2}$, that is, a 90 degree angle. The error is 3.8% when starting at $\frac{\pi}{4}$, a 45 degree angle. At a 5 degree initial angle, the error is only 0.048%.

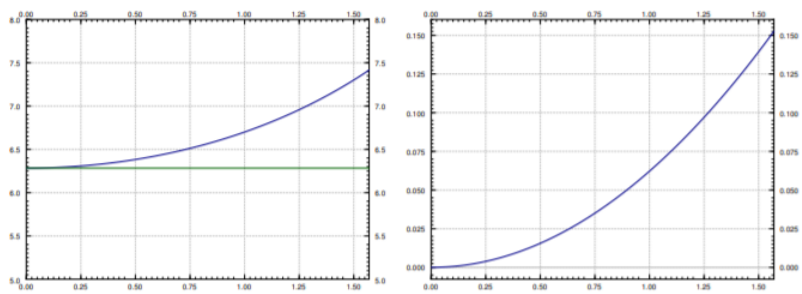


Figure 10.3.4: The plot of T and T_{linear} with $\frac{g}{L} = 1$ (left), and the plot of the relative error $\frac{T - T_{\text{linear}}}{T}$ (right), for θ_0 between 0 and $\frac{\pi}{2}$.

While it is not immediately obvious from the formula, it is true that

$$\lim_{\theta_0 \uparrow \pi} T = \infty.$$

That is, the period goes to infinity as the initial angle approaches the unstable equilibrium point. So if we put the pendulum almost upside down it may take a very long time before it gets down. This is consistent with the limiting behavior, where the exactly upside down pendulum never makes an oscillation, so we could think of that as infinite period.

10.3.2: Predator-Prey or Lotka-Volterra Systems

One of the most common simple applications of nonlinear systems are the so-called *predator-prey* or*Lotka-Volterra*[1] systems. For example, these systems arise when two species interact, one as the prey and one as the predator. It is then no surprise that the equations also see applications in economics. The system also arises in chemical reactions. In biology, this system of equations explains the natural periodic variations of populations of different species in nature. Before the application of differential equations, these periodic variations in the population baffled biologists.

We keep with the classical example of hares and foxes in a forest, it is the easiest to understand.

$$\begin{aligned} x &= \#\text{ of hares (the prey)}, \\ y &= \#\text{ of foxes (the predator)}. \end{aligned} \tag{10.3.1}$$

When there are a lot of hares, there is plenty of food for the foxes, so the fox population grows. However, when the fox population grows, the foxes eat more hares, so when there are lots of foxes, the hare population should go down, and vice versa. The Lotka–Volterra model proposes that this behavior is described by the system of equations

$$x' = (a - by)x,$$
$$y' = (cx - d)y, \tag{10.3.2}$$

where a, b, c, d are some parameters that describe the interaction of the foxes and hares[2]. In this model, these are all positive numbers.

Let us analyze the idea behind this model. The model is a slightly more complicated idea based on the exponential population model. First expand,

$$x' = (a - by)x = ax - byx.$$

The hares are expected to simply grow exponentially in the absence of foxes, that is where the ax term comes in, the growth in population is proportional to the population itself. We are assuming the hares always find enough food and have enough space to reproduce. However, there is another component $-byx$, that is, the population also is decreasing proportionally to the number of foxes. Together we can write the equation as $(a - by)x$, so it is like exponential growth or decay but the constant depends on the number of foxes.

The equation for foxes is very similar, expand again

$$y' = (cx - d)y = cxy - dy.$$

The foxes need food (hares) to reproduce: the more food, the bigger the rate of growth, hence the cxy term. On the other hand, there are natural deaths in the fox population, and hence the $-dy$ term.

Without further delay, let us start with an explicit example. Suppose the equations are

$$x' = (0.4 - 0.01y)x, \qquad y' = (0.003x - 0.3)y.$$

See Figure $10.3.5$ for the phase portrait. In this example it makes sense to also plot x and y as graphs with respect to time. Therefore the second graph in Figure $10.3.5$ is the graph of x and y on the vertical axis (the prey x is the thinner line with taller peaks), against time on the horizontal axis. The particular solution graphed was with initial conditions of 20 foxes and 50 hares.

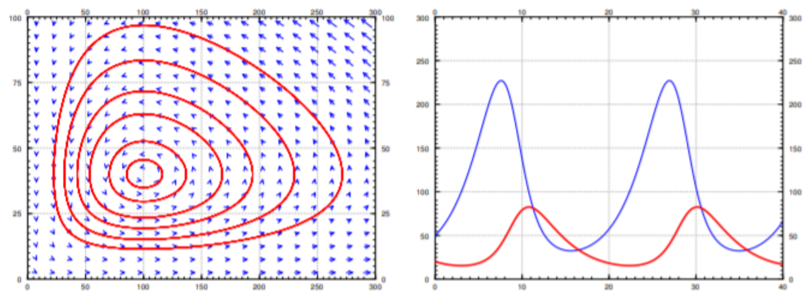


Figure 10.3.5: The phase portrait (left) and graphs of x and y for a sample solution (right).

Let us analyze what we see on the graphs. We work in the general setting rather than putting in specific numbers. We start with finding the critical points. Set $(a - by)x = 0$, and $(cx - d)y = 0$. The first equation is satisfied if either $x = 0$ or $y = \frac{a}{b}$. If $x = 0$, the second equation implies $y = 0$. If $y = \frac{a}{b}$, the second equation implies $x = \frac{d}{c}$. There are two equilibria: at $(0, 0)$ when there are no animals at all, and at $(\frac{d}{c}, \frac{a}{b})$. In our specific example $x = \frac{d}{c} = 100$, and $y = \frac{a}{b} = 40$. This is the point where there are 100 hares and 40 foxes.

We compute the Jacobian matrix:

$$\begin{bmatrix} a - by & -bx \\ cy & cx - d \end{bmatrix}.$$

At the origin $(0, 0)$ we get the matrix $\begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$, so the eigenvalues are a and $-d$, hence real and of opposite signs. So the critical point at the origin is a saddle. This makes sense. If you started with some foxes but no hares, then the foxes would go extinct, that is, you would approach the origin. If you started with no foxes and a few hares, then the hares would keep multiplying without check, and so you would go away from the origin.

OK, how about the other critical point at $(\frac{d}{c}, \frac{a}{b})$. Here the Jacobian matrix becomes

$$\begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}.$$

The eigenvalues satisfy $\lambda^2 + ad = 0$. In other words, $\lambda = \pm i\sqrt{ad}$. The eigenvalues being purely imaginary, we are in the case where we cannot quite decide using only linearization. We could have a stable center, spiral sink, or a spiral source. That is, the equilibrium could be asymptotically stable, stable, or unstable. Of course I gave you a picture above that seems to imply it is a stable center. But never trust a picture only. Perhaps the oscillations are getting larger and larger, but only *very* slowly. Of course this would be bad as it would imply something will go wrong with our population sooner or later. And I only graphed a very specific example with very specific trajectories.

How can we be sure we are in the stable situation? As we said before, in the case of purely imaginary eigenvalues, we have to do a bit more work. Previously we found that for conservative systems, there was a certain quantity that was conserved on the trajectories, and hence the trajectories had to go in closed loops. We can use a similar technique here. We just have to figure out what is the conserved quantity. After some trial and error we find the constant

$$C = \frac{y^a x^d}{e^{cx+by}} = y^a x^d e^{-cx-by}$$

is conserved. Such a quantity is called the *constant of motion*. Let us check C really is a constant of motion. How do we check, you say? Well, a constant is something that does not change with time, so let us compute the derivative with respect to time:

$$C' = ay^{a-1}y'x^d e^{-cx-by} + y^a dx^{d-1}x'e^{-cx-by} + y^a x^d e^{-cx-by}(-cx' - by').$$

Our equations give us what x' and y' are so let us plug those in:

$$\begin{aligned} C' &= ay^{a-1}(cx-d)yx^d e^{-cx-by} + y^a dx^{d-1}(a-by)xe^{-cx-by} \\ &\quad + y^a x^d e^{-cx-by}\left(-c(a-by)x - b(cx-d)y\right) \\ &= y^a x^d e^{-cx-by}\Big(a(cx-d) + d(a-by) + \big(-c(a-by)x - b(cx-d)y\big)\Big) \\ &= 0. \end{aligned} \tag{10.3.3}$$

So along the trajectories C is constant. In fact, the expression $C = \frac{y^a x^d}{e^{cx+by}}$ gives us an implicit equation for the trajectories. In any case, once we have found this constant of motion, it must be true that the trajectories are simple curves, that is, the level curves of $\frac{y^a x^d}{e^{cx+by}}$. It turns out, the critical point at $(\frac{d}{c}, \frac{a}{b})$ is a maximum for C (left as an exercise). So $(\frac{d}{c}, \frac{a}{b})$ is a stable equilibrium point, and we do not have to worry about the foxes and hares going extinct or their populations exploding.

One blemish on this wonderful model is that the number of foxes and hares are discrete quantities and we are modeling with continuous variables. Our model has no problem with there being 0.1 fox in the forest for example, while in reality that makes no sense. The approximation is a reasonable one as long as the number of foxes and hares are large, but it does not make much sense for small numbers. One must be careful in interpreting any results from such a model.

An interesting consequence (perhaps counterintuitive) of this model is that adding animals to the forest might lead to extinction, because the variations will get too big, and one of the populations will get close to zero. For example, suppose there are 20 foxes and 50 hares as before, but now we bring in more foxes, bringing their number to 200. If we run the computation, we find the number of hares will plummet to just slightly more than 1 hare in the whole forest. In reality that most likely means the hares die out, and then the foxes will die out as well as they will have nothing to eat.

Showing that a system of equations has a stable solution can be a very difficult problem. When Isaac Newton put forth his laws of planetary motions, he proved that a single planet orbiting a single sun is a stable system. But any solar system with more than 1 planet proved very difficult indeed. In fact, such a system behaves chaotically (see Section 8.5), meaning small changes in initial conditions lead to very different long-term outcomes. From numerical experimentation and measurements, we know the earth will not fly out into the empty space or crash into the sun, for at least some millions of years or so. But we do not know what happens beyond that.

10.3.3: Footnotes

[1] Named for the American mathematician, chemist, and statistician Alfred James Lotka (1880–1949) and the Italian mathematician and physicist Vito Volterra (1860–1940).

[2] This interaction does not end well for the hare.

10.4: Limit cycles

For nonlinear systems, trajectories do not simply need to approach or leave a single point. They may in fact approach a larger set, such as a circle or another closed curve.

Example 10.4.1: The Van der Pol oscillator

The *Van der Pol oscillator*[1] is the following equation

$$x'' - \mu(1-x^2)x' + x = 0,$$

where μ is some positive constant. The Van der Pol oscillator originated with electrical circuits, but finds applications in diverse fields such as biology, seismology, and other physical sciences.

For simplicity, let us use $\mu = 1$. A phase diagram is given in the left hand plot in Figure 10.4.1. Notice how the trajectories seem to very quickly settle on a closed curve. On the right hand plot we have the plot of a single solution for $t = 0$ to $t = 30$ with initial conditions $x(0) = 0.1$ and $x'(0) = 0.1$. Notice how the solution quickly tends to a periodic solution.

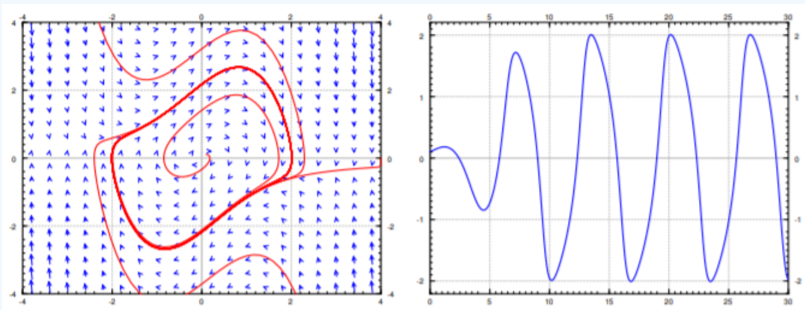


Figure 10.4.1: The phase portrait (left) and a graph of a sample solution of the Van der Pol oscillator.

The Van der Pol oscillator is an example of so-called **relaxation oscillation.** The word relaxation comes from the sudden jump (the very steep part of the solution). For larger μ the steep part becomes even more pronounced, for small μ the limit cycle looks more like a circle. In fact setting $\mu = 0$, we get $x'' + x = 0$, which is a linear system with a center and all trajectories become circles.

The closed curve in the phase portrait above is called a **limit cycle**. A limit cycle is a closed trajectory such that at least one other trajectory spirals into it (or spirals out of it). If all trajectories that start near the limit cycle spiral into it, the limit cycle is called **asymptotically stable**. The limit cycle in the Van der Pol oscillator is asymptotically stable.

Given a limit cycle on an autonomous system, any solution that starts on it is periodic. In fact, this is true for any trajectory that is a closed curve (a so-called *closed trajectory*). Such a curve is called a *periodic orbit*. More precisely, if $\big(x(t), y(t)\big)$ is a solution such that for some t_0 the point $\big(x(t_0), y(t_0)\big)$ lies on a periodic orbit, then both $x(t)$ and $y(t)$ are periodic functions (with the same period). That is, there is some number P such that $x(t) = x(t+P)$ and $y(t) = y(t+P)$.

Consider the system

$$x' = f(x,y), \qquad y' = g(x,y), \tag{10.4.1}$$

where the functions f and g have continuous derivatives in some region R in the plane.

Theorem 10.4.1: Poincarè-Bendixson Theorem

Suppose R is a closed bounded region (a region in the plane that includes its boundary and does not have points arbitrarily far from the origin). Suppose $\big(x(t), y(t)\big)$ is a solution of (10.4.1) in R that exists for all $t \geq t_0$. Then either the solution is a periodic function, or the solution spirals towards a periodic solution in R.

The main point of the theorem is that if you find one solution that exists for all t large enough (that is, as t goes to infinity) and stays within a bounded region, then you have found either a periodic orbit, or a solution that spirals towards a limit cycle or tends to a critical point. That is, in the long term, the behavior is very close to a periodic function. Note that a constant solution at a critical point is periodic (with any period). The theorem is more a qualitative statement rather than something to help us in computations. In practice it is hard to find analytic solutions and so hard to show rigorously that they exist for all time. But if we think the solution exists we numerically solve for a large time to approximate the limit cycle. Another caveat is that the theorem only works in two dimensions. In three dimensions and higher, there is simply too much room.

The theorem applies to all solutions in the Van der Pol oscillator. Solutions that start at any point except the origin $(0, 0)$ will tend to the periodic solution around the limit cycle, and if the initial condition of $(0, 0)$ will lead to the constant solution $x = 0$, $y = 0$.

Example $10.4.2$

Consider

$$x' = y + (x^2 + y^2 - 1)^2 x, \qquad y' = -x + (x^2 + y^2 - 1)^2 y.$$

A vector field along with solutions with initial conditions $(1.02, 0)$, $(0.9, 0)$, and $(0.1, 0)$ are drawn in Figure $10.4.2$.

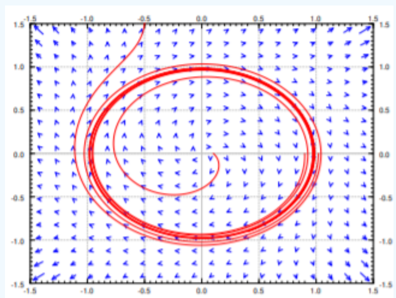


Figure 10.4.2: Unstable limit cycle example.

Notice that points on the unit circle (distance one from the origin) satisfy $x^2 + y^2 - 1 = 0$. And $x(t) = \sin(t)$, $y = \cos(t)$ is a solution of the system. Therefore we have a closed trajectory. For points off the unit circle, the second term in x' pushes the solution further away from the y-axis than the system $x' = y$, $y' = -x$, and y' pushes the solution further away from the x-axis than the linear system $x' = y$, $y' = -x$. In other words for all other initial conditions the trajectory will spiral out.

This means that for initial conditions inside the unit circle, the solution spirals out towards the periodic solution on the unit circle, and for initial conditions outside the unit circle the solutions spiral off towards infinity. Therefore the unit circle is a limit cycle, but not an asymptotically stable one. The Poincaré–Bendixson Theorem applies to the initial points inside the unit circle, as those solutions stay bounded, but not to those outside, as those solutions go off to infinity.

A very similar analysis applies to the system

$$x' = y + (x^2 + y^2 - 1)x, \qquad y' = -x + (x^2 + y^2 - 1)y.$$

We still obtain a closed trajectory on the unit circle, and points outside the unit circle spiral out to infinity, but now points inside the unit circle spiral towards the critical point at the origin. So this system does not have a limit cycle, even though it has a closed trajectory.

Due to the Picard theorem (3.1.1) we find that no matter where we are in the plane we can always find a solution a little bit further in time, as long as f and g have continuous derivatives. So if we find a closed trajectory in an autonomous system, then for every initial point inside the closed trajectory, the solution will exist for all time and it will stay bounded (it will stay inside the closed trajectory). So the moment we found the solution above going around the unit circle, we knew that for every initial point inside the circle, the solution exists for all time and the Poincaré–Bendixson theorem applies.

Let us next look for conditions when limit cycles (or periodic orbits) do not exist. We assume the equation $(10.4.1)$ is defined on a *simply connected region*, that is, a region with no holes we can go around. For example the entire plane is a simply connected region, and so is the inside of the unit disc. However, the entire plane minus a point is not a simply connected domain as it has a at the origin.

Theorem 10.4.2: Bendixson-Dulac Theorem

Suppose f and g are defined in a simply connected region R. If the expression[4]

$$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}$$

is either always positive or always negative on R (except perhaps a small set such as on isolated points or curves) then the system (10.4.1) has no closed trajectory inside R.

The theorem gives us a way of ruling out the existence of a closed trajectory, and hence a way of ruling out limit cycles. The exception about points or lines really means that we can allow the expression to be zero at a few points, or perhaps on a curve, but not on any larger set.

Example 10.4.3

Let us look at $x'=y+y^2e^x$, $y'=x$ in the entire plane (see Example 8.2.2.) The entire plane is simply connected and so we can apply the theorem. We compute $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}=y^2e^x+0$. The function y^2e^x is always positive except on the line $y=0$. Therefore, via the theorem, the system has no closed trajectories.

In some books (or the internet) the theorem is not stated carefully and it concludes there are no periodic solutions. That is not quite right. The above example has two critical points and hence it has constant solutions, and constant functions are periodic. The conclusion of the theorem should be that there exist no trajectories that form closed curves. Another way to state the conclusion of the theorem would be to say that there exist no nonconstant periodic solutions that stay in R.

Example 10.4.4

Let us look at a somewhat more complicated example. Take the system $x'=-y-x^2$, $y'=-x+y^2$ (see Example 8.2.1). We compute $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}=2x+2y$. This expression takes on both signs, so if we are talking about the whole plane we cannot simply apply the theorem. However, we could apply it on the set where $x+y>0$. Via the theorem, there is no closed trajectory in that set. Similarly, there is no closed trajectory in the set $x+y<0$. We cannot conclude (yet) that there is no closed trajectory in the entire plane. Perhaps half of it is in the set where $x+y>0$ and the other half is in the set where $x+y<0$.

The key is to look at the set $x+y=0$, or $x=-y$. Let us make a substitution $x=z$ and $y=-z$ (so that $x=-y$). Both equations become $z'=z-z^2$. So any solution of $z'=z-z^2$, gives us a solution $x(t)=z(t)$, $y(t)=-z(t)$. In particular, any solution that starts out on the line $x+y=0$, stays on the line $x+y=0$. In other words, there cannot be a closed trajectory that starts on the set where $x+y>0$ and goes through the set where $x+y<0$, as it would have to pass through $x+y=0$.

Example 10.4.5

Consider $x'=y+(x^2+y^2-1)x$, $y'=-x+(x^2+y^2-1)y$, and consider the region R given by $x^2+y^2>\frac{1}{2}$. That is, R is the region outside a circle of radius $\frac{1}{\sqrt{2}}$ centered at the origin. Then there is a closed trajectory in R, namely $x=\cos(t)$, $y=\sin(t)$. Furthermore,

$$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}=4x^2+4y^2-2,$$

which is always positive on R. So what is going on? The Bendixson–Dulac theorem does not apply since the region R is not simply connected—it has a hole, the circle we cut out!

10.4.1: Footnotes

[1] Named for the Dutch physicist Balthasar van der Pol (1889–1959).

[2] Ivar Otto Bendixson (1861–1935) was a Swedish mathematician.

[3] Henri Dulac (1870–1955) was a French mathematician.

[4] Usually the expression in the Bendixson–Dulac Theorem is $\frac{\partial(\varphi f)}{\partial x} + \frac{\partial(\varphi g)}{\varphi y}$ for some continuously differentiable function φ. For simplicity, let us just consider the case $\varphi = 1$.

10.5: Chaos

You have surely heard the story about the flap of a butterfly wing in the Amazon causing hurricanes in the North Atlantic. In a prior section, we mentioned that a small change in initial conditions of the planets can lead to very different configuration of the planets in the long term. These are examples of chaotic systems. Mathematical chaos is not really chaos, there is precise order behind the scenes. Everything is still deterministic. However a chaotic system is extremely sensitive to initial conditions. This also means even small errors induced via numerical approximation create large errors very quickly, so it is almost impossible to numerically approximate for long times. This is large part of the trouble as chaotic systems cannot be in general solved analytically.

Take the weather for example. As a small change in the initial conditions (the temperature at every point of the atmosphere for example) produces drastically different predictions in relatively short time, we cannot accurately predict weather. This is because we do not actually know the exact initial conditions, we measure temperatures at a few points with some error and then we somehow estimate what is in between. There is no way we can accurately measure the effects of every butterfly wing. Then we will solve numerically introducing new errors. That is why you should not trust weather prediction more than a few days out.

The idea of chaotic behavior was first noticed by Edward Lorenz in the 1960s when trying to model thermally induced air convection (movement). The equations Lorentz was looking at form the relatively simple looking system:

$$x' = -10x + 10y, \quad y' = 28x - y - xz, \quad z' = -\frac{8}{3}z + xy.$$

A small change in the initial conditions yield a very different solution after a reasonably short time.

Figure 10.5.1

A very simple example the reader can experiment with, which displays chaotic behavior, is a double pendulum. The equations that govern this system are somewhat complicated and their derivation is quite tedious, so we will not bother to write them down. The idea is to put a pendulum on the end of another pendulum. If you look at the movement of the bottom mass, the movement will appear chaotic. This type of system is a basis for a whole number of office novelty desk toys. It is very simple to build a version. Take a piece of a string, and tie two heavy nuts at different points of the string; one at the end, and one a bit above. Now give the bottom nut a little push, as long as the swings are not too big and the string stays tight, you have a double pendulum system.

10.5.1: Duffing Equation and Strange Attractors

Let us study the so-called *Duffing equation*:

$$x'' + ax' + bx + cx^3 = C\cos(\omega t).$$

Here a, b, c, C, and ω are constants. You will recognize that except for the cx^3 term, this equation looks like a forced mass-spring system. The cx^3 term comes up when the spring does not exactly obey Hooke's law (which no real-world spring actually does obey exactly). When c is not zero, the equation does not have a nice closed form solution, so we have to resort to numerical solutions as is usual for nonlinear systems. Not all choices of constants and initial conditions will exhibit chaotic behavior. Let us study

$$x'' + 0.05x' + x^3 = 8\cos(t).$$

The equation is not autonomous, so we will not be able to draw the vector field in the phase plane. We can still draw the trajectories however. In Figure $10.5.2$we plot trajectories for t going from 0 to 15, for two very close initial conditions $(2,3)$ and $(2,2.9)$, and also the solutions in the (x,t) space. The two trajectories are close at first, but after a while diverge significantly. This sensitivity to initial conditions is precisely what we mean by the system behaving chaotically.

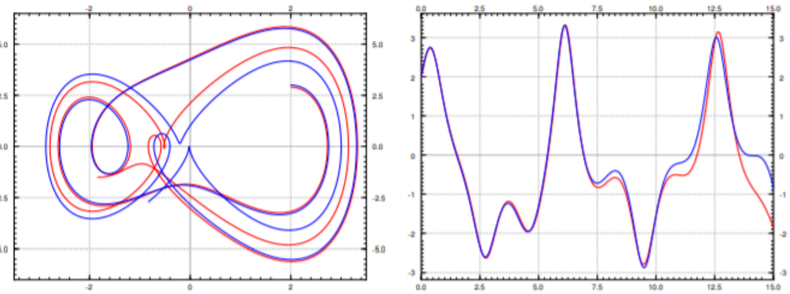


Figure 10.5.2: On left, two trajectories in phase space for $0 \leq t \leq 15$, for the Duffing equation one with initial conditions $(2, 3)$ and the other with $(2, 2.9)$. On right the two solutions in (x, t)-space.

Let us see the long term behavior. In Figure $10.5.3$, we plot the behavior of the system for initial conditions $(2, 3)$, but for much longer period of time. Note that for this period of time it was necessary to use a ridiculously large number of steps[1] in the numerical algorithm used to produce the graph, as even small errors quickly propagate. From the graph it is hard to see any particular pattern in the shape of the solution except that it seems to oscillate, but each oscillation appears quite unique. The oscillation is expected due to the forcing term.

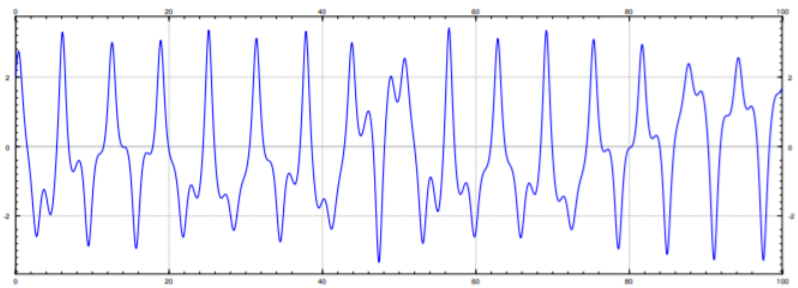


Figure 10.5.3: The solution to the given Duffing equation for t from 0 to 100.

In general it is very difficult to analyze chaotic systems, or to find the order behind the madness, but let us try to do something that we did for the standard mass-spring system. One way we analyzed what happens is that we figured out what was the long term behavior (not dependent on initial conditions). From the figure above it is clear that we will not get a nice description of the long term behavior, but perhaps we can figure out some order to what happens on each "oscillation" and what do these oscillations have in common.

The concept we will explore is that of a *Poincaré section*[2]. Instead of looking at t in a certain interval, we will look at where the system is at a certain sequence of points in time. Imagine flashing a strobe at a certain fixed frequency and drawing the points where the solution is during the flashes. The right strobing frequency depends on the system in question. The correct frequency to use for the forced Duffing equation (and other similar systems) is the frequency of the forcing term. For the Duffing equation above, find a solution $\big(x(t), y(t)\big)$, and look at the points

$$\big(x(0), y(0)\big), \quad \big(x(2\pi), y(2\pi)\big), \quad \big(x(4\pi), y(4\pi)\big), \quad \big(x(6\pi), y(6\pi)\big), \quad \ldots$$

As we are really not interested in the transient part of the solution, that is, the part of the solution that depends on the initial condition we skip some number of steps in the beginning. For example, we might skip the first 100 such steps and start plotting points at $t = 100(2\pi)$, that is

$$\big(x(200\pi), y(200\pi)\big), \quad \big(x(202\pi), y(202\pi)\big), \quad \big(x(204\pi), y(204\pi)\big), \quad \big(x(206\pi), y(206\pi)\big), \quad \ldots$$

The plot of these points is the Poincaré section. After plotting enough points, a curious pattern emerges in Figure $10.5.4$ (the left hand picture), a so-called *strange attractor*.

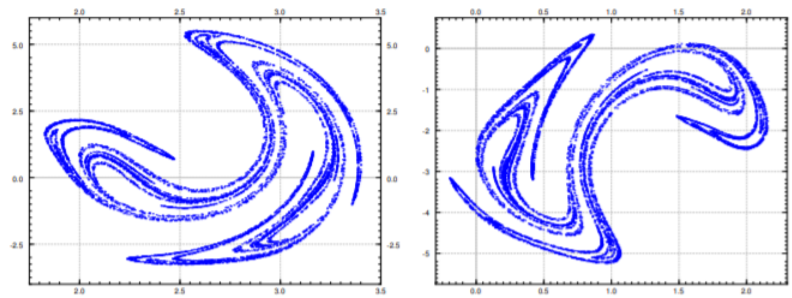


Figure 10.5.4: Strange attractor. The left plot is with no phase shift, the right plot has phase shift $\frac{\pi}{4}$.

If we have a sequence of points, then an *attractor* is a set towards which the points in the sequence eventually get closer and closer to, that is, they are attracted. The Poincaré section above is not really the attractor itself, but as the points are very close to it, we can see its shape. The strange attractor in the figure is a very complicated set, and it in fact has fractal structure, that is, if you would zoom in as far as you want, you would keep seeing the same complicated structure.

The initial condition does not really make any difference. If we started with different initial condition, the points would eventually gravitate towards the attractor, and so as long as we throw away the first few points, we always get the same picture.

An amazing thing is that a chaotic system such as the Duffing equation is not random at all. There is a very complicated order to it, and the strange attractor says something about this order. We cannot quite say what state the system will be in eventually, but given a fixed strobing frequency we can narrow it down to the points on the attractor.

If you would use a phase shift, for example $\frac{\pi}{4}$, and look at the times

$$\frac{\pi}{4}, \quad 2\pi+\frac{\pi}{4}, \quad 4\pi+\frac{\pi}{4}, \quad 6\pi+\frac{\pi}{4}, \quad \ldots$$

you would obtain a slightly different looking attractor. The picture is the right hand side of Figure $10.5.4$ It is as if we had rotated, distorted slightly, and then moved the original. Therefore for each phase shift you can find the set of points towards which the system periodically keeps coming back to.

You should study the pictures and notice especially the scales---where are these attractors located in the phase plane. Notice the regions where the strange attractor lives and compare it to the plot of the trajectories in Figure $10.5.2$

Let us compare the discussion in this section to the discussion in Section 2.6 about forced oscillations. Take the equation

$$x''+2px'+\omega_0^2 x=\frac{F_0}{m}\cos(\omega t).$$

This is like the Duffing equation, but with no x^3 term. The steady periodic solution is of the form

$$x=C\cos(\omega t+\gamma).$$

Strobing using the frequency ω we would obtain a single point in the phase space. So the attractor in this setting is a single point---an expected result as the system is not chaotic. In fact it was the opposite of chaotic. Any difference induced by the initial conditions dies away very quickly, and we settle into always the same steady periodic motion.

10.5.2: Lorenz System

In two dimensions to have the kind of chaotic behavior we are looking for, we have to study forced, or non-autonomous, systems such as the Duffing equation. Due to the Poincaré-Bendoxson Theorem, if an autonomous two-dimensional system has a solution that exists for all time in the future and does not go towards infinity, then we obtain a limit cycle or a closed trajectory. Hardly the chaotic behavior we are looking for.

In three dimensions even autonomous systems can be chaotic. Let us very briefly return to the *Lorenz system*

$$x'=-10x+10y, \qquad y'=28x-y-xz, \qquad z'=-\frac{8}{3}z+xy.$$

The Lorenz system is an autonomous system in three dimensions exhibiting chaotic behavior. See the Figure $10.5.5$ for a sample trajectory, which is now a curve in three-dimensional space.

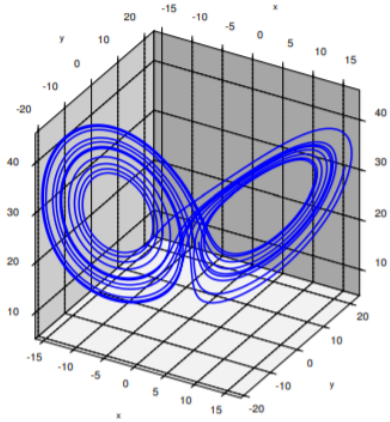


Figure 10.5.5: A trajectory in the Lorenz system.

The solutions will tend to an *attractor* in space, the so-called *Lorenz attractor*. In this case no strobing is necessary. Again we cannot quite see the attractor itself, but if we try to follow a solution for long enough, as in the figure, we will get a pretty good picture of what the attractor looks like. The Lorenz attractor is also a strange attractor and has a complicated fractal structure. And, just as for the Duffing equation, what we want to draw is not the whole trajectory, but start drawing the trajectory after a while, once it is close to the attractor.

The path is not just a repeating figure-eight. The trajectory will spin some seemingly random number of times on the left, then spin a number of times on the right, and so on. As this system arose in weather prediction, one can perhaps imagine a few days of warm weather and then a few days of cold weather, where it is not easy to predict when the weather will change, just as it is not really easy to predict far in advance when the solution will jump onto the other side. See Figure $10.5.6$ for a plot of the x component of the solution drawn above. A negative x corresponds to the left "loop" and a positive x corresponds to the right "loop".

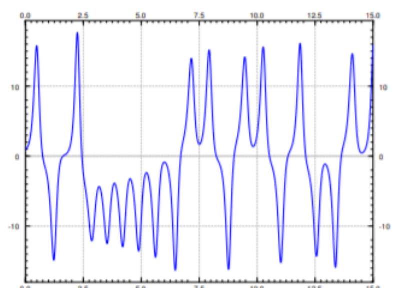


Figure 10.5.6: Graph of the $x(t)$ component of the solution.

Most of the mathematics we studied in this book is quite classical and well understood. On the other hand, chaos, including the Lorenz system, continues to be the subject of current research. Furthermore, chaos has found applications not just in the sciences, but also in art.

10.5.3: Footnotes

[1] In fact for reference, 30,000 steps were used with the Runge–Kutta algorithm, see exercises in Section 1.7.

[2] Named for the French polymath Jules Henri Poincaré (1854-1912).

10.E: Nonlinear Equations (Exercises)

10.E.1: 8.1: Linearization, critical points, and equilibria

Exercise $10.E.8.1.1$

Sketch the phase plane vector field for:

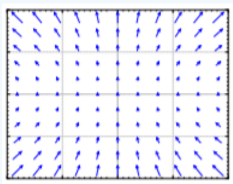
a. $x' = x^2,\ \ y' = y^2$,
b. $x' = (x-y)^2,\ \ y' = -x$,
c. $x' = e^y,\ \ y' = e^x$.

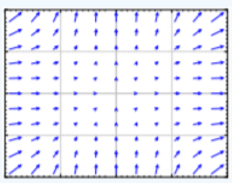
Exercise $10.E.8.1.2$

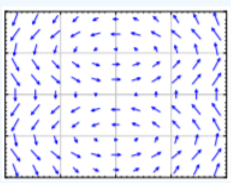
Match systems

1. $x' = x^2$, $y' = y^2$,
2. $x' = xy$, $y' = 1 + y^2$,
3. $x' = \sin(\pi y)$, $y' = x$,

to the vector fields below. Justify.

a. 

b. 

c. 

Exercise $10.E.8.1.3$

Find the critical points and linearizations of the following systems.

a. $x' = x^2 - y^2$, $y' = x^2 + y^2 - 1$,
b. $x' = -y$, $y' = 3x + yx^2$,
c. $x' = x^2 + y$, $y' = y^2 + x$.

Exercise $10.E.8.1.4$

For the following systems, verify they have critical point at $(0,0)$, and find the linearization at $(0,0)$.

a. $x' = x + 2y + x^2 - y^2$, $y' = 2y - x^2$
b. $x' = -y$, $y' = x - y^3$
c. $x' = ax + by + f(x,y)$, $y' = cx + dy + g(x,y)$, where $f(0,0) = 0$, $g(0,0) = 0$, and all first partial derivatives of f and g are also zero at $(0,0)$, that is, $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = \frac{\partial g}{\partial x}(0,0) = \frac{\partial g}{\partial y}(0,0) = 0$.

? Exercise $10.E.8.1.5$

Take $x' = (x-y)^2$, $y' = (x+y)^2$.

a. Find the set of critical points.
b. Sketch a phase diagram and describe the behavior near the critical point(s).
c. Find the linearization. Is it helpful in understanding the system?

? Exercise $10.E.8.1.6$

Take $x' = x^2$, $y' = x^3$.

a. Find the set of critical points.
b. Sketch a phase diagram and describe the behavior near the critical point(s).
c. Find the linearization. Is it helpful in understanding the system?

? Exercise $10.E.8.1.7$

Find the critical points and linearizations of the following systems.

a. $x' = \sin(\pi y) + (x-1)^2$, $y' = y^2 - y$,
b. $x' = x + y + y^2$, $y' = x$,
c. $x' = (x-1)^2 + y$, $y' = x^2 + y$.

Answer

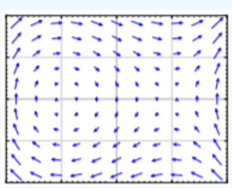
a. Critical points $(0,0)$ and $(0,1)$. At $(0,0)$ using $u = x$, $v = y$ the linearization is $u' = -2u - \left(\frac{1}{\pi}\right) v$, $v' = -v$. At $(0,1)$ using $u = x$, $v = y - 1$ the linearization is $u' = -2u + \left(\frac{1}{\pi}\right) v$ $v' = v$.
b. Critical point $(0,0)$. Using $u = x$, $v = y$ the linearization is $u' = u + v$, $v' = u$.
c. Critical point $\left(\frac{1}{2}, -\frac{1}{4}\right)$. Using $u = x - \frac{1}{2}$, $v = y + \frac{1}{4}$ the linearization is $u' = -u + v$, $v' = u + v$.

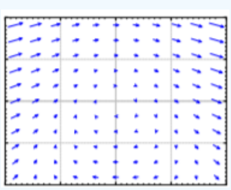
? Exercise $10.E.8.1.8$

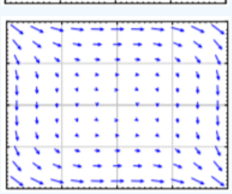
Match systems

a. $x' = y^2$, $y' = -x^2$,
b. $x' = y$, $y' = (x-1)(x+1)$,
c. $x' = y + x^2$, $y' = -x$,

to the vector fields below. Justify.

a. 

b. 

c. 

Answer

a. is c),
b. is a),
c. is b)

Exercise $10.E.8.1.9$

The idea of critical points and linearization works in higher dimensions as well. You simply make the Jacobian matrix bigger by adding more functions and more variables. For the following system of 3 equations find the critical points and their linearizations:

$x' = x + z^2, \quad y' = z^2 - y, \quad z' = z + x^2.$

Answer

Critical points are $(0,0,0)$, and $(-1,1,-1)$. The linearization at the origin using variables $u = x$, $v = y$, $w = z$ is $u' = u$, $v' = -v$, $z' = w$. The linearization at the point $(-1,1,-1)$ using variables $u = x+1$, $v = y-1$, $w = z+1$ is $u' = u - 2$, $v' = -v - 2w$, $w' = w - 2u$.

Exercise $10.E.8.1.10$

Any two-dimensional non-autonomous system $x' = f(x,y,t)$, $y' = g(x,y,t)$ can be written as a three-dimensional autonomous system (three equations). Write down this autonomous system using the variables u, v, w.

Answer

$u' = f(u,v,w)$, $v' = g(u,v,w)$, $w' = 1$.

10.E.2: 8.2: Stability and classification of isolated critical points

Exercise $10.E.8.2.1$

For the systems below, find and classify the critical points, also indicate if the equilibria are stable, asymptotically stable, or unstable.

a. $x' = -x + 3x^2, y' = -y$
b. $x' = x^2 + y^2 - 1$, $y' = x$
c. $x' = ye^x$, $y' = y - x + y^2$

Exercise $10.E.8.2.2$

Find the implicit equations of the trajectories of the following conservative systems. Next find their critical points (if any) and classify them.

a. $x'' + x + x^3 = 0$
b. $\theta'' + \sin\theta = 0$
c. $z'' + (z-1)(z+1) = 0$
d. $x'' + x^2 + 1 = 0$

Exercise $10.E.8.2.3$

Find and classify the critical point(s) of $x' = -x^2$, $y' = -y^2$.

? Exercise $10.E.8.2.4$

Suppose $x' = -xy, y' = x^2 - 1 - y$.

a. Show there are two spiral sinks at $(-1, 0)$ and $(1, 0)$.
b. For any initial point of the form $(0, y_0)$,find what is the trajectory.
c. Can a trajectory starting at (x_0, y_0) where $x_0 > 0$ spiral into the critical point at $(-1, 0)$? Why or why not?

? Exercise $10.E.8.2.5$

In the example $x' = y, y' = y^3 - x$ show that for any trajectory, the distance from the origin is an increasing function. Conclude that the origin behaves like is a spiral source. Hint: Consider $f(t) = \left(x(t)\right)^2 + \left(y(t)\right)^2$ and show it has positive derivative.

? Exercise $10.E.8.2.6$

Suppose f is always positive. Find the trajectories of $x'' + f(x') = 0$. Are there any critical points?

? Exercise $10.E.8.2.7$

Suppose that $x' = f(x, y), y' = g(x, y)$. Suppose that $g(x, y) > 1$ for all x and y. Are there any critical points? What can we say about the trajectories at t goes to infinity?

? Exercise $10.E.8.2.8$

For the systems below, find and classify the critical points.

a. $x' = -x + x^2, y' = y$
b. $x' = y - y^2 - x, y' = -x$
c. $x' = xy, y' = x + y - 1$

Answer

a. $(0, 0)$: saddle (unstable), $(1, 0)$: source (unstable),
b. $(0, 0)$: spiral sink (asymptotically stable), $(0, 1)$: saddle (unstable),
c. $(1, 0)$: saddle (unstable), $(0, 1)$: source (unstable)

? Exercise $10.E.8.2.9$

Find the implicit equations of the trajectories of the following conservative systems. Next find their critical points (if any) and classify them.

a. $x'' + x^2 = 4$
b. $x'' + e^x = 0$
c. $x'' + (x + 1)e^x = 0$

Answer

a. $\frac{1}{2}y^2 + \frac{1}{3}x^3 - 4x = C$, critical points: $(-2, 0)$, an unstable saddle, and $(2, 0)$, a stable center.
b. $\frac{1}{2}y^2 + e^x = C$, no critical points.
c. $\frac{1}{2}y^2 + xe^x = C$, critical point at $(-1, 0)$ is a stable center.

Exercise $10.E.8.2.10$

The conservative system $x''+x^3=0$ is not almost linear. Classify its critical point(s) nonetheless.

Answer

Critical point at $(0,0)$. Trajectories are $y=\pm\sqrt{2C-\left(\frac{1}{2}\right)x^4}$, for $C>0$, these give closed curves around the origin, so the critical point is a stable center.

Exercise $10.E.8.2.11$

Derive an analogous classification of critical points for equations in one dimension, such as $x'=f(x)$ based on the derivative. A point x_0 is critical when $f(x_0)=0$ and almost linear if in addition $f'(x_0)\neq 0$. Figure out if the critical point is stable or unstable depending on the sign of $f'(x_0)$. Explain. Hint: see Section 1.6.

Answer

A critical point x_0 is stable if $f'(x_0)<0$ and unstable when $f'(x_0)<0$.

10.E.3: 8.3: Applications of nonlinear systems

Exercise $10.E.8.3.1$

Take the *damped nonlinear pendulum equation* $\theta''+\mu\theta'+(\frac{g}{L})\sin\theta=0$ for some $\mu>0$ (that is, there is some friction).

a. Suppose $\mu=1$ and $\frac{g}{L}=1$ for simplicity, find and classify the critical points.
b. Do the same for any $\mu>0$ and any g and L, but such that the damping is small, in particular, $\mu^2<4(\frac{g}{L})$.
c. Explain what your findings mean, and if it agrees with what you expect in reality.

Exercise $10.E.8.3.2$

Suppose the hares do not grow exponentially, but logistically. In particular consider

$$x'=(0.4-0.01y)x-\gamma x^2, \quad y'=(0.003x-0.3)y. \tag{10.E.1}$$

For the following two values of γ, find and classify all the critical points in the positive quadrant, that is, for $x\geq 0$ and $y\geq 0$. Then sketch the phase diagram. Discuss the implication for the long term behavior of the population.

a. $\gamma=0.001$,
b. $\gamma=0.01$.

Exercise $10.E.8.3.3$

a. Suppose x and y are positive variables. Show $\frac{yx}{e^{x+y}}$ attains a maximum at $(1,1)$.
b. Suppose a,b,c,d are positive constants, and also suppose x and y are positive variables. Show $\frac{y^a x^d}{e^{cx+by}}$ attains a maximum at $(\frac{d}{c},\frac{a}{b})$.

Exercise $10.E.8.3.4$

Suppose that for the pendulum equation we take a trajectory giving the spinning-around motion, for example $\omega=\sqrt{\frac{2g}{L}\cos\theta+\frac{2g}{L}+\omega_0^2}$. This is the trajectory where the lowest angular velocity is ω_0^2. Find an integral expression for how long it takes the pendulum to go all the way around.

Exercise $10.E.8.3.5$: (challenging)

Take the pendulum, suppose the initial position is $\theta = 0$.

a. Find the expression for ω giving the trajectory with initial condition $(0, \omega_0)$. Hint: Figure out what C should be in terms of ω_0.
b. Find the crucial angular velocity ω_1, such that for any higher initial angular velocity, the pendulum will keep going around its axis, and for any lower initial angular velocity, the pendulum will simply swing back and forth. Hint: When the pendulum doesn't go over the top the expression for ω will be undefined for some θs.
c. What do you think happens if the initial condition is $(0, \omega_1)$, that is, the initial angle is 0, and the initial angular velocity is exactly ω_1.

Exercise $10.E.8.3.6$

Take the damped nonlinear pendulum equation $\theta'' + \mu\theta' + (\frac{g}{L})\sin\theta = 0$ for some $\mu > 0$ (that is, there is friction). Suppose the friction is large, in particular $\mu^2 > 4(\frac{g}{L})$.

a. Find and classify the critical points.
b. Explain what your findings mean, and if it agrees with what you expect in reality.

Answer

a. Critical points are $\omega = 0$, $\theta = k\pi$ for any integer k. When k is odd, we have a saddle point. When k is even we get a sink.
b. The findings mean the pendulum will simply go to one of the sinks, for example $(0, 0)$ and it will not swing back and forth. The friction is too high for it to oscillate, just like an overdamped mass-spring system.

Exercise $10.E.8.3.7$

Suppose we have the system predator-prey system where the foxes are also killed at a constant rate h (h foxes killed per unit time): $x' = (a - by)x,\ y' = (cx - d)y - h$.

a. Find the critical points and the Jacobin matrices of the system.
b. Put in the constants $a = 0.4$, $b = 0.01$, $c = 0.003$, $d = 0.3$, $h = 10$. Analyze the critical points. What do you think it says about the forest?

Answer

a. Solving for the critical points we get $(0, -\frac{h}{d})$ and $\left(\frac{bh+ad}{ac}, \frac{a}{b}\right)$. The Jacobian matrix at $(0, -\frac{h}{d})$ is $\begin{bmatrix} a + \frac{bh}{d} & 0 \\ -\frac{cd}{d} & -d \end{bmatrix}$ whose eigenvalues are $a + \frac{bh}{d}$ and $-d$. The eigenvalues are real of opposite signs and we get a saddle. (In the application, however, we are only looking at the positive quadrant so this critical point is irrelevant.) At $\left(\frac{bh+ad}{ac}, \frac{a}{b}\right)$ we get Jacobian matrix $\begin{bmatrix} 0 & -\frac{b(bh+ad)}{ac} \\ \frac{ac}{b} & \frac{bh+ad}{a} - d \end{bmatrix}$.
b. For the specific numbers given, the second critical point is $\left(\frac{550}{3}, 40\right)$ the matrix is $\begin{bmatrix} 0 & -\frac{11}{6} \\ \frac{3}{25} & \frac{1}{4} \end{bmatrix}$, which has eigenvalues $\frac{5 \pm i\sqrt{327}}{40}$. Therefore there is a spiral source; the solution spirals outwards. The solution eventually hits one of the axes, $x = 0$ or $y = 0$, so something will die out in the forest.

Exercise $10.E.8.3.8$: (challenging)

Suppose the foxes never die. That is, we have the system $x' = (a - by)x, \; y' = cxy$. Find the critical points and notice they are not isolated. What will happen to the population in the forest if it starts at some positive numbers. Hint: Think of the constant of motion.

Answer

The critical points are on the line $x = 0$. In the positive quadrant the y' is always positive and so the fox population always grows. The constant of motion is $C = y^a e^{-cx-by}$, for any C this curve must hit the y-axis (why?), so the trajectory will simply approach a point on the y axis somewhere and the number of hares will go to zero.

10.E.4: 8.4: Limit cycles

Exercise $10.E.8.4.1$

Show that the following systems have no closed trajectories.

a. $x' = x^3 + y, y' = y^3 + x^2$,
b. $x' = e^{x-y}, y' = e^{x+y}$,
c. $x' = x + 3y^2 - y^3, y' = y^3 + x^2$.

Exercise $10.E.8.4.2$

Formulate a condition for a 2-by-2 linear system $\vec{x}\,' = A\vec{x}$ to not be a center using the Bendixson-Dulac theorem. That is, the theorem says something about certain elements of A.

Exercise $10.E.8.4.3$

Explain why the Bendixson-Dulac Theorem does not apply for any conservative system $x'' + h(x) = 0$.

Exercise $10.E.8.4.4$

A system such as $x' = x, y' = y$ has solutions that exist for all time t, yet there are no closed trajectories or other limit cycles. Explain why the Poincare-Bendixson Theorem does not apply.

Exercise $10.E.8.4.5$

Differential equations can also be given in different coordinate systems. Suppose we have the system $r' = 1 - r^2$, $\theta' = 1$ given in polar coordinates. Find all the closed trajectories and check if they are limit cycles and if so, if they are asymptotically stable or not.

Exercise $10.E.8.4.6$

Show that the following systems have no closed trajectories.

a. $x' = x + y^2$, $y' = y + x^2$,
b. $x' = -x\sin^2(y), y' = e^x$,
c. $x' = xy, y' = x + x^2$.

Answer

Use Bendixson–Dulac Theorem.

a. $f_x + g_y = 1 + 1 > 0$, so no closed trajectories.
b. $f_x + g_y = -\sin^2(y) + 0 < 0$ for all x, y except the lines given by $y = k\pi$ (where we get zero), so no closed trajectories.

c. $f_x + g_y = y + 0 > 0$ for all x, y except the line given by $y = 0$ (where we get zero), so no closed trajectories.

Exercise $10.E.8.4.7$

Suppose an autonomous system in the plane has a solution $x = \cos(t) + e^{-t}$, $y = \sin(t) + e^{-t}$. What can you say about the system (in particular about limit cycles and periodic solutions)?

Answer

Using Poincaré–Bendixson Theorem, the system has a limit cycle, which is the unit circle centered at the origin, as $x = \cos(t) + e^{-t}$, $y = \sin(t) + e^{-t}$ gets closer and closer to the unit circle. Thus $x = \cos(t)$, $y = \sin(t)$ is the periodic solution.

Exercise $10.E.8.4.8$

Show that the limit cycle of the Van der Pol oscillator (for $\mu > 0$) must not lie completely in the set where $-\sqrt{\frac{1+\mu}{\mu}} < x < \sqrt{\frac{1+\mu}{\mu}}$. Compare with Figure 8.4.1.

Answer

$f(x, y) = y,\ g(x, y) = \mu(1 - x^2)y - x$. So $f_x + g_y = \mu(1 - x^2)$. The Bendixson–Dulac Theorem says there is no closed trajectory lying entirely in the set $x^2 < 1$.

Exercise $10.E.8.4.9$

Suppose we have the system $r' = \sin(r)$, $\theta' = 1$ given in polar coordinates. Find all the closed trajectories.

Answer

The closed trajectories are those where $\sin(r) = 0$, therefore, all the circles centered at the origin with radius that is a multiple of π are closed trajectories.

10.E.5: 8.5: Chaos

Exercise $10.E.8.5.1$

For the non-chaotic equation $x'' + 2px' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$, suppose we strobe with frequency ω as we mentioned above. Use the known steady periodic solution to find precisely the point which is the attractor for the Poincarè section.

Exercise $10.E.8.5.2$: (project)

A simple fractal attractor can be drawn via the following chaos game. Draw three points of a triangle (just the vertices) and number them, say p_1, p_2 and p_3. Start with some random point p (does not have to be one of the three points above) and draw it. Roll a die, and use it to pick of the p_1, p_2, or p_3 randomly (for example 1 and 4 mean p_1, 2 and 5 mean p_2, and 3 and 6 mean p_3). Suppose we picked p_2, then let p_{new} be the point exactly halfway between p and p_2. Draw this point and let p now refer to this new point p_{new}. Rinse, repeat. Try to be precise and draw as many iterations as possible. Your points should be attracted to the so-called *Sierpinski triangle*. A computer was used to run the game for 10,000 iterations to obtain the picture in Figure $10.E.1$.

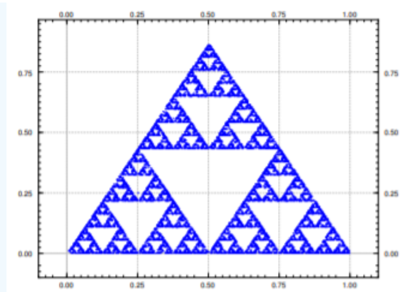


Figure $10.E.1$: $10,000$ iterations of the chaos game producing the Sierpinski triangle.

Exercise $10.E.8.5.3$: (project)

Construct the double pendulum described in the text with a string and two nuts (or heavy beads). Play around with the position of the middle nut, and perhaps use different weight nuts. Describe what you find.

Exercise $10.E.8.5.4$: (computer project)

Use a computer software (such as Matlab, Octave, or perhaps even a spreadsheet), plot the solution of the given forced Duffing equation with Euler's method. Plotting the solution for t from 0 to 100 with several different (small) step sizes. Discuss.

Exercise $10.E.8.5.5$

Find critical points of the Lorenz system and the associated linearizations.

Answer

Critical points: $(0,0,0)$, $(3\sqrt{8},3\sqrt{8},27)$, $(-3\sqrt{8},-3\sqrt{8},27)$. Linearization at $(0,0,0)$ using $u=x$, $v=y$, $w=z$ is $u'=-10u+10v$, $v'=28u-v$, $w'=-\left(\frac{8}{3}\right)w$. Linearization at $(3\sqrt{8},3\sqrt{8},27)$ using $u=x-3\sqrt{8}$, $v=y-3\sqrt{8}$, $w=z-27$ is $u'=-10u+10v$, $v'=u-v-3\sqrt{8}w$, $w'=3\sqrt{8}u+3\sqrt{8}v-\left(\frac{8}{3}\right)w$. Linearization at $(-3\sqrt{8},-3\sqrt{8},27)$ using $u=x+3\sqrt{8}$, $v=y+3\sqrt{8}$, $w=z-27$ is $u'=-10u+10v$, $v'=u-v+3\sqrt{8}w$, $w'=-3\sqrt{8}u-3\sqrt{8}v-\left(\frac{8}{3}\right)w$.

CHAPTER OVERVIEW

11: Appendix A- Linear Algebra

11.1: A.1- Vectors, Mappings, and Matrices

In real life, there is most often more than one variable. We wish to organize dealing with multiple variables in a consistent manner, and in particular organize dealing with linear equations and linear mappings, as those are both rather useful and rather easy to handle. Mathematicians joke that And well, they (the engineers) are not wrong. Quite often, solving an engineering problem is figuring out the right finite-dimensional linear problem to solve, which is then solved with some matrix manipulation. Most importantly, linear problems are the ones that we know how to solve, and we have many tools to solve them. For engineers, mathematicians, physicists, and anybody else in a technical field, it is absolutely vital to learn linear algebra.

As motivation, suppose we wish to solve

$$\begin{aligned} x-y&=2,\\ 2x+y&=4, \end{aligned} \tag{11.1.1}$$

for x and y. That is, we desire numbers x and y such that the two equations are satisfied. Let us perhaps start by adding the equations together to find

$$x+2x-y+y=2+4, \qquad \text{or} \qquad 3x=6.$$

In other words, $x=2$. Once we have that, we plug $x=2$ into the first equation to find $2-y=2$, so $y=0$. OK, that was easy. What is all this fuss about linear equations. Well, try doing this if you have 5000 unknowns[1]. Also, we may have such equations not just of numbers, but of functions and derivatives of functions in differential equations. Clearly we need a systematic way of doing things. A nice consequence of making things systematic and simpler to write down is that it becomes easier to have computers do the work for us. Computers are rather stupid, they do not think, but are very good at doing lots of repetitive tasks precisely, as long as we figure out a systematic way for them to perform the tasks.

11.1.1: Vectors and operations on Vectors

Consider n real numbers as an n-tuple:

$$(x_1, x_2, \ldots, x_n).$$

The set of such n-tuples is the so-called *n-dimensional space*, often denoted by $\mathbb{R}^n$. Sometimes we call this the n-dimensional *euclidean space*[2]. In two dimensions, $\mathbb{R}^2$ is called the *cartesian plane*[3]. Each such n-tuple represents a point in the n-dimensional space. For example, the point $(1,2)$ in the plane $\mathbb{R}^2$ is one unit to the right and two units up from the origin.

When we do algebra with these n-tuples of numbers we call them *vectors*[4]. Mathematicians are keen on separating what is a vector and what is a point of the space or in the plane, and it turns out to be an important distinction, however, for the purposes of linear algebra we can think of everything being represented by a vector. A way to think of a vector, which is especially useful in calculus and differential equations, is an arrow. It is an object that has a *direction* and a *magnitude*. For instance, the vector $(1,2)$ is the arrow from the origin to the point $(1,2)$ in the plane. The magnitude is the length of the arrow. See Figure $11.1.1$. If we think of vectors as arrows, the arrow doesn't always have to start at the origin. If we do move it around, however, it should always keep the same direction and the same magnitude.

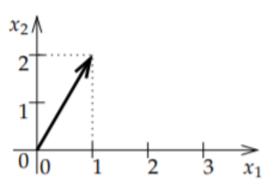


Figure 11.1.1: The vector $(1,2)$ drawn as an arrow from the origin to the point $(1,2)$.

As vectors are arrows, when we want to give a name to a vector, we draw a little arrow above it:

$$\vec{x}$$

Another popular notation is $\mathbf{x}$, although we will use the little arrows. It may be easy to write a bold letter in a book, but it is not so easy to write it by hand on paper or on the board. Mathematicians often don't even write the arrows. A mathematician would write x and just remember that x is a vector and not a number. Just like you remember that Jose is your uncle, and you don't have to keep repeating and you can just say In this book, however, we will call Jose and write vectors with the little arrows.

The *magnitude* can be computed using the Pythagorean theorem. The vector $(1,2)$ drawn in the figure has magnitude $\sqrt{1^2+2^2}=\sqrt{5}$. The magnitude is denoted by $\|\vec{x}\|$, and, in any number of dimensions, it can be computed in the same way:

$$\|\vec{x}\| = \|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}\,.$$

For reasons that will become clear in the next section, we often write vectors as so-called *column vectors*:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Don't worry. It is just a different way of writing the same thing, and it will be useful later. For example, the vector $(1,2)$ can be written as

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The fact that we write arrows above vectors allows us to write several vectors $\vec{x}_1$, $\vec{x}_2$, etc., without confusing these with the components of some other vector $\vec{x}$.

So where is the *algebra* from *linear algebra*? Well, arrows can be added, subtracted, and multiplied by numbers. First we consider *addition*. If we have two arrows, we simply move along one, and then along the other. See Figure $11.1.2$

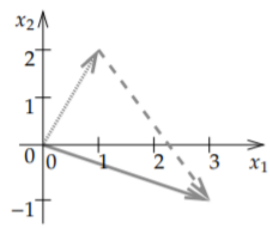


Figure 11.1.2: Adding the vectors $(1,2)$, drawn dotted, and $(2,-3)$, drawn dashed. The result, $(3,-1)$, is drawn as a solid arrow.

It is rather easy to see what it does to the numbers that represent the vectors. Suppose we want to add $(1,2)$ to $(2,-3)$ as in the figure. We travel along $(1,2)$ and then we travel along $(2,-3)$. What we did was travel one unit right, two units up, and then we travelled two units right, and three units down (the negative three). That means that we ended up at $\big(1+2, 2+(-3)\big) = (3,-1)$. And that's how addition always works:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}.$$

Subtracting is similar. What $\vec{x}-\vec{y}$ means visually is that we first travel along $\vec{x}$, and then we travel backwards along $\vec{y}$. See Figure $11.1.3$ It is like adding $\vec{x}+(-\vec{y})$ where $-\vec{y}$ is the arrow we obtain by erasing the arrow head from one side and drawing it on the other side, that is, we reverse the direction. In terms of the numbers, we simply go backwards both horizontally and vertically, so we negate both numbers. For instance, if $\vec{y}$ is $(-2,1)$, then $-\vec{y}$ is $(2,-1)$.

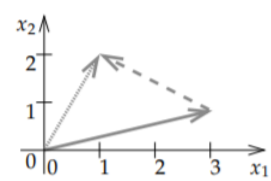


Figure 11.1.3: Subtraction, the vector $(1,2)$, drawn dotted, minus $(-2,1)$, drawn dashed. The result, $(3,1)$, is drawn as a solid arrow.

Another intuitive thing to do to a vector is to *scale* it. We represent this by multiplication of a number with a vector. Because of this, when we wish to distinguish between vectors and numbers, we call the numbers *scalars*. For example, suppose we want to travel three times further. If the vector is $(1,2)$, traveling 3 times further means going 3 units to the right and 6 units up, so we get the vector $(3,6)$. We just multiply each number in the vector by 3. If α is a number, then

$$\alpha\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}\alpha x_1\\\alpha x_2\\\vdots\\\alpha x_n\end{bmatrix}.$$

Scaling (by a positive number) multiplies the magnitude and leaves direction untouched. The magnitude of $(1,2)$ is $\sqrt{5}$. The magnitude of 3 times $(1,2)$, that is, $(3,6)$, is $3\sqrt{5}$.

When the scalar is negative, then when we multiply a vector by it, the vector is not only scaled, but it also switches direction. Multiplying $(1,2)$ by -3 means we should go 3 times further but in the opposite direction, so 3 units to the left and 6 units down, or in other words, $(-3,-6)$. As we mentioned above, $-\vec{y}$ is a reverse of $\vec{y}$, and this is the same as $(-1)\vec{y}$.

In Figure $11.1.4$, you can see a couple of examples of what scaling a vector means visually.

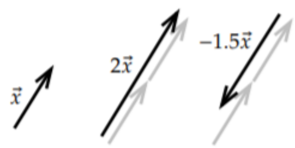


Figure 11.1.4: A vector $\vec{x}$, the vector $2\vec{x}$ (same direction, double the magnitude), and the vector $-1.5\vec{x}$ (opposite direction, 1.5 times the magnitude).

We put all of these operations together to work out more complicated expressions. Let us compute a small example:

$$3\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}-4\\-1\end{bmatrix}-3\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}3(1)+2(-4)-3(-2)\\3(2)+2(-1)-3(2)\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}.$$

As we said a vector is a direction and a magnitude. Magnitude is easy to represent, it is just a number. The *direction* is usually given by a vector with magnitude one. We call such a vector a *unit vector*. That is, $\vec{u}$ is a unit vector when $\|\vec{u}\|=1$. For instance, the vectors $(1,0)$, $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, and $(0,-1)$ are all unit vectors.

To represent the direction of a vector $\vec{x}$, we need to find the unit vector in the same direction. To do so, we simply rescale $\vec{x}$ by the reciprocal of the magnitude, that is $\frac{1}{\|\vec{x}\|}\vec{x}$, or more concisely $\frac{\vec{x}}{\|\vec{x}\|}$.

As an example, the unit vector in the direction of $(1,2)$ is the vector

$$\frac{1}{\sqrt{1^2+2^2}}(1,2)=\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right).$$

11.1.2: Linear Mappings and Matrices

A *vector-valued function* F is a rule that takes a vector $\vec{x}$ and returns another vector $\vec{y}$. For example, F could be a scaling that doubles the size of vectors:

$$F(\vec{x})=2\vec{x}.$$

Applied to say $(1,3)$ we get

$$F\left(\begin{bmatrix}1\\3\end{bmatrix}\right)=2\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}.$$

If F is a mapping that takes vectors in $\mathbb{R}^2$ to $\mathbb{R}^2$ (such as the above), we write

$$F\colon\mathbb{R}^2\to\mathbb{R}^2.$$

The words *function* and *mapping* are used rather interchangeably, although more often than not, *mapping* is used when talking about a vector-valued function, and the word *function* is often used when the function is scalar-valued.

A beginning student of mathematics (and many a seasoned mathematician), that sees an expression such as

$$f(3x+8y)$$

yearns to write

$$3f(x)+8f(y).$$

After all, who hasn't wanted to write $\sqrt{x+y}=\sqrt{x}+\sqrt{y}$ or something like that at some point in their mathematical lives. Wouldn't life be simple if we could do that? Of course we can't always do that (for example, not with the square roots!) But there are many other functions where we can do exactly the above. Such functions are called *linear*.

A mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if

$$F(\vec{x}+\vec{y})=F(\vec{x})+F(\vec{y}),$$

for any vectors $\vec{x}$ and $\vec{y}$, and also

$$F(\alpha\vec{x})=\alpha F(\vec{x}),$$

for any scalar α. The F we defined above that doubles the size of all vectors is linear. Let us check:

$$F(\vec{x}+\vec{y})=2(\vec{x}+\vec{y})=2\vec{x}+2\vec{y}=F(\vec{x})+F(\vec{y}),$$

and also

$$F(\alpha\vec{x})=2\alpha\vec{x}=\alpha 2\vec{x}=\alpha F(\vec{x}).$$

We also call a linear function a *linear transformation*. If you want to be really fancy and impress your friends, you can call it a *linear operator*. When a mapping is linear we often do not write the parentheses. We write simply

$$F\vec{x}$$

instead of $F(\vec{x})$. We do this because linearity means that the mapping F behaves like multiplying $\vec{x}$ by That something is a matrix.

A *matrix* is an $m\times n$ array of numbers (m rows and n columns). A 3×5 matrix is

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}.$$

The numbers a_{ij} are called *elements* or *entries*.

A column vector is simply an $m\times 1$ matrix. Similarly to a column vector there is also a *row vector*, which is a $1\times n$ matrix. If we have an $n\times n$ matrix, then we say that it is a *square matrix*.

Now how does a matrix A relate to a linear mapping? Well a matrix tells you where certain special vectors go. Let's give a name to those certain vectors. The *standard basis of vectors* of $\mathbb{R}^n$ are

$$\vec{e}_1=\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}, \quad \vec{e}_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \quad \vec{e}_3=\begin{bmatrix}0\\0\\1\\\vdots\\0\end{bmatrix}, \quad \cdots, \quad \vec{e}_n=\begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}.$$

In $\mathbb{R}^3$ these vectors are

$$\vec{e}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{e}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \vec{e}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

You may recall from calculus of several variables that these are sometimes called $\vec{\imath}, \vec{\jmath}, \vec{k}$.

The reason these are called a *basis* is that every other vector can be written as a *linear combination* of them. For example, in $\mathbb{R}^3$ the vector $(4,5,6)$ can be written as

$$4\vec{e}_1+5\vec{e}_2+6\vec{e}_3=4\begin{bmatrix}1\\0\\0\end{bmatrix}+5\begin{bmatrix}0\\1\\0\end{bmatrix}+6\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}4\\5\\6\end{bmatrix}.$$

So how does a matrix represent a linear mapping? Well, the columns of the matrix are the vectors where A as a linear mapping takes $\vec{e}_1$, $\vec{e}_2$, etc. For instance, consider

$$M=\begin{bmatrix}1&2\\3&4\end{bmatrix}.$$

As a linear mapping $M:\mathbb{R}^2\to\mathbb{R}^2$ takes $\vec{e}_1=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$ and $\vec{e}_2=\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$. In other words,

$$M\vec{e}_1=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix},\qquad\text{and}\qquad M\vec{e}_2=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}.$$

More generally, if we have an $n\times m$ matrix A, that is, we have n rows and m columns, then the mapping $A:\mathbb{R}^m\to\mathbb{R}^n$ takes $\vec{e}_j$ to the j^{th} column of A. For example,

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}&a_{15}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}\\a_{31}&a_{32}&a_{33}&a_{34}&a_{35}\end{bmatrix}$$

represents a mapping from $\mathbb{R}^5$ to $\mathbb{R}^3$ that does

$$A\vec{e}_1=\begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix},\qquad A\vec{e}_2=\begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix},\qquad A\vec{e}_3=\begin{bmatrix}a_{13}\\a_{23}\\a_{33}\end{bmatrix},\qquad A\vec{e}_4=\begin{bmatrix}a_{14}\\a_{24}\\a_{34}\end{bmatrix},\qquad A\vec{e}_5=\begin{bmatrix}a_{15}\\a_{25}\\a_{35}\end{bmatrix}.$$

What about another vector $\vec{x}$, which isn't in the standard basis? Where does it go? We use linearity. First, we write the vector as a linear combination of the standard basis vectors:

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}+x_5\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}=x_1\vec{e}_1+x_2\vec{e}_2+x_3\vec{e}_3+x_4\vec{e}_4+x_5\vec{e}_5.$$

Then

$$A\vec{x}=A(x_1\vec{e}_1+x_2\vec{e}_2+x_3\vec{e}_3+x_4\vec{e}_4+x_5\vec{e}_5)=x_1A\vec{e}_1+x_2A\vec{e}_2+x_3A\vec{e}_3+x_4A\vec{e}_4+x_5A\vec{e}_5.$$

If we know where A takes all the basis vectors, we know where it takes all vectors.

Suppose M is the 2×2 matrix from above, then

$$M\begin{bmatrix}-2\\0.1\end{bmatrix}=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}-2\\0.1\end{bmatrix}=-2\begin{bmatrix}1\\3\end{bmatrix}+0.1\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}-1.8\\-5.6\end{bmatrix}.$$

Every linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ can be represented by an $n\times m$ matrix. You just figure out where it takes the standard basis vectors. Conversely, every $n\times m$ matrix represents a linear mapping. Hence, we may think of matrices being linear mappings, and linear mappings being matrices.

Or can we? In this book we study mostly linear differential operators, and linear differential operators are linear mappings, although they are not acting on $\mathbb{R}^n$, but on an infinite-dimensional space of functions:

$$Lf=g.$$

For a function f we get a function g, and L is linear in the sense that

$$L(f+h)=Lf+Lh,\qquad\text{and}\qquad L(\alpha f)=\alpha Lf.$$

for any number (scalar) α and all functions f and h.

So the answer is not really. But if we consider vectors in finite-dimensional spaces $\mathbb{R}^n$ then yes, every linear mapping is a matrix. We have mentioned at the beginning of this section, that we can That's not strictly true, but it is true approximately. Those spaces of functions can be approximated by a finite-dimensional space, and then linear operators are just matrices. So approximately, this is true. And as far as actual computations that we can do on a computer, we can work only with finitely many dimensions anyway. If you ask a computer or your calculator to plot a function, it samples the function at finitely many points and then connects the dots[5]. It does not actually give you infinitely many values. The way that you have been using the computer or your calculator so far has already been a certain approximation of the space of functions by a finite-dimensional space.

To end the section, we notice how $A\vec{x}$ can be written more succintly. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad \text{and} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot (-1) \\ 3 \cdot 2 + 4 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

That is, you take the entries in a row of the matrix, you multiply them by the entries in your vector, you add things up, and that's the corresponding entry in the resulting vector.

11.1.3: Footnotes

[1] One of the downsides of making everything look like a linear problem is that the number of variables tends to become huge.

[2] Named after the ancient Greek mathematician Euclid of Alexandria (around 300 BC), possibly the most famous of mathematicians; even small towns often have Euclid Street or Euclid Avenue.

[3] Named after the French mathematician René Descartes (1596–1650). It is as his name in Latin is Renatus Cartesius.

[4] A common notation to distinguish vectors from points is to write $(1,2)$ for the point and $\langle 1,2\rangle$ for the vector. We write both as $(1,2)$.

[5] If you have ever used Matlab, you may have noticed that to plot a function, we take a vector of inputs, ask Matlab to compute the corresponding vector of values of the function, and then we ask it to plot the result.

11.2: A.2- Matrix Algebra

11.2.1: One-by-One Matrices

Let us motivate what we want to achieve with matrices. Real-valued linear mappings of the real line, linear functions that eat numbers and spit out numbers, are just multiplications by a number. Consider a mapping defined by multiplying by a number. Let's call this number α. The mapping then takes x to αx. We can *add* such mappings: If we have another mapping β, then

$$\alpha x+\beta x=(\alpha+\beta)x.$$

We get a new mapping $\alpha+\beta$ that multiplies x by, well, $\alpha+\beta$. If D is a mapping that doubles its input, $Dx=2x$, and T is a mapping that triples, $Tx=3x$, then $D+T$ is a mapping that multiplies by 5, $(D+T)x=5x$.

Similarly we can *compose* such mappings, that is, we could apply one and then the other. We take x, we run it through the first mapping α to get α times x, then we run αx through the second mapping β. In other words,

$$\beta(\alpha x)=(\beta\alpha)x.$$

We just multiply those two numbers. Using our doubling and tripling mappings, if we double and then triple, that is $T(Dx)$ then we obtain $3(2x)=6x$. The composition TD is the mapping that multiplies by 6. For larger matrices, composition also ends up being a kind of multiplication.

11.2.2: Matrix Addition and Scalar Multiplication

The mappings that multiply numbers by numbers are just 1×1 matrices. The number α above could be written as a matrix $[\alpha]$. Perhaps we would want to do to all matrices the same things that we did to those 1×1 matrices at the start of this section above. First, let us add matrices. If we have a matrix A and a matrix B that are of the same size, say $m\times n$, then they are mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. The mapping $A+B$ should also be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, and it should do the following to vectors:

$$(A+B)\vec{x}=A\vec{x}+B\vec{x}.$$

It turns out you just add the matrices element-wise: If the ij^{th} entry of A is a_{ij}, and the ij^{th} entry of B is b_{ij}, then the ij^{th} entry of $A+B$ is $a_{ij}+b_{ij}$. If

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad \text{and} \qquad B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix},$$

then

$$A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}.$$

Let us illustrate on a more concrete example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}+\begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & -1 \end{bmatrix}=\begin{bmatrix} 1+7 & 2+8 \\ 3+9 & 4+10 \\ 5+11 & 6-1 \end{bmatrix}=\begin{bmatrix} 8 & 10 \\ 12 & 14 \\ 16 & 5 \end{bmatrix}.$$

Let's check that this does the right thing to a vector. Let's use some of the vector algebra that we already know, and regroup things:

$$\begin{aligned}\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}+\begin{bmatrix}7&8\\9&10\\11&-1\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix} &= \left(2\begin{bmatrix}1\\3\\5\end{bmatrix}-\begin{bmatrix}2\\4\\6\end{bmatrix}\right)+\left(2\begin{bmatrix}7\\9\\11\end{bmatrix}-\begin{bmatrix}8\\10\\-1\end{bmatrix}\right)\\ &= 2\left(\begin{bmatrix}1\\3\\5\end{bmatrix}+\begin{bmatrix}7\\9\\11\end{bmatrix}\right)-\left(\begin{bmatrix}2\\4\\6\end{bmatrix}+\begin{bmatrix}8\\10\\-1\end{bmatrix}\right)\\ &= 2\begin{bmatrix}1+7\\3+9\\5+11\end{bmatrix}-\begin{bmatrix}2+8\\4+10\\6-1\end{bmatrix}=2\begin{bmatrix}8\\12\\16\end{bmatrix}-\begin{bmatrix}10\\14\\5\end{bmatrix}\\ &= \begin{bmatrix}8&10\\12&14\\16&5\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}\quad\left(=\begin{bmatrix}2(8)-10\\2(12)-14\\2(16)-5\end{bmatrix}=\begin{bmatrix}6\\10\\27\end{bmatrix}\right).\end{aligned}\tag{11.2.1}$$

If we replaced the numbers by letters that would constitute a proof! You'll notice that we didn't really have to even compute what the result is to convince ourselves that the two expressions were equal.

If the sizes of the matrices do not match, then addition is not defined. If A is 3×2 and B is 2×5, then we cannot add these matrices. We don't know what that could possibly mean.

It is also useful to have a matrix that when added to any other matrix does nothing. This is the zero matrix, the matrix of all zeros:

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}1&2\\3&4\end{bmatrix}.$$

We often denote the zero matrix by 0 without specifying size. We would then just write $A+0$, where we just assume that 0 is the zero matrix of the same size as A.

There are really two things we can multiply matrices by. We can multiply matrices by scalars or we can multiply by other matrices. Let us first consider multiplication by scalars. For a matrix A and a scalar α, we want αA to be the matrix that accomplishes

$$(\alpha A)\vec{x}=\alpha(A\vec{x}).$$

That is just scaling the result by α. If you think about it, scaling every term in A by α achieves just that: If

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix},\qquad\text{then}\qquad \alpha A=\begin{bmatrix}\alpha a_{11}&\alpha a_{12}&\alpha a_{13}\\\alpha a_{21}&\alpha a_{22}&\alpha a_{23}\end{bmatrix}.$$

For example,

$$2\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}=\begin{bmatrix}2&4&6\\8&10&12\end{bmatrix}.$$

Let us list some properties of matrix addition and scalar multiplication. Denote by 0 the zero matrix, by α, β scalars, and by A, B, C matrices. Then:

$$\begin{aligned}A+0&=A=0+A,\\A+B&=B+A,\\(A+B)+C&=A+(B+C),\\\alpha(A+B)&=\alpha A+\alpha B,\\(\alpha+\beta)A&=\alpha A+\beta A.\end{aligned}\tag{11.2.2}$$

These rules should look very familiar.

11.2.3: Matrix Multiplication

As we mentioned above, composition of linear mappings is also a multiplication of matrices. Suppose A is an $m\times n$ matrix, that is, A takes $\mathbb{R}^n$ to $\mathbb{R}^m$, and B is an $n\times p$ matrix, that is, B takes $\mathbb{R}^p$ to $\mathbb{R}^n$. The composition AB should work as follows

$$AB\vec{x}=A(B\vec{x}).$$

First, a vector $\vec{x}$ in $\mathbb{R}^p$ gets taken to the vector $B\vec{x}$ in $\mathbb{R}^n$. Then the mapping A takes it to the vector $A(B\vec{x})$ in $\mathbb{R}^m$. In other words, the composition AB should be an $m \times p$ matrix. In terms of sizes we should have

$$\text{"} \quad [m \times n]\,[n \times p] = [m \times p]. \quad \text{"}$$

Notice how the middle size must match.

OK, now we know what sizes of matrices we should be able to multiply, and what the product should be. Let us see how to actually compute matrix multiplication. We start with the so-called *dot product* (or *inner product*) of two vectors. Usually this is a row vector multiplied with a column vector of the same size. Dot product multiplies each pair of entries from the first and the second vector and sums these products. The result is a single number. For example,

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3.$$

And similarly for larger (or smaller) vectors. A dot product is really a product of two matrices: a $1 \times n$ matrix and an $n \times 1$ matrix resulting in a 1×1 matrix, that is, a number.

Armed with the dot product we define the *product of matrices*. We denote by $\text{row}_i(A)$ the i^{th} row of A and by $\text{column}_j(A)$ the j^{th} column of A. For an $m \times n$ matrix A and an $n \times p$ matrix B we can compute the product AB: The matrix AB is an $m \times p$ matrix whose ij^{th} entry is the dot product

$$\text{row}_i(A) \cdot \text{column}_j(B).$$

For example, given a 2×3 and a 3×2 matrix we should end up with a 2×2 matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}, \tag{11.2.3}$$

or with some numbers:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -7 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-1) + 2 \cdot (-7) + 3 \cdot 1 & 1 \cdot 2 + 2 \cdot 0 + 3 \cdot (-1) \\ 4 \cdot (-1) + 5 \cdot (-7) + 6 \cdot 1 & 4 \cdot 2 + 5 \cdot 0 + 6 \cdot (-1) \end{bmatrix} = \begin{bmatrix} -12 & -1 \\ -33 & 2 \end{bmatrix}.$$

A useful consequence of the definition is that the evaluation $A\vec{x}$ for a matrix A and a (column) vector $\vec{x}$ is also matrix multiplication. That is really why we think of vectors as column vectors, or $n \times 1$ matrices. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot (-1) \\ 3 \cdot 2 + 4 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

If you look at the last section, that is precisely the last example we gave.

You should stare at the computation of multiplication of matrices AB and the previous definition of $A\vec{y}$ as a mapping for a moment. What we are doing with matrix multiplication is applying the mapping A to the columns of B. This is usually written as follows. Suppose we write the $n \times p$ matrix $B = [\vec{b}_1 \; \vec{b}_2 \; \cdots \; \vec{b}_p]$, where $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p$ are the columns of B. Then for an $m \times n$ matrix A,

$$AB = A[\vec{b}_1 \; \vec{b}_2 \; \cdots \; \vec{b}_p] = [A\vec{b}_1 \; A\vec{b}_2 \; \cdots \; A\vec{b}_p].$$

The columns of the $m \times p$ matrix AB are the vectors $A\vec{b}_1, A\vec{b}_2, \ldots, A\vec{b}_p$. For example, in (11.2.3), the columns of

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

are

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix}.$$

This is a very useful way to understand what matrix multiplication is. It should also make it easier to remember how to perform matrix multiplication.

11.2.4: Rules of Matrix Algebra

For multiplication we want an analogue of a 1. That is, we desire a matrix that just leaves everything as it found it. This analogue is the so-called *identity matrix*. The identity matrix is a square matrix with 1s on the main diagonal and zeros everywhere else. It is usually denoted by I. For each size we have a different identity matrix and so sometimes we may denote the size as a subscript. For example, I_3 is the 3×3 identity matrix

$$I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us see how the matrix works on a smaller example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} \cdot 1 + a_{12} \cdot 0 & a_{11} \cdot 0 + a_{12} \cdot 1 \\ a_{21} \cdot 1 + a_{22} \cdot 0 & a_{21} \cdot 0 + a_{22} \cdot 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Multiplication by the identity from the left looks similar, and also does not touch anything.

We have the following rules for matrix multiplication. Suppose that A, B, C are matrices of the correct sizes so that the following make sense. Let α denote a scalar (number). Then

$$\begin{aligned} A(BC) &= (AB)C && \text{(associative law)}, \\ A(B+C) &= AB+AC && \text{(distributive law)}, \\ (B+C)A &= BA+CA && \text{(distributive law)}, \\ \alpha(AB) &= (\alpha A)B = A(\alpha B), && \\ IA &= A = AI && \text{(identity)}. \end{aligned} \tag{11.2.4}$$

Example 11.2.1

Let us demonstrate a couple of these rules. For example, the associative law:

$$\underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_{A} \Bigg(\underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} -1 & 4 \\ 5 & 2 \end{bmatrix}}_{C} \Bigg) = \underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 16 & 24 \\ -16 & -2 \end{bmatrix}}_{BC} = \underbrace{\begin{bmatrix} -96 & -78 \\ 64 & 52 \end{bmatrix}}_{A(BC)},$$

and

$$\Bigg(\underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_{B} \Bigg) \underbrace{\begin{bmatrix} -1 & 4 \\ 5 & 2 \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} -9 & -21 \\ 6 & 14 \end{bmatrix}}_{AB} \underbrace{\begin{bmatrix} -1 & 4 \\ 5 & 2 \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} -96 & -78 \\ 64 & 52 \end{bmatrix}}_{(AB)C}.$$

Or how about multiplication by scalars:

$$10 \Bigg(\underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_{B} \Bigg) = 10 \underbrace{\begin{bmatrix} -9 & -21 \\ 6 & 14 \end{bmatrix}}_{AB} = \underbrace{\begin{bmatrix} -90 & -210 \\ 60 & 140 \end{bmatrix}}_{10(AB)}, \tag{11.2.5}$$

$$\Bigg(10 \underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_{A} \Bigg) \underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} -30 & 30 \\ 20 & -20 \end{bmatrix}}_{10A} \underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} -90 & -210 \\ 60 & 140 \end{bmatrix}}_{(10A)B}, \tag{11.2.6}$$

and

$$\underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_{A} \left(10 \underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_{B}\right) = \underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 40 & 40 \\ 10 & -30 \end{bmatrix}}_{10B} = \underbrace{\begin{bmatrix} -90 & -210 \\ 60 & 140 \end{bmatrix}}_{A(10B)}.$$

A multiplication rule, one you have used since primary school on numbers, is quite conspicuously missing for matrices. That is, matrix multiplication is not commutative. Firstly, just because AB makes sense, it may be that BA is not even defined. For example, if A is 2×3, and B is 3×4, the we can multiply AB but not BA.

Even if AB and BA are both defined, does not mean that they are equal. For example, take $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$:

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \qquad \neq \qquad \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = BA.$$

11.2.5: Inverse

A couple of other algebra rules you know for numbers do not quite work on matrices:

i. $AB = AC$ does not necessarily imply $B = C$, even if A is not 0.
ii. $AB = 0$ does not necessarily mean that $A = 0$ or $B = 0$.

For example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

To make these rules hold, we do not just need one of the matrices to not be zero, we would need to by a matrix. This is where the *matrix inverse* comes in. Suppose that A and B are $n \times n$ matrices such that

$$AB = I = BA.$$

Then we call B the inverse of A and we denote B by A^{-1}. Perhaps not surprisingly, $(A^{-1})^{-1} = A$, since if the inverse of A is B, then the inverse of B is A. If the inverse of A exists, then we say A is *invertible*. If A is not invertible, we say A is *singular*.

If $A = [a]$ is a 1×1 matrix, then A^{-1} is $a^{-1} = \frac{1}{a}$. That is where the notation comes from. The computation is not nearly as simple when A is larger.

The proper formulation of the cancellation rule is:

If A is invertible, then $AB = AC$ implies $B = C$.

The computation is what you would do in regular algebra with numbers, but you have to be careful never to commute matrices:

$$\begin{aligned} AB &= AC, \\ A^{-1}AB &= A^{-1}AC, \\ IB &= IC, \\ B &= C. \end{aligned} \tag{11.2.7}$$

And similarly for cancellation on the right:

If A is invertible, then $BA = CA$ implies $B = C$.

The rule says, among other things, that the inverse of a matrix is unique if it exists: If $AB = I = AC$, then A is invertible and $B = C$.

We will see later how to compute an inverse of a matrix in general. For now, let us note that there is a simple formula for the inverse of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For example:

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 4 - 1 \cdot 2} \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}.$$

Let's try it:

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & \frac{-1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 2 & \frac{-1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Just as we cannot divide by every number, not every matrix is invertible. In the case of matrices however we may have singular matrices that are not zero. For example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

is a singular matrix. But didn't we just give a formula for an inverse? Let us try it:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 2 - 1 \cdot 2} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} =?$$

We get into a bit of trouble; we are trying to divide by zero.

So a 2×2 matrix A is invertible whenever

$$ad - bc \neq 0$$

and otherwise it is singular. The expression $ad - bc$ is called the *determinant* and we will look at it more carefully in a later section. There is a similar expression for a square matrix of any size.

11.2.6: Diagonal Matrices

A simple (and surprisingly useful) type of a square matrix is a so-called *diagonal matrix*. It is a matrix whose entries are all zero except those on the main diagonal from top left to bottom right. For example a 4×4 diagonal matrix is of the form

$$\begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}.$$

Such matrices have nice properties when we multiply by them. If we multiply them by a vector, they multiply the k^{th} entry by d_k. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 \\ 2 \cdot 5 \\ 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 18 \end{bmatrix}.$$

Similarly, when they multiply another matrix from the left, they multiply the k^{th} row by d_k. For example,

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix}.$$

On the other hand, multiplying on the right, they multiply the columns:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 2 & 3 & -1 \\ 2 & 3 & -1 \end{bmatrix}.$$

And it is really easy to multiply two diagonal matrices together—we multiply the entries:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 & 0 & 0 \\ 0 & 2 \cdot 3 & 0 \\ 0 & 0 & 3 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

For this last reason, they are easy to invert, you simply invert each diagonal element:

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}^{-1} = \begin{bmatrix} d_1^{-1} & 0 & 0 \\ 0 & d_2^{-1} & 0 \\ 0 & 0 & d_3^{-1} \end{bmatrix}.$$

Let us check an example

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}^{-1}}_{A^{-1}} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}.$$

It is no wonder that the way we solve many problems in linear algebra (and in differential equations) is to try to reduce the problem to the case of diagonal matrices.

11.2.7: Transpose

Vectors do not always have to be column vectors, that is just a convention. Swapping rows and columns is from time to time needed. The operation that swaps rows and columns is the so-called *transpose*. The transpose of A is denoted by A^T. Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Transpose takes an $m \times n$ matrix to an $n \times m$ matrix.

A key feature of the transpose is that if the product AB makes sense, then $B^T A^T$ also makes sense, at least from the point of view of sizes. In fact, we get precisely the transpose of AB. That is:

$$(AB)^T = B^T A^T.$$

For example,

$$\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & -2 \end{bmatrix} \right)^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

It is left to the reader to verify that computing the matrix product on the left and then transposing is the same as computing the matrix product on the right.

If we have a column vector $\vec{x}$ to which we apply a matrix A and we transpose the result, then the row vector $\vec{x}^T$ applies to A^T from the left:

$$(A\vec{x})^T = \vec{x}^T A^T.$$

Another place where transpose is useful is when we wish to apply the dot product[1] to two column vectors:

$$\vec{x} \cdot \vec{y} = \vec{y}^T \vec{x}.$$

That is the way that one often writes the dot product in software.

We say a matrix A is *symmetric* if $A = A^T$. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

is a symmetric matrix. Notice that a symmetric matrix is always square, that is, $n \times n$. Symmetric matrices have many nice properties[2], and come up quite often in applications.

11.2.8: Footnotes

[1] As a side note, mathematicians write $\vec{y}^T \vec{x}$ and physicists write $\vec{x}^T \vec{y}$. Shhh…don't tell anyone, but the physicists are probably right on this.

[2] Although so far we have not learned enough about matrices to really appreciate them.

11.3: A.3- Elimination

11.3.1: Linear Systems of Equations

One application of matrices is to solve systems of linear equations[1]. Consider the following system of linear equations

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + x_2 + 3x_3 &= 5, \qquad (11.3.1) \\ x_1 + 4x_2 + x_3 &= 10. \qquad (11.3.2) \end{aligned}$$

There is a systematic procedure called *elimination* to solve such a system. In this procedure, we attempt to eliminate each variable from all but one equation. We want to end up with equations such as $x_3 = 2$, where we can just read off the answer.

We write a system of linear equations as a matrix equation:

$$A\vec{x} = \vec{b}.$$

The system (11.3.1) is written as

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}}_{\vec{b}}.$$

If we knew the inverse of A, then we would be done; we would simply solve the equation:

$$\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}.$$

Well, but that is part of the problem, we do not know how to compute the inverse for matrices bigger than 2×2. We will see later that to compute the inverse we are really solving $A\vec{x} = \vec{b}$ for several different $\vec{b}$. In other words, we will need to do elimination to find A^{-1}. In addition, we may wish to solve $A\vec{x} = \vec{b}$ if A is not invertible, or perhaps not even square.

Let us return to the equations themselves and see how we can manipulate them. There are a few operations we can perform on the equations that do not change the solution. First, perhaps an operation that may seem stupid, we can swap two equations in (11.3.1):

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 5, \\ 2x_1 + 2x_2 + 2x_3 &= 2, \qquad (11.3.3) \\ x_1 + 4x_2 + x_3 &= 10. \end{aligned}$$

Clearly these new equations have the same solutions x_1, x_2, x_3. A second operation is that we can multiply an equation by a nonzero number. For example, we multiply the third equation in (11.3.1) by 3:

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + x_2 + 3x_3 &= 5, \qquad (11.3.4) \\ 3x_1 + 12x_2 + 3x_3 &= 30. \end{aligned}$$

Finally, we can add a multiple of one equation to another equation. For instance, we add 3 times the third equation in (11.3.1) to the second equation:

$$\begin{aligned} 2x_1 + \quad 2x_2 + \quad 2x_3 &= 2, \\ (1+3)x_1 + (1+12)x_2 + (3+3)x_3 &= 5+30, \qquad (11.3.5) \\ x_1 + \quad 4x_2 + \quad x_3 &= 10. \end{aligned}$$

The same x_1, x_2, x_3 should still be solutions to the new equations. These were just examples; we did not get any closer to the solution. We must to do these three operations in some more logical manner, but it turns out these three operations suffice to solve every linear equation.

The first thing is to write the equations in a more compact manner. Given

$$A\vec{x} = \vec{b},$$

we write down the so-called *augmented matrix*

$$[A \mid \vec{b}\,],$$

where the vertical line is just a marker for us to know where the of the equation starts. For the system (11.3.1) the augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array}\right].$$

The entire process of elimination, which we will describe, is often applied to any sort of matrix, not just an augmented matrix. Simply think of the matrix as the 3×4 matrix

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{bmatrix}.$$

11.3.2: Echelon Form and Elementary Operations

We apply the three operations above to the matrix. We call these the *elementary operations* or *elementary row operations*. Translating the operations to the matrix setting, the operations become:

i. Swap two rows.
ii. Multiply a row by a nonzero number.
iii. Add a multiple of one row to another row.

We run these operations until we get into a state where it is easy to read off the answer, or until we get into a contradiction indicating no solution.

More specifically, we run the operations until we obtain the so-called *row echelon form*. Let us call the first (from the left) nonzero entry in each row the *leading entry*. A matrix is in *row echelon form* if the following conditions are satisfied:

i. The leading entry in any row is strictly to the right of the leading entry of the row above.
ii. Any zero rows are below all the nonzero rows.
iii. All leading entries are 1.

A matrix is in *reduced row echelon form* if furthermore the following condition is satisfied.

iv. All the entries above a leading entry are zero.

Note that the definition applies to matrices of any size.

Example 11.3.1

The following matrices are in row echelon form. The leading entries are marked:

$$\begin{bmatrix} \boxed{1} & 2 & 9 & 3 \\ 0 & 0 & \boxed{1} & 5 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & -1 & -3 \\ 0 & \boxed{1} & 5 \\ 0 & 0 & \boxed{1} \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \boxed{1} & -5 & 2 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

None of the matrices above are in *reduced* row echelon form. For example, in the first matrix none of the entries above the second and third leading entries are zero; they are 9, 3, and 5. The following matrices are in reduced row echelon form. The leading entries are marked:

$$\begin{bmatrix} \boxed{1} & 3 & 0 & 8 \\ 0 & 0 & \boxed{1} & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The procedure we will describe to find a reduced row echelon form of a matrix is called *Gauss-Jordan elimination*. The first part of it, which obtains a row echelon form, is called *Gaussian elimination* or *row reduction*. For some problems, a row echelon form is sufficient, and it is a bit less work to only do this first part.

To attain the row echelon form we work systematically. We go column by column, starting at the first column. We find topmost entry in the first column that is not zero, and we call it the *pivot*. If there is no nonzero entry we move to the next column. We swap rows to put the row with the pivot as the first row. We divide the first row by the pivot to make the pivot entry be a 1. Now look at all the rows below and subtract the correct multiple of the pivot row so that all the entries below the pivot become zero.

After this procedure we forget that we had a first row (it is now fixed), and we forget about the column with the pivot and all the preceding zero columns. Below the pivot row, all the entries in these columns are just zero. Then we focus on the smaller matrix and we repeat the steps above.

It is best shown by example, so let us go back to the example from the beginning of the section. We keep the vertical line in the matrix, even though the procedure works on any matrix, not just an augmented matrix. We start with the first column and we locate the pivot, in this case the first entry of the first column.

$$\left[\begin{array}{ccc|c} \boxed{2} & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array}\right]$$

We multiply the first row by $\frac{1}{2}$.

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array}\right]$$

We subtract the first row from the second and third row (two elementary operations).

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 9 \end{array}\right]$$

We are done with the first column and the first row for now. We almost pretend the matrix doesn't have the first column and the first row.

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & 0 & 2 & 4 \\ * & 3 & 0 & 9 \end{array}\right]$$

OK, look at the second column, and notice that now the pivot is in the third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & \boxed{3} & 0 & 9 \end{array}\right]$$

We swap rows.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{3} & 0 & 9 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

And we divide the pivot row by 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

We do not need to subtract anything as everything below the pivot is already zero. We move on, we again start ignoring the second row and second column and focus on

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & 2 & 4 \end{array}\right].$$

We find the pivot, then divide that row by 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & 4 \end{array}\right] \qquad \rightarrow \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

The matrix is now in row echelon form.

The equation corresponding to the last row is $x_3 = 2$. We know x_3 and we could substitute it into the first two equations to get equations for x_1 and x_2. Then we could do the same thing with x_2, until we solve for all 3 variables. This procedure is called *backsubstitution* and we can achieve it via elementary operations. We start from the lowest pivot (leading entry in the row echelon form) and subtract the right multiple from the row above to make all the entries above this pivot zero. Then we move to the next pivot and so on. After we are done, we will have a matrix in reduced row echelon form.

We continue our example. Subtract the last row from the first to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

The entry above the pivot in the second row is already zero. So we move onto the next pivot, the one in the second row. We subtract this row from the top row to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

The matrix is in reduced row echelon form.

If we now write down the equations for x_1, x_2, x_3, we find

$$x_1 = -4, \qquad x_2 = 3, \qquad x_3 = 2.$$

In other words, we have solved the system.

11.3.3: Non-Unique Solutions and Inconsistent Systems

It is possible that the solution of a linear system of equations is not unique, or that no solution exists. Suppose for a moment that the row echelon form we found was

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Then we have an equation $0 = 1$ coming from the last row. That is impossible and the equations are what we call *inconsistent*. There is no solution to $A\vec{x} = \vec{b}$.

On the other hand, if we find a row echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

then there is no issue with finding solutions. In fact, we will find way too many. Let us continue with backsubstitution (subtracting 3 times the third row from the first) to find the reduced row echelon form and let's mark the pivots.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & -5 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row is all zeros; it just says $0 = 0$ and we ignore it. The two remaining equations are

$$x_1 + 2x_2 = -5, \qquad x_3 = 3.$$

Let us solve for the variables that corresponded to the pivots, that is x_1 and x_3 as there was a pivot in the first column and in the third column:

$$\begin{aligned} x_1 &= -2x_2 - 5, \\ x_3 &= 3. \end{aligned} \tag{11.3.6}$$

The variable x_2 can be anything you wish and we still get a solution. The x_2 is called a *free variable*. There are infinitely many solutions, one for every choice of x_2. If we pick $x_2 = 0$, then $x_1 = -5$, and $x_3 = 3$ give a solution. But we also get a solution by picking say $x_2 = 1$, in which case $x_1 = -9$ and $x_3 = 3$, or by picking $x_2 = -5$ in which case $x_1 = 5$ and $x_3 = 3$.

The general idea is that if any row has all zeros in the columns corresponding to the variables, but a nonzero entry in the column corresponding to the right-hand side $\vec{b}$, then the system is inconsistent and has no solutions. In other words, the system is inconsistent if you find a pivot on the right side of the vertical line drawn in the augmented matrix. Otherwise, the system is *consistent*, and at least one solution exists.

Suppose the system is consistent (at least one solution exists):

i. If every column corresponding to a variable has a pivot element, then the solution is unique.
ii. If there are columns corresponding to variables with no pivot, then those are *free variables* that can be chosen arbitrarily, and there are infinitely many solutions.

When $\vec{b} = \vec{0}$, we have a so-called *homogeneous matrix equation*

$$A\vec{x} = \vec{0}.$$

There is no need to write an augmented matrix in this case. As the elementary operations do not do anything to a zero column, it always stays a zero column. Moreover, $A\vec{x} = \vec{0}$ always has at least one solution, namely $\vec{x} = \vec{0}$. Such a system is always consistent. It may have other solutions: If you find any free variables, then you get infinitely many solutions.

The set of solutions of $A\vec{x} = \vec{0}$ comes up quite often so people give it a name. It is called the *nullspace* or the *kernel* of A. One place where the kernel comes up is invertibility of a square matrix A. If the kernel of A contains a nonzero vector, then it contains infinitely many vectors (there was a free variable). But then it is impossible to invert $\vec{0}$, since infinitely many vectors go to $\vec{0}$, so there is no unique vector that A takes to $\vec{0}$. So if the kernel is nontrivial, that is, if there are any nonzero vectors in the kernel, in other words, if there are any free variables, or in yet other words, if the row echelon form of A has columns without pivots, then A is not invertible. We will return to this idea later.

11.3.4: Linear Independence and Rank

If rows of a matrix correspond to equations, it may be good to find out how many equations we really need to find the same set of solutions. Similarly, if we find a number of solutions to a linear equation $A\vec{x} = \vec{0}$, we may ask if we found enough so that all other solutions can be formed out of the given set. The concept we want is that of linear independence. That same concept is useful for differential equations, for example in Chapter 2.

Given row or column vectors $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n$, a *linear combination* is an expression of the form

$$\alpha_1\vec{y}_1 + \alpha_2\vec{y}_2 + \cdots + \alpha_n\vec{y}_n,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all scalars. For example, $3\vec{y}_1 + \vec{y}_2 - 5\vec{y}_3$ is a linear combination of $\vec{y}_1$, $\vec{y}_2$, and $\vec{y}_3$.

We have seen linear combinations before. The expression

$$A\vec{x}$$

is a linear combination of the columns of A, while

$$\vec{x}^T A = (A^T \vec{x})^T$$

is a linear combination of the rows of A.

The way linear combinations come up in our study of differential equations is similar to the following computation. Suppose that $\vec{x}_1$, $\vec{x}_2$, …, $\vec{x}_n$ are solutions to $A\vec{x}_1 = \vec{0}$, $A\vec{x}_2 = \vec{0}$, …, $A\vec{x}_n = \vec{0}$. Then the linear combination

$$\vec{y} = \alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \cdots + \alpha_n\vec{x}_n$$

is a solution to $A\vec{y} = \vec{0}$:

$$\begin{aligned} A\vec{y} &= A(\alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \cdots + \alpha_n\vec{x}_n) \\ &= \alpha_1 A\vec{x}_1 + \alpha_2 A\vec{x}_2 + \cdots + \alpha_n A\vec{x}_n = \alpha_1\vec{0} + \alpha_2\vec{0} + \cdots + \alpha_n\vec{0} = \vec{0}. \end{aligned} \tag{11.3.7}$$

So if you have found enough solutions, you have them all. The question is, when did we find enough of them?

We say the vectors $\vec{y}_1$, $\vec{y}_2$, …, $\vec{y}_n$ are *linearly independent* if the only solution to

$$\alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \cdots + \alpha_n\vec{x}_n = \vec{0}$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Otherwise, we say the vectors are *linearly dependent*.

For example, the vectors $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ are linearly independent. Let's try:

$$\alpha_1 \begin{bmatrix}1\\2\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\alpha_1\\2\alpha_1+\alpha_2\end{bmatrix} = \vec{0} = \begin{bmatrix}0\\0\end{bmatrix}.$$

So $\alpha_1 = 0$, and then it is clear that $\alpha_2 = 0$ as well. In other words, the two vectors are linearly independent.

If a set of vectors is linearly dependent, that is, some of the α_js are nonzero, then we can solve for one vector in terms of the others. Suppose $\alpha_1 \neq 0$. Since $\alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \cdots + \alpha_n\vec{x}_n = \vec{0}$, then

$$\vec{x}_1 = \frac{-\alpha_2}{\alpha_1}\vec{x}_2 - \frac{-\alpha_3}{\alpha_1}\vec{x}_3 + \cdots + \frac{-\alpha_n}{\alpha_1}\vec{x}_n.$$

For example,

$$2\begin{bmatrix}1\\2\\3\end{bmatrix} - 4\begin{bmatrix}1\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

and so

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

You may have noticed that solving for those α_js is just solving linear equations, and so you may not be surprised that to check if a set of vectors is linearly independent we use row reduction.

Given a set of vectors, we may not be interested in just finding if they are linearly independent or not, we may be interested in finding a linearly independent subset. Or perhaps we may want to find some other vectors that give the same linear combinations and are linearly independent. The way to figure this out is to form a matrix out of our vectors. If we have row vectors we consider them as rows of a matrix. If we have column vectors we consider them columns of a matrix. The set of all linear combinations of a set of vectors is called their *span*.

$$\operatorname{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\} = \{\text{Set of all linear combinations of } \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}.$$

Given a matrix A, the maximal number of linearly independent rows is called the *rank* of A, and we write for the rank. For example,

$$\operatorname{rank}\begin{bmatrix}1&1&1\\2&2&2\\-1&-1&-1\end{bmatrix}=1.$$

The second and third row are multiples of the first one. We cannot choose more than one row and still have a linearly independent set. But what is

$$\operatorname{rank}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \quad = \quad ?$$

That seems to be a tougher question to answer. The first two rows are linearly independent (neither is a multiple of the other), so the rank is at least two. If we would set up the equations for the α_1, α_2, and α_3, we would find a system with infinitely many solutions. One solution is

$$[1 \quad 2 \quad 3]-2[4 \quad 5 \quad 6]+[7 \quad 8 \quad 9]=[0 \quad 0 \quad 0].$$

So the set of all three rows is linearly dependent, the rank cannot be 3. Therefore the rank is 2.

But how can we do this in a more systematic way? We find the row echelon form!

$$\text{Row echelon form of} \quad \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \quad \text{is} \quad \begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix}.$$

The elementary row operations do not change the set of linear combinations of the rows (that was one of the main reasons for defining them as they were). In other words, the span of the rows of the A is the same as the span of the rows of the row echelon form of A. In particular, the number of linearly independent rows is the same. And in the row echelon form, all nonzero rows are linearly independent. This is not hard to see. Consider the two nonzero rows in the example above. Suppose we tried to solve for the α_1 and α_2 in

$$\alpha_1[1 \quad 2 \quad 3]+\alpha_2[0 \quad 1 \quad 2]=[0 \quad 0 \quad 0].$$

Since the first column of the row echelon matrix has zeros except in the first row means that $\alpha_1=0$. For the same reason, α_2 is zero. We only have two nonzero rows, and they are linearly independent, so the rank of the matrix is 2.

The span of the rows is called the *row space*. The row space of A and the row echelon form of A are the same. In the example,

$$\begin{aligned}\text{row space of } \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} &= \operatorname{span}\{[1 \quad 2 \quad 3],[4 \quad 5 \quad 6],[7 \quad 8 \quad 9]\} \\ &= \operatorname{span}\{[1 \quad 2 \quad 3],[0 \quad 1 \quad 2]\}.\end{aligned} \tag{11.3.8}$$

Similarly to row space, the span of columns is called the *column space*.

$$\text{column space of } \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix}1\\4\\7\end{bmatrix},\begin{bmatrix}2\\5\\8\end{bmatrix},\begin{bmatrix}3\\6\\9\end{bmatrix}\right\}.$$

So it may also be good to find the number of linearly independent columns of A. One way to do that is to find the number of linearly independent rows of A^T. It is a tremendously useful fact that the number of linearly independent columns is always the same as the number of linearly independent rows:

Theorem 11.3.1

$\operatorname{rank} A = \operatorname{rank} A^T$

In particular, to find a set of linearly independent columns we need to look at where the pivots were. If you recall above, when solving $A\vec{x}=\vec{0}$ the key was finding the pivots, any non-pivot columns corresponded to free variables. That means we can solve for the non-pivot columns in terms of the pivot columns. Let's see an example. First we reduce some random matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix}.$$

We find a pivot and reduce the rows below:

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & -1 & -2 \\ 3 & 6 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix}.$$

We find the next pivot, make it one, and rinse and repeat:

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{\text{-1}} & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The final matrix is the row echelon form of the matrix. Consider the pivots that we marked. The pivot columns are the first and the third column. All other columns correspond to free variables when solving $A\vec{x} = \vec{0}$, so all other columns can be solved in terms of the first and the third column. In other words

$$\text{column space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \right\}.$$

We could perhaps use another pair of columns to get the same span, but the first and the third are guaranteed to work because they are pivot columns.

The discussion above could be expanded into a proof of the theorem if we wanted. As each nonzero row in the row echelon form contains a pivot, then the rank is the number of pivots, which is the same as the maximal number of linearly independent columns.

The idea also works in reverse. Suppose we have a bunch of column vectors and we just need to find a linearly independent set. For example, suppose we started with the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}.$$

These vectors are not linearly independent as we saw above. In particular, the span of $\vec{v}_1$ and $\vec{v}_3$ is the same as the span of all four of the vectors. So $\vec{v}_2$ and $\vec{v}_4$ can both be written as linear combinations of $\vec{v}_1$ and $\vec{v}_3$. A common thing that comes up in practice is that one gets a set of vectors whose span is the set of solutions of some problem. But perhaps we get way too many vectors, we want to simplify. For example above, all vectors in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ can be written $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \alpha_4\vec{v}_4$ for some numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. But it is also true that every such vector can be written as $a\vec{v}_1 + b\vec{v}_3$ for two numbers a and b. And one has to admit, that looks much simpler. Moreover, these numbers a and b are unique. More on that in the next section.

To find this linearly independent set we simply take our vectors and form the matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$, that is, the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix}.$$

We crank up the row-reduction machine, feed this matrix into it, find the pivot columns, and pick those. In this case, $\vec{v}_1$ and $\vec{v}_3$.

11.3.5: Computing the Inverse

If the matrix A is square and there exists a unique solution $\vec{x}$ to $A\vec{x} = \vec{b}$ for any $\vec{b}$ (there are no free variables), then A is invertible. This is equivalent to the $n \times n$ matrix A being of rank n.

In particular, if $A\vec{x} = \vec{b}$ then $\vec{x} = A^{-1}\vec{b}$. Now we just need to compute what A^{-1} is. We can surely do elimination every time we want to find $A^{-1}\vec{b}$, but that would be ridiculous. The mapping A^{-1} is linear and hence given by a matrix, and we have seen that to figure out the matrix we just need to find where A^{-1} takes the standard basis vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$.

That is, to find the first column of A^{-1}, we solve $A\vec{x} = \vec{e}_1$, because then $A^{-1}\vec{e}_1 = \vec{x}$. To find the second column of A^{-1}, we solve $A\vec{x} = \vec{e}_2$. And so on. It is really just n eliminations that we need to do. But it gets even easier. If you think about it, the elimination is the same for everything on the left side of the augmented matrix. Doing n eliminations separately we would redo most of the computations. Best is to do all at once.

Therefore, to find the inverse of A, we write an $n \times 2n$ augmented matrix $[\,A \mid I\,]$, where I is the identity matrix, whose columns are precisely the standard basis vectors. We then perform row reduction until we arrive at the reduced row echelon form. If A is invertible, then pivots can be found in every column of A, and so the reduced row echelon form of $[\,A \mid I\,]$ looks like $[\,I \mid A^{-1}\,]$. We then just read off the inverse A^{-1}. If you do not find a pivot in every one of the first n columns of the augmented matrix, then A is not invertible.

This is best seen by example. Suppose we wish to invert the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}.$$

We write the augmented matrix and we start reducing:

$$\begin{aligned}
&\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to
&&\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -5 & -2 & 1 & 0 \\ 0 & -5 & -9 & -3 & 0 & 1 \end{array}\right] \to \\
\to &\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & \frac{5}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -5 & -9 & -3 & 0 & 1 \end{array}\right] \to
&&\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & \frac{5}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{-11}{4} & \frac{-1}{2} & \frac{-5}{4} & 1 \end{array}\right] \to \\
\to &\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & \frac{5}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \boxed{1} & \frac{2}{11} & \frac{5}{11} & \frac{-4}{11} \end{array}\right] \to
&&\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & \frac{5}{11} & \frac{-5}{11} & \frac{12}{11} \\ 0 & \boxed{1} & 0 & \frac{3}{11} & \frac{-9}{11} & \frac{5}{11} \\ 0 & 0 & \boxed{1} & \frac{2}{11} & \frac{5}{11} & \frac{-4}{11} \end{array}\right] \to \\
\to &\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & \frac{-1}{11} & \frac{3}{11} & \frac{2}{11} \\ 0 & \boxed{1} & 0 & \frac{3}{11} & \frac{-9}{11} & \frac{5}{11} \\ 0 & 0 & \boxed{1} & \frac{2}{11} & \frac{5}{11} & \frac{-4}{11} \end{array}\right].
\end{aligned} \tag{11.3.9}$$

So

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{-1}{11} & \frac{3}{11} & \frac{2}{11} \\ \frac{3}{11} & \frac{-9}{11} & \frac{5}{11} \\ \frac{2}{11} & \frac{5}{11} & \frac{-4}{11} \end{bmatrix}.$$

Not too terrible, no? Perhaps harder than inverting a 2×2 matrix for which we had a simple formula, but not too bad. Really in practice this is done efficiently by a computer.

11.3.6: Footnotes

[1] Although perhaps we have this backwards, quite often we solve a linear system of equations to find out something about matrices, rather than vice versa.

11.4: A.4- Subspaces, Dimension, and The Kernel

11.4.1: Subspaces, Basis, and Dimension

We often find ourselves looking at the set of solutions of a linear equation $L\vec{x} = \vec{0}$ for some matrix L, that is, we are interested in the kernel of L. The set of all such solutions has a nice structure: It looks and acts a lot like some euclidean space $\mathbb{R}^k$.

We say that a set S of vectors in $\mathbb{R}^n$ is a *subspace* if whenever $\vec{x}$ and $\vec{y}$ are members of S and α is a scalar, then

$$\vec{x} + \vec{y}, \qquad \text{and} \qquad \alpha\vec{x}$$

are also members of S. That is, we can add and multiply by scalars and we still land in S. So every linear combination of vectors of S is still in S. That is really what a subspace is. It is a subset where we can take linear combinations and still end up being in the subset. Consequently the span of a number of vectors is automatically a subspace.

Example 11.4.1

If we let $S = \mathbb{R}^n$, then this S is a subspace of $\mathbb{R}^n$. Adding any two vectors in $\mathbb{R}^n$ gets a vector in $\mathbb{R}^n$, and so does multiplying by scalars.

The set $S' = \{\vec{0}\}$, that is, the set of the zero vector by itself, is also a subspace of $\mathbb{R}^n$. There is only one vector in this subspace, so we only need to verify the definition for that one vector, and everything checks out: $\vec{0} + \vec{0} = \vec{0}$ and $\alpha\vec{0} = \vec{0}$.

The set S'' of all the vectors of the form (a, a) for any real number a, such as $(1, 1)$, $(3, 3)$, or $(-0.5, -0.5)$ is a subspace of $\mathbb{R}^2$. Adding two such vectors, say $(1, 1) + (3, 3) = (4, 4)$ again gets a vector of the same form, and so does multiplying by a scalar, say $8(1, 1) = (8, 8)$.

If S is a subspace and we can find k linearly independent vectors in S

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k,$$

such that every other vector in S is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$, then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is called a *basis* of S. In other words, S is the span of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$. We say that S has dimension k, and we write

$$\dim S = k.$$

Theorem 11.4.1

If $S \subset \mathbb{R}^n$ is a subspace and S is not the trivial subspace $\{\vec{0}\}$, then there exists a unique positive integer k (the dimension) and a (not unique) basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, such that every $\vec{w}$ in S can be uniquely represented by

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_k\vec{v}_k,$$

for some scalars α_1, α_2, ..., α_k.

Just as a vector in $\mathbb{R}^k$ is represented by a k-tuple of numbers, so is a vector in a k-dimensional subspace of $\mathbb{R}^n$ represented by a k-tuple of numbers. At least once we have fixed a basis. A different basis would give a different k-tuple of numbers for the same vector.

We should reiterate that while k is unique (a subspace cannot have two different dimensions), the set of basis vectors is not at all unique. There are lots of different bases for any given subspace. Finding just the right basis for a subspace is a large part of what one does in linear algebra. In fact, that is what we spend a lot of time on in linear differential equations, although at first glance it may not seem like that is what we are doing.

Example 11.4.2

The standard basis

$$\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n,$$

is a basis of $\mathbb{R}^n$, (hence the name). So as expected

$$\dim \mathbb{R}^n = n.$$

On the other hand the subspace $\{\vec{0}\}$ is of dimension 0.

The subspace S'' from Example 11.4.1, that is, the set of vectors (a, a), is of dimension 1. One possible basis is simply $\{(1,1)\}$, the single vector $(1,1)$: every vector in S'' can be represented by $a(1,1) = (a,a)$. Similarly another possible basis would be $\{(-1,-1)\}$. Then the vector (a,a) would be represented as $(-a)(1,1)$.

Row and column spaces of a matrix are also examples of subspaces, as they are given as the span of vectors. We can use what we know about rank, row spaces, and column spaces from the previous section to find a basis.

Example 11.4.3

In the last section, we considered the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix}.$$

Using row reduction to find the pivot columns, we found

$$\text{column space of } A \left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix} \right) = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \right\}.$$

What we did was we found the basis of the column space. The basis has two elements, and so the column space of A is two dimensional. Notice that the rank of A is two.

We would have followed the same procedure if we wanted to find the basis of the subspace X spanned by

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}.$$

We would have simply formed the matrix A with these vectors as columns and repeated the computation above. The subspace X is then the column space of A.

Example 11.4.4

Consider the matrix

$$L = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}.$$

Conveniently, the matrix is in reduced row echelon form. The matrix is of rank 3. The column space is the span of the pivot columns. It is the 3-dimensional space

$$\text{column space of } L = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^3.$$

The row space is the 3-dimensional space

$$\text{row space of } L = \operatorname{span}\{ [1 \quad 2 \quad 0 \quad 0 \quad 3], [0 \quad 0 \quad 1 \quad 0 \quad 4], [0 \quad 0 \quad 0 \quad 1 \quad 5] \}.$$

As these vectors have 5 components, we think of the row space of L as a subspace of $\mathbb{R}^5$.

The way the dimensions worked out in the examples is not an accident. Since the number of vectors that we needed to take was always the same as the number of pivots, and the number of pivots is the rank, we get the following result.

Theorem 11.4.2

Rank

The dimension of the column space and the dimension of the row space of a matrix A are both equal to the rank of A.

11.4.2: Kernel

The set of solutions of a linear equation $L\vec{x} = \vec{0}$, the kernel of L, is a subspace: If $\vec{x}$ and $\vec{y}$ are solutions, then

$$L(\vec{x}+\vec{y}) = L\vec{x}+L\vec{y} = \vec{0}+\vec{0} = \vec{0}, \qquad \text{and} \qquad L(\alpha\vec{x}) = \alpha L\vec{x} = \alpha\vec{0} = \vec{0}.$$

So $\vec{x}+\vec{y}$ and $\alpha\vec{x}$ are solutions. The dimension of the kernel is called the *nullity* of the matrix.

The same sort of idea governs the solutions of linear differential equations. We try to describe the kernel of a linear differential operator, and as it is a subspace, we look for a basis of this kernel. Much of this book is dedicated to finding such bases.

The kernel of a matrix is the same as the kernel of its reduced row echelon form. For a matrix in reduced row echelon form, the kernel is rather easy to find. If a vector $\vec{x}$ is applied to a matrix L, then each entry in $\vec{x}$ corresponds to a column of L, the column that the entry multiplies. To find the kernel, pick a non-pivot column make a vector that has a -1 in the entry corresponding to this non-pivot column and zeros at all the other entries corresponding to the other non-pivot columns. Then for all the entries corresponding to pivot columns make it precisely the value in the corresponding row of the non-pivot column to make the vector be a solution to $L\vec{x} = \vec{0}$. This procedure is best understood by example.

Example 11.4.5

Consider

$$L = \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 5 \end{bmatrix}.$$

This matrix is in reduced row echelon form, the pivots are marked. There are two non-pivot columns, so the kernel has dimension 2, that is, it is the span of 2 vectors. Let us find the first vector. We look at the first non-pivot column, the 2^{nd} column, and we put a -1 in the 2^{nd} entry of our vector. We put a 0 in the 5^{th} entry as the 5^{th} column is also a non-pivot column:

$$\begin{bmatrix} ? \\ -1 \\ ? \\ ? \\ 0 \end{bmatrix}.$$

Let us fill the rest. When this vector hits the first row, we get a -2 and 1 times whatever the first question mark is. So make the first question mark 2. For the second and third rows, it is sufficient to make it the question marks zero. We are really filling in the non-pivot column into the remaining entries. Let us check while marking which numbers went where:

$$\begin{bmatrix} 1 & \boxed{2} & 0 & 0 & 3 \\ 0 & \boxed{0} & 1 & 0 & 4 \\ 0 & \boxed{0} & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} \boxed{2} \\ -1 \\ \boxed{0} \\ \boxed{0} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Yay! How about the second vector. We start with

$$\begin{bmatrix} ? \\ 0 \\ ? \\ ? \\ -1. \end{bmatrix}$$

We set the first question mark to 3, the second to 4, and the third to 5. Let us check, marking things as previously,

$$\begin{bmatrix} 1 & 2 & 0 & 0 & \boxed{3} \\ 0 & 0 & 1 & 0 & \boxed{4} \\ 0 & 0 & 0 & 1 & \boxed{5} \end{bmatrix} \begin{bmatrix} \boxed{3} \\ 0 \\ \boxed{4} \\ \boxed{5} \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

There are two non-pivot columns, so we only need two vectors. We have found the basis of the kernel. So,

$$\text{kernel of } L = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \\ 5 \\ -1 \end{bmatrix} \right\}$$

What we did in finding a basis of the kernel is we expressed all solutions of $L\vec{x} = \vec{0}$ as a linear combination of some given vectors.

The procedure to find the basis of the kernel of a matrix L:

i. Find the reduced row echelon form of L.
ii. Write down the basis of the kernel as above, one vector for each non-pivot column.

The rank of a matrix is the dimension of the column space, and that is the span on the pivot columns, while the kernel is the span of vectors one for each non-pivot column. So the two numbers must add to the number of columns.

Theorem 11.4.3

Rank-Nullity

If a matrix A has n columns, rank r, and nullity k (dimension of the kernel), then

$$n = r + k.$$

The theorem is immensely useful in applications. It allows one to compute the rank r if one knows the nullity k and vice versa, without doing any extra work.

Let us consider an example application, a simple version of the so-called *Fredholm alternative*. A similar result is true for differential equations. Consider

$$A\vec{x} = \vec{b},$$

where A is a square $n \times n$ matrix. There are then two mutually exclusive possibilities:

i. A nonzero solution $\vec{x}$ to $A\vec{x} = \vec{0}$ exists.
ii. The equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for every $\vec{b}$.

How does the Rank–Nullity theorem come into the picture? Well, if A has a nonzero solution $\vec{x}$ to $A\vec{x} = \vec{0}$, then the nullity k is positive. But then the rank $r = n - k$ must be less than n. It means that the column space of A is of dimension less than n, so it is a subspace that does not include everything in $\mathbb{R}^n$. So $\mathbb{R}^n$ has to contain some vector $\vec{b}$ not in the column space of A. In fact, most vectors in $\mathbb{R}^n$ are not in the column space of A.

11.5: A.5- Inner Product and Projections

11.5.1: Inner Product and Orthogonality

To do basic geometry, we need length, and we need angles. We have already seen the euclidean length, so let us figure out how to compute angles. Mostly, we are worried about the right angle[1].

Given two (column) vectors in $\mathbb{R}^n$, we define the (standard) *inner product* as the dot product:

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \vec{y}^T \vec{x} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Why do we seemingly give a new notation for the dot product? Because there are other possible inner products, which are not the dot product, although we will not worry about others here. An inner product can even be defined on spaces of functions as we do in Chapter 4:

$$\langle f(t), g(t) \rangle = \int_a^b f(t) g(t)\, dt.$$

But we digress.

The inner product satisfies the following rules:

i. $\langle \vec{x}, \vec{x} \rangle \geq 0$, and $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = 0$,
ii. $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$,
iii. $\langle a\vec{x}, \vec{y} \rangle = \langle \vec{x}, a\vec{y} \rangle = a\langle \vec{x}, \vec{y} \rangle$,
iv. $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$ and $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$.

Anything that satisfies the properties above can be called an inner product, although in this section we are concerned with the standard inner product in $\mathbb{R}^n$.

The standard inner product gives the euclidean length:

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\,.$$

How does it give angles?

You may recall from multivariable calculus, that in two or three dimensions, the standard inner product (the dot product) gives you the angle between the vectors:

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos\theta.$$

That is, θ is the angle that $\vec{x}$ and $\vec{y}$ make when they are based at the same point.

In $\mathbb{R}^n$ (any dimension), we are simply going to say that θ from the formula is what the angle is. This makes sense as any two vectors based at the origin lie in a 2-dimensional plane (subspace), and the formula works in 2 dimensions. In fact, one could even talk about angles between functions this way, and we do in Chapter 4, where we talk about orthogonal functions (functions at right angle to each other).

To compute the angle we compute

$$\cos\theta = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \|\vec{y}\|}.$$

Our angles are always in radians. We are computing the cosine of the angle, which is really the best we can do. Given two vectors at an angle θ, we can give the angle as $-\theta$, $2\pi - \theta$, etc., see Figure $11.5.1$. Fortunately, $\cos\theta = \cos(-\theta) = \cos(2\pi - \theta)$. If we solve for θ using the inverse cosine $\cos^{-1}$, we can just decree that $0 \leq \theta \leq \pi$.

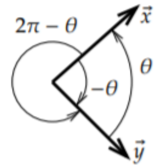


Figure 11.5.1: Angle between vectors.

> **Example 11.5.1**
>
> Let us compute the angle between the vectors $(3,0)$ and $(1,1)$ in the plane. Compute
>
> $$\cos\theta = \frac{\langle (3,0),(1,1)\rangle}{\|(3,0)\|\|(1,1)\|} = \frac{3+0}{3\sqrt{2}} = \frac{1}{\sqrt{2}}.$$
>
> Therefore $\theta = \frac{\pi}{4}$.

As we said, the most important angle is the right angle. A right angle is $\frac{\pi}{2}$ radians, and $\cos(\frac{\pi}{2}) = 0$, so the formula is particularly easy in this case. We say vectors $\vec{x}$ and $\vec{y}$ are *orthogonal* if they are at right angles, that is if

$$\langle \vec{x}, \vec{y}\rangle = 0.$$

The vectors $(1,0,0,1)$ and $(1,2,3,-1)$ are orthogonal. So are $(1,1)$ and $(1,-1)$. However, $(1,1)$ and $(1,2)$ are not orthogonal as their inner product is 3 and not 0.

11.5.2: Orthogonal Projection

A typical application of linear algebra is to take a difficult problem, write everything in the right basis, and in this new basis the problem becomes simple. A particularly useful basis is an orthogonal basis, that is a basis where all the basis vectors are orthogonal. When we draw a coordinate system in two or three dimensions, we almost always draw our axes as orthogonal to each other.

Generalizing this concept to functions, it is particularly useful in Chapter 4 to express a function using a particular orthogonal basis, the Fourier series.

To express one vector in terms of an orthogonal basis, we need to first *project* one vector onto another. Given a nonzero vector $\vec{v}$, we define the *orthogonal projection* of $\vec{w}$ onto $\vec{v}$ as

$$\operatorname{proj}_{\vec{v}}(\vec{w}) = \left(\frac{\langle \vec{w},\vec{v}\rangle}{\langle \vec{v},\vec{v}\rangle}\right)\vec{v}.$$

For the geometric idea, see Figure $11.5.2$ That is, we find the "shadow of $\vec{w}$" on the line spanned by $\vec{v}$ if the direction of the sun's rays were exactly perpendicular to the line. Another way of thinking about it is that the tip of the arrow of $\operatorname{proj}_{\vec{v}}(\vec{w})$ is the closest point on the line spanned by $\vec{v}$ to the tip of the arrow of $\vec{w}$. In terms of euclidean distance, $\vec{u} = \operatorname{proj}_{\vec{v}}(\vec{w})$ minimizes the distance $\|\vec{w}-\vec{u}\|$ among all vectors $\vec{u}$ that are multiples of $\vec{v}$. Because of this, this projection comes up often in applied mathematics in all sorts of contexts we cannot solve a problem exactly: We can't always solve "Find $\vec{w}$ as a multiple of $\vec{v}$" but $\operatorname{proj}_{\vec{v}}(\vec{w})$ is the best "solution."

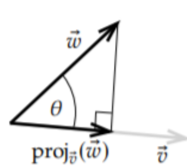


Figure 11.5.2: Orthogonal projection.

The formula follows from basic trigonometry. The length of $\operatorname{proj}_{\vec{v}}(\vec{w})$ should be $\cos\theta$ times the length of $\vec{w}$, that is $(\cos\theta)\|\vec{w}\|$. We take the unit vector in the direction of $\vec{v}$, that is, $\frac{\vec{v}}{\|\vec{v}\|}$ and we multiply it by the length of the projection. In other words,

$$\operatorname{proj}_{\vec{v}}(\vec{w}) = (\cos\theta)\|\vec{w}\|\frac{\vec{v}}{\|\vec{v}\|} = \frac{(\cos\theta)\|\vec{w}\|\|\vec{v}\|}{\|\vec{v}\|^2}\vec{v} = \frac{\langle \vec{w},\vec{v}\rangle}{\langle \vec{v},\vec{v}\rangle}\vec{v}.$$

Example 11.5.2

Suppose we wish to project the vector $(3,2,1)$ onto the vector $(1,2,3)$. Compute

$$\begin{aligned}\operatorname{proj}_{(1,2,3)}\big((3,2,1)\big)=\frac{\langle(3,2,1),(1,2,3)\rangle}{\langle(1,2,3),(1,2,3)\rangle}(1,2,3)&=\frac{3\cdot 1+2\cdot 2+1\cdot 3}{1\cdot 1+2\cdot 2+3\cdot 3}(1,2,3)\\&=\frac{10}{14}(1,2,3)=\left(\frac{5}{7},\frac{10}{7},\frac{15}{7}\right).\end{aligned} \tag{11.5.1}$$

Let us double check that the projection is orthogonal. That is $\vec{w}-\operatorname{proj}_{\vec{v}}(\vec{w})$ ought to be orthogonal to $\vec{v}$, see the right angle in Figure $11.5.2$ That is,

$$(3,2,1)-\operatorname{proj}_{(1,2,3)}\big((3,2,1)\big)=\left(3-\frac{5}{7},2-\frac{10}{7},1-\frac{15}{7}\right)=\left(\frac{16}{7},\frac{4}{7},\frac{-8}{7}\right)$$

ought to be orthogonal to $(1,2,3)$. We compute the inner product and we had better get zero:

$$\left\langle\left(\frac{16}{7},\frac{4}{7},\frac{-8}{7}\right),(1,2,3)\right\rangle=\frac{16}{7}\cdot 1+\frac{4}{7}\cdot 2-\frac{8}{7}\cdot 3=0.$$

11.5.3: Orthogonal Basis

As we said, a basis $\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n$ is an *orthogonal basis* if all vectors in the basis are orthogonal to each other, that is, if

$$\langle\vec{v}_j,\vec{v}_k\rangle=0$$

for all choices of j and k where $j\neq k$ (a nonzero vector cannot be orthogonal to itself). A basis is furthermore called an *orthonormal basis* if all the vectors in a basis are also unit vectors, that is, if all the vectors have magnitude 1. For example, the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$is an orthonormal basis of $\mathbb{R}^3$: Any pair is orthogonal, and each vector is of unit magnitude.

The reason why we are interested in orthogonal (or orthonormal) bases is that they make it really simple to represent a vector (or a projection onto a subspace) in the basis. The simple formula for the orthogonal projection onto a vector gives us the coefficients. In Chapter 4, we use the same idea by finding the correct orthogonal basis for the set of solutions of a differential equation. We are then able to find any particular solution by simply applying the orthogonal projection formula, which is just a couple of a inner products.

Let us come back to linear algebra. Suppose that we have a subspace and an orthogonal basis $\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n$. We wish to express $\vec{x}$ in terms of the basis. If $\vec{x}$ is not in the span of the basis (when it is not in the given subspace), then of course it is not possible, but the following formula gives us at least the orthogonal projection onto the subspace, or in other words, the best approximation in the subspace.

First suppose that $\vec{x}$ is in the span. Then it is the sum of the orthogonal projections:

$$\vec{x}=\operatorname{proj}_{\vec{v}_1}(\vec{x})+\operatorname{proj}_{\vec{v}_2}(\vec{x})+\cdots+\operatorname{proj}_{\vec{v}_n}(\vec{x})=\frac{\langle\vec{x},\vec{v}_1\rangle}{\langle\vec{v}_1,\vec{v}_1\rangle}\vec{v}_1+\frac{\langle\vec{x},\vec{v}_2\rangle}{\langle\vec{v}_2,\vec{v}_2\rangle}\vec{v}_2+\cdots+\frac{\langle\vec{x},\vec{v}_n\rangle}{\langle\vec{v}_n,\vec{v}_n\rangle}\vec{v}_n.$$

In other words, if we want to write $\vec{x}=a_1\vec{v}_1+a_2\vec{v}_2+\cdots+a_n\vec{v}_n$, then

$$a_1=\frac{\langle\vec{x},\vec{v}_1\rangle}{\langle\vec{v}_1,\vec{v}_1\rangle},\quad a_2=\frac{\langle\vec{x},\vec{v}_2\rangle}{\langle\vec{v}_2,\vec{v}_2\rangle},\quad \ldots,\quad a_n=\frac{\langle\vec{x},\vec{v}_n\rangle}{\langle\vec{v}_n,\vec{v}_n\rangle}.$$

Another way to derive this formula is to work in reverse. Suppose that $\vec{x}=a_1\vec{v}_1+a_2\vec{v}_2+\cdots+a_n\vec{v}_n$. Take an inner product with $\vec{v}_j$, and use the properties of the inner product:

$$\begin{aligned}\langle\vec{x},\vec{v}_j\rangle&=\langle a_1\vec{v}_1+a_2\vec{v}_2+\cdots+a_n\vec{v}_n,\vec{v}_j\rangle\\&=a_1\langle\vec{v}_1,\vec{v}_j\rangle+a_2\langle\vec{v}_2,\vec{v}_j\rangle+\cdots+a_n\langle\vec{v}_n,\vec{v}_j\rangle.\end{aligned} \tag{11.5.2}$$

As the basis is orthogonal, then $\langle\vec{v}_k,\vec{v}_j\rangle=0$ whenever $k\neq j$. That means that only one of the terms, the j^{th} one, on the right hand side is nonzero and we get

$$\langle \vec{x}, \vec{v}_j \rangle = a_j \langle \vec{v}_j, \vec{v}_j \rangle.$$

Solving for a_j we find $a_j = \frac{\langle \vec{x}, \vec{v}_j \rangle}{\langle \vec{v}_j, \vec{v}_j \rangle}$ as before.

Example 11.5.3

The vectors $(1,1)$ and $(1,-1)$ form an orthogonal basis of $\mathbb{R}^2$. Suppose we wish to represent $(3,4)$ in terms of this basis, that is, we wish to find a_1 and a_2 such that

$$(3,4) = a_1(1,1) + a_2(1,-1).$$

We compute:

$$a_1 = \frac{\langle (3,4),(1,1) \rangle}{\langle (1,1),(1,1) \rangle} = \frac{7}{2}, \qquad a_2 = \frac{\langle (3,4),(1,-1) \rangle}{\langle (1,-1),(1,-1) \rangle} = \frac{-1}{2}.$$

So

$$(3,4) = \frac{7}{2}(1,1) + \frac{-1}{2}(1,-1).$$

If the basis is orthonormal rather than orthogonal, then all the denominators are one. It is easy to make a basis orthonormal—divide all the vectors by their size. If you want to decompose many vectors, it may be better to find an orthonormal basis. In the example above, the orthonormal basis we would thus create is

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right).$$

Then the computation would have been

$$\begin{aligned} (3,4) &= \left\langle (3,4), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + \left\langle (3,4), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\rangle \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \\ &= \frac{7}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + \frac{-1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right). \end{aligned} \tag{11.5.3}$$

Maybe the example is not so awe inspiring, but given vectors in $\mathbb{R}^{20}$ rather than $\mathbb{R}^2$, then surely one would much rather do 20 inner products (or 40 if we did not have an orthonormal basis) rather than solving a system of twenty equations in twenty unknowns using row reduction of a 20×21 matrix.

As we said above, the formula still works even if $\vec{x}$ is not in the subspace, although then it does not get us the vector $\vec{x}$ but its projection. More concretely, suppose that S is a subspace that is the span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and $\vec{x}$ is any vector. Let $\operatorname{proj}_S(\vec{x})$ be the vector in S that is the closest to $\vec{x}$. Then

$$\operatorname{proj}_S(\vec{x}) = \frac{\langle \vec{x}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 + \frac{\langle \vec{x}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}\vec{v}_2 + \cdots + \frac{\langle \vec{x}, \vec{v}_n \rangle}{\langle \vec{v}_n, \vec{v}_n \rangle}\vec{v}_n.$$

Of course, if $\vec{x}$ is in S, then $\operatorname{proj}_S(\vec{x}) = \vec{x}$, as the closest vector in S to $\vec{x}$ is $\vec{x}$ itself. But true utility is obtained when $\vec{x}$ is not in S. In much of applied mathematics, we cannot find an exact solution to a problem, but we try to find the best solution out of a small subset (subspace). The partial sums of Fourier series from Chapter 4 are one example. Another example is least square approximation to fit a curve to data. Yet another example is given by the most commonly used numerical methods to solve partial differential equations, the finite element methods.

Example 11.5.4

The vectors $(1,2,3)$ and $(3,0,-1)$ are orthogonal, and so they are an orthogonal basis of a subspace S:

$$S = \operatorname{span}\{(1,2,3),(3,0,-1)\}.$$

Let us find the vector in S that is closest to $(2,1,0)$. That is, let us find $\operatorname{proj}_S\big((2,1,0)\big)$.

$$\begin{aligned} \operatorname{proj}_S\big((2,1,0)\big) &= \frac{\langle (2,1,0),(1,2,3)\rangle}{\langle (1,2,3),(1,2,3)\rangle}(1,2,3) + \frac{\langle (2,1,0),(3,0,-1)\rangle}{\langle (3,0,-1),(3,0,-1)\rangle}(3,0,-1) \\ &= \frac{2}{7}(1,2,3) + \frac{3}{5}(3,0,-1) \\ &= \left(\frac{73}{35}, \frac{4}{7}, \frac{9}{35}\right). \end{aligned} \tag{11.5.4}$$

11.5.4: Gram–Schmidt Process

Before leaving orthogonal bases, let us note a procedure for manufacturing them out of any old basis. It may not be difficult to come up with an orthogonal basis for a 2-dimensional subspace, but for a 20-dimensional subspace, it seems a daunting task. Fortunately, the orthogonal projection can be used to "project away" the bits of the vectors that are making them not orthogonal. It is called the *Gram-Schmidt process*.

We start with a basis of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We construct an orthogonal basis $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ as follows.

$$\begin{aligned} \vec{w}_1 &= \vec{v}_1, \\ \vec{w}_2 &= \vec{v}_2 - \operatorname{proj}_{\vec{w}_1}(\vec{v}_2), \\ \vec{w}_3 &= \vec{v}_3 - \operatorname{proj}_{\vec{w}_1}(\vec{v}_3) - \operatorname{proj}_{\vec{w}_2}(\vec{v}_3), \\ \vec{w}_4 &= \vec{v}_4 - \operatorname{proj}_{\vec{w}_1}(\vec{v}_4) - \operatorname{proj}_{\vec{w}_2}(\vec{v}_4) - \operatorname{proj}_{\vec{w}_3}(\vec{v}_4), \\ &\vdots \\ \vec{w}_n &= \vec{v}_n - \operatorname{proj}_{\vec{w}_1}(\vec{v}_n) - \operatorname{proj}_{\vec{w}_2}(\vec{v}_n) - \cdots - \operatorname{proj}_{\vec{w}_{n-1}}(\vec{v}_n). \end{aligned} \tag{11.5.5}$$

What we do is at the k^{th} step, we take $\vec{v}_k$ and we subtract the projection of $\vec{v}_k$ to the subspace spanned by $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{k-1}$.

Example 11.5.5

Consider the vectors $(1,2,-1)$, and $(0,5,-2)$ and call S the span of the two vectors. Let us find an orthogonal basis of S:

$$\begin{aligned} \vec{w}_1 &= (1,2,-1), \\ \vec{w}_2 &= (0,5,-2) - \operatorname{proj}_{(1,2,-1)}\big((0,2,-2)\big) \\ &= (0,1,-1) - \frac{\langle (0,5,-2),(1,2,-1)\rangle}{\langle (1,2,-1),(1,2,-1)\rangle}(1,2,-1) = (0,5,-2) - 2(1,2,-1) = (-2,1,0). \end{aligned} \tag{11.5.6}$$

So $(1,2,-1)$ and $(-2,1,0)$ span S and are orthogonal. Let us check: $\langle (1,2,-1),(-2,1,0)\rangle = 0$.

Suppose we wish to find an orthonormal basis, not just an orthogonal one. Well, we simply make the vectors into unit vectors by dividing them by their magnitude. The two vectors making up the orthonormal basis of S are:

$$\frac{1}{\sqrt{6}}(1,2,-1) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right), \qquad \frac{1}{\sqrt{5}}(-2,1,0) = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right).$$

11.5.5: Footnotes

[1] When Euclid defined angles in his *Elements*, the only angle he ever really defined was the right angle.

11.6: A.6- Determinant

For square matrices we define a useful quantity called the *determinant*. Define the determinant of a 1×1 matrix as the value of its only entry

$$\det([\,a\,]) \overset{\text{def}}{=} a.$$

For a 2×2 matrix, define

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \overset{\text{def}}{=} ad - bc.$$

Before defining the determinant for larger matrices, we note the meaning of the determinant. An $n \times n$ matrix gives a mapping of the n-dimensional euclidean space $\mathbb{R}^n$ to itself. So a 2×2 matrix A is a mapping of the plane to itself. The determinant of A is the factor by which the area of objects changes. If we take the unit square (square of side 1) in the plane, then A takes the square to a parallelogram of area $|\det(A)|$. The sign of $\det(A)$ denotes a change of orientation (negative if the axes get flipped). For example, let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then $\det(A) = 1 + 1 = 2$. Let us see where A sends the unit square—the square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. The point $(0,0)$ gets sent to $(0,0)$.

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The image of the square is another square with vertices $(0,0)$, $(1,-1)$, $(1,1)$, and $(2,0)$. The image square has a side of length $\sqrt{2}$, and it is therefore of area 2. See Figure $11.6.1$.

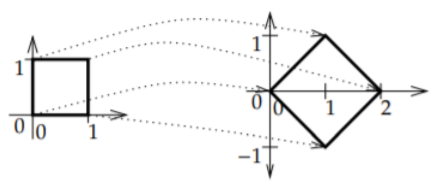


Figure 11.6.1: Image of the unit square via the mapping A.

In general, the image of a square is going to be a parallelogram. In high school geometry, you may have seen a formula for computing the area of a with vertices $(0,0)$, (a,c), (b,d) and $(a+b, c+d)$. The area is

$$\left|\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right| = |ad - bc|.$$

The vertical lines above mean absolute value. The matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ carries the unit square to the given parallelogram.

There are a number of ways to define the determinant for an $n \times n$ matrix. Let us use the so-called *cofactor expansion*. We define A_{ij} as the matrix A with the i^{th} row and the j^{th} column deleted. For example, if

$$\text{If} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \qquad \text{then} \qquad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \qquad \text{and} \qquad A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}.$$

We now define the determinant recursively

$$\det(A) \overset{\text{def}}{=} \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}),$$

or in other words

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots \begin{cases} +a_{1n}\det(A_{1n}) & \text{if } n \text{ is odd,} \\ -a_{1n}\det(A_{1n}) & \text{if } n \text{ even.} \end{cases}$$

For a 3×3 matrix, we get $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$. For example,

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) &= 1 \cdot \det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right) - 2 \cdot \det\left(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}\right) + 3 \cdot \det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\right) \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0. \end{aligned} \tag{11.6.1}$$

It turns out that we did not have to necessarily use the first row. That is for any i,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

It is sometimes useful to use a row other than the first. In the following example it is more convenient to expand along the second row. Notice that for the second row we are starting with a negative sign.

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}\right) &= -0 \cdot \det\left(\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}\right) + 5 \cdot \det\left(\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}\right) - 0 \cdot \det\left(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}\right) \\ &= 0 + 5(1 \cdot 9 - 3 \cdot 7) + 0 = -60. \end{aligned} \tag{11.6.2}$$

Let us check if it is really the same as expanding along the first row,

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}\right) &= 1 \cdot \det\left(\begin{bmatrix} 5 & 0 \\ 8 & 9 \end{bmatrix}\right) - 2 \cdot \det\left(\begin{bmatrix} 0 & 0 \\ 7 & 9 \end{bmatrix}\right) + 3 \cdot \det\left(\begin{bmatrix} 0 & 5 \\ 7 & 8 \end{bmatrix}\right) \\ &= 1(5 \cdot 9 - 0 \cdot 8) - 2(0 \cdot 9 - 0 \cdot 7) + 3(0 \cdot 8 - 5 \cdot 7) = -60. \end{aligned} \tag{11.6.3}$$

In computing the determinant, we alternately add and subtract the determinants of the submatrices A_{ij} multiplied by a_{ij} for a fixed i and all j. The numbers $(-1)^{i+j}\det(A_{ij})$ are called *cofactors* of the matrix. And that is why this method of computing the determinant is called the *cofactor expansion*.

Similarly we do not need to expand along a row, we can expand along a column. For any j,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

A related fact is that

$$\det(A) = \det(A^T).$$

A matrix is *upper triangular* if all elements below the main diagonal are 0. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

is upper triangular. Similarly a *lower triangular* matrix is one where everything above the diagonal is zero. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}.$$

The determinant for triangular matrices is very simple to compute. Consider the lower triangular matrix. If we expand along the first row, we find that the determinant is 1 times the determinant of the lower triangular matrix $\left[\begin{smallmatrix} 5 & 0 \\ 8 & 9 \end{smallmatrix}\right]$. So the deteriminant is just the product of the diagonal entries:

$$\det\left(\begin{bmatrix}1 & 0 & 0\\ 4 & 5 & 0\\ 7 & 8 & 9\end{bmatrix}\right) = 1\cdot 5\cdot 9 = 45.$$

Similarly for upper triangular matrices

$$\det\left(\begin{bmatrix}1 & 2 & 3\\ 0 & 5 & 6\\ 0 & 0 & 9\end{bmatrix}\right) = 1\cdot 5\cdot 9 = 45.$$

In general, if A is triangular, then

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

If A is diagonal, then it is also triangular (upper and lower), so same formula applies. For example,

$$\det\left(\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{bmatrix}\right) = 2\cdot 3\cdot 5 = 30.$$

In particular, the identity matrix I is diagonal, and the diagonal entries are all 1. Thus,

$$\det(I) = 1.$$

The determinant is telling you how geometric objects scale. If B doubles the sizes of geometric objects and A triples them, then AB (which applies B to an object and then it applies A) should make size go up by a factor of 6. This is true in general:

Theorem 11.6.1

$$\det(AB) = \det(A)\det(B).$$

This property is one of the most useful, and it is employed often to actually compute determinants. A particularly interesting consequence is to note what it means for the existence of inverses. Take A and B to be inverses, that is $AB = I$. Then

$$\det(A)\det(B) = \det(AB) = \det(I) = 1.$$

Neither $\det(A)$ nor $\det(B)$ can be zero. This fact is an extremely useful property of the determinant, and one which is used often in this book:

Theorem 11.6.2

An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

In fact, $\det(A^{-1})\det(A) = 1$ says that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

So we know what the determinant of A^{-1} is without computing A^{-1}.

Let us return to the formula for the inverse of a 2×2 matrix:

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}.$$

Notice the determinant of the matrix $\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]$ in the denominator of the fraction. The formula only works if the determinant is nonzero, otherwise we are dividing by zero.

A common notation for the determinant is a pair of vertical lines:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

Personally, I find this notation confusing as vertical lines usually mean a positive quantity, while determinants can be negative. Also think about how to write the absolute value of a determinant. This notation is not used in this book.

11.7: A.E- Linear Algebra (Exercises)

11.7.1: A.1: Vectors, Mappings, and Matrices

? Exercise $11.7.A.1.1$

On a piece of graph paper draw the vectors:

a. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$
b. $\begin{bmatrix} -2 \\ -4 \end{bmatrix}$
c. $(3, -4)$

? Exercise $11.7.A.1.2$

On a piece of graph paper draw the vector $(1, 2)$ starting at (based at) the given point:

a. based at $(0, 0)$
b. based at $(1, 2)$
c. based at $(0, -1)$

? Exercise $11.7.A.1.3$

On a piece of graph paper draw the following operations. Draw and label the vectors involved in the operations as well as the result:

a. $\begin{bmatrix} 1 \\ -4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
b. $\begin{bmatrix} -3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
c. $3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

? Exercise $11.7.A.1.4$

Compute the magnitude of

a. $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$
b. $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$
c. $(1, 3, -4)$

? Exercise $11.7.A.1.5$

Compute

a. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ -8 \end{bmatrix}$
b. $\begin{bmatrix} -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 6 \\ -4 \end{bmatrix}$
c. $-\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

d. $4\begin{bmatrix}-1\\5\end{bmatrix}$
e. $5\begin{bmatrix}1\\0\end{bmatrix}+9\begin{bmatrix}0\\1\end{bmatrix}$
f. $3\begin{bmatrix}1\\-8\end{bmatrix}-2\begin{bmatrix}3\\-1\end{bmatrix}$

? Exercise $11.7.A.1.6$

Find the unit vector in the direction of the given vector

a. $\begin{bmatrix}1\\-3\end{bmatrix}$
b. $\begin{bmatrix}2\\1\\-1\end{bmatrix}$
c. $(3,1,-2)$

? Exercise $11.7.A.1.7$

If $\vec{x}=(1,2)$ and $\vec{y}$ are added together, we find $\vec{x}+\vec{y}=(0,2)$. What is $\vec{y}$?

? Exercise $11.7.A.1.8$

Write $(1,2,3)$ as a linear combination of the standard basis vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$.

? Exercise $11.7.A.1.9$

If the magnitude of $\vec{x}$ is 4, what is the magnitude of

a. $0\vec{x}$
b. $3\vec{x}$
c. $-\vec{x}$
d. $-4\vec{x}$
e. $\vec{x}+\vec{x}$
f. $\vec{x}-\vec{x}$

? Exercise $11.7.A.1.10$

Suppose a linear mapping $F\colon \mathbb{R}^2\to\mathbb{R}^2$ takes $(1,0)$ to $(2,-1)$ and it takes $(0,1)$ to $(3,3)$. Where does it take

a. $(1,1)$
b. $(2,0)$
c. $(2,-1)$

? Exercise $11.7.A.1.11$

Suppose a linear mapping $F\colon \mathbb{R}^3\to\mathbb{R}^2$ takes $(1,0,0)$ to $(2,1)$, it takes $(0,1,0)$ to $(3,4)$, and it takes $(0,0,1)$ to $(5,6)$. Write down the matrix representing the mapping F.

? Exercise $11.7.A.1.12$

Suppose that a mapping $F\colon \mathbb{R}^2\to\mathbb{R}^2$ takes $(1,0)$ to $(1,2)$, $(0,1)$ to $(3,4)$, and $(1,1)$ to $(0,-1)$. Explain why F is not linear.

? Exercise $11.7.A.1.13$: (challenging)

Let $\mathbb{R}^3$ represent the space of quadratic polynomials in t: a point (a_0, a_1, a_2) in $\mathbb{R}^3$ represents the polynomial $a_0 + a_1 t + a_2 t^2$. Consider the derivative $\frac{d}{dt}$ as a mapping of $\mathbb{R}^3$ to $\mathbb{R}^3$, and note that $\frac{d}{dt}$ is linear. Write down $\frac{d}{dt}$ as a 3×3 matrix.

? Exercise $11.7.A.1.14$

Compute the magnitude of

a. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$
b. $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$
c. $(-2, 1, -2)$

Answer

a. $\sqrt{10}$
b. $\sqrt{14}$
c. 3

? Exercise $11.7.A.1.15$

Find the unit vector in the direction of the given vector

a. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
b. $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$
c. $(2, -5, 2)$

Answer

a. $\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
b. $\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$
c. $\left(\frac{2}{\sqrt{33}}, \frac{-5}{\sqrt{33}}, \frac{2}{\sqrt{33}}\right)$

? Exercise $11.7.A.1.16$

Compute

a. $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \end{bmatrix}$
b. $\begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
c. $-\begin{bmatrix} -5 \\ 3 \end{bmatrix}$

d. $2\begin{bmatrix} -2 \\ 4 \end{bmatrix}$
e. $3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
f. $2\begin{bmatrix} 2 \\ -3 \end{bmatrix} - 6\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Answer

a. $\begin{bmatrix} 9 \\ -2 \end{bmatrix}$
b. $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$
c. $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$
d. $\begin{bmatrix} -4 \\ 8 \end{bmatrix}$
e. $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$
f. $\begin{bmatrix} -8 \\ 3 \end{bmatrix}$

Exercise $11.7.A.1.17$

If the magnitude of $\vec{x}$ is 5, what is the magnitude of

a. $4\vec{x}$
b. $-2\vec{x}$
c. $-4\vec{x}$

Answer

a. 20
b. 10
c. 20

Exercise $11.7.A.1.18$

Suppose a linear mapping $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ takes $(1,0)$ to $(1,-1)$ and it takes $(0,1)$ to $(2,0)$. Where does it take

a. $(1,1)$
b. $(0,2)$
c. $(1,-1)$

Answer

a. $(3,-1)$
b. $(4,0)$
c. $(-1,-1)$

11.7.2: A.2: Matrix Algebra

Exercise $11.7.A.2.1$

Add the following matrices

a. $\begin{bmatrix} -1 & 2 & 2 \\ 5 & 8 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 3 \\ 8 & 3 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -8 & -3 \\ 3 & 1 & 0 \\ 6 & -4 & 1 \end{bmatrix}$

? Exercise $11.7.A.2.2$

Compute

a. $3\begin{bmatrix} 0 & 3 \\ -2 & 2 \end{bmatrix} + 6\begin{bmatrix} 1 & 5 \\ -1 & 5 \end{bmatrix}$

b. $2\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} - 3\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

? Exercise $11.7.A.2.3$

Multiply the following matrices

a. $\begin{bmatrix} -1 & 2 \\ 3 & 1 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 3 & -1 & 3 & 1 \\ 8 & 3 & 2 & -3 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & 7 \\ 1 & 2 & 3 & -1 \\ 1 & -1 & 3 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 1 & 6 & 3 \\ 5 & 6 & 5 & 0 \\ 4 & 6 & 6 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 2 \\ 3 & 5 \\ 5 & 6 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 6 & 4 \end{bmatrix}$

? Exercise $11.7.A.2.4$

Compute the inverse of the given matrices

a. $\begin{bmatrix} -3 \end{bmatrix}$

b. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$

? Exercise $11.7.A.2.5$

Compute the inverse of the given matrices

a. $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

Exercise $11.7.A.2.6$

Add the following matrices

a. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 6 & -2 & 3 \\ 7 & 3 & 3 \\ 8 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -1 & -3 \\ 6 & 7 & 3 \\ -9 & 4 & -1 \end{bmatrix}$

Answer

Add texts here. Do not delete this text first.

Exercise $11.7.A.2.7$

Compute

a. $2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 3\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$

b. $3\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

Answer

Add texts here. Do not delete this text first.

Exercise $11.7.A.2.8$

Multiply the following matrices

a. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 6 & 3 \\ 3 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 3 & 3 \\ 2 & -2 & 1 \\ 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 & 6 & 2 \\ 4 & 6 & 0 \\ 2 & 0 & 4 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 4 & 1 \\ 2 & -1 & 0 \\ 4 & -1 & 5 \end{bmatrix} \begin{bmatrix} 0 & 2 & 5 & 0 \\ 2 & 0 & 5 & 2 \\ 3 & 6 & 1 & 6 \end{bmatrix}$

d. $\begin{bmatrix} -2 & -2 \\ 5 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 3 \end{bmatrix}$

Answer

Add texts here. Do not delete this text first.

? Exercise $11.7.A.2.9$

Compute the inverse of the given matrices

a. $\begin{bmatrix} 2 \end{bmatrix}$
b. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
c. $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$
d. $\begin{bmatrix} 4 & 2 \\ 4 & 4 \end{bmatrix}$

Answer

Add texts here. Do not delete this text first.

? Exercise $11.7.A.2.10$

Compute the inverse of the given matrices

a. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
b. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
c. $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$

Answer

Add texts here. Do not delete this text first.

11.7.3: A.3: Elimination

? Exercise $11.7.A.3.1$

Compute the reduced row echelon form for the following matrices:

a. $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
b. $\begin{bmatrix} 3 & 3 \\ 6 & -3 \end{bmatrix}$
c. $\begin{bmatrix} 3 & 6 \\ -2 & -3 \end{bmatrix}$
d. $\begin{bmatrix} 6 & 6 & 7 & 7 \\ 1 & 1 & 0 & 1 \end{bmatrix}$
e. $\begin{bmatrix} 9 & 3 & 0 & 2 \\ 8 & 6 & 3 & 6 \\ 7 & 9 & 7 & 9 \end{bmatrix}$
f. $\begin{bmatrix} 2 & 1 & 3 & -3 \\ 6 & 0 & 0 & -1 \\ -2 & 4 & 4 & 3 \end{bmatrix}$

g. $\begin{bmatrix} 6 & 6 & 5 \\ 0 & -2 & 2 \\ 6 & 5 & 6 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 2 & 0 & -1 \\ 6 & 6 & -3 & 3 \\ 6 & 2 & -3 & 5 \end{bmatrix}$

? Exercise $11.7.A.3.2$

Compute the inverse of the given matrices

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

? Exercise $11.7.A.3.3$

Solve (find all solutions), or show no solution exists

a. $\begin{aligned} 4x_1 + 3x_2 &= -2 \\ -x_1 + \ \ x_2 &= 4 \end{aligned}$

b. $\begin{aligned} x_1 + 5x_2 + 3x_3 &= 7 \\ 8x_1 + 7x_2 + 8x_3 &= 8 \\ 4x_1 + 8x_2 + 6x_3 &= 4 \end{aligned}$

c. $\begin{aligned} 4x_1 + 8x_2 + 2x_3 &= 3 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 4x_1 + 8x_2 \qquad &= 2 \end{aligned}$

d. $\begin{aligned} x + 2y + 3z &= 4 \\ 2x - \ \ y + 3z &= 1 \\ 3x + \ \ y + 6z &= 6 \end{aligned}$

? Exercise $11.7.A.3.4$

By computing the inverse, solve the following systems for $\vec{x}$.

a. $\begin{bmatrix} 4 & 1 \\ -1 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 13 \\ 26 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

? Exercise $11.7.A.3.5$

Compute the rank of the given matrices

a. $\begin{bmatrix} 6 & 3 & 5 \\ 1 & 4 & 1 \\ 7 & 7 & 6 \end{bmatrix}$

b. $\begin{bmatrix} 5 & -2 & -1 \\ 3 & 0 & 6 \\ 2 & 4 & 5 \end{bmatrix}$
c. $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}$

? Exercise $11.7.A.3.6$

For the matrices in Exercise $11.7.A.3.5$ find a linearly independent set of row vectors that span the row space (they don't need to be rows of the matrix).

? Exercise $11.7.A.3.7$

For the matrices in Exercise $11.7.A.3.5$ find a linearly independent set of columns that span the column space. That is, find the pivot columns of the matrices.

? Exercise $11.7.A.3.8$

Find a linearly independent subset of the following vectors that has the same span.

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

? Exercise $11.7.A.3.9$

Compute the reduced row echelon form for the following matrices:

a. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
b. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
c. $\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$
d. $\begin{bmatrix} 1 & -3 & 1 \\ 4 & 6 & -2 \\ -2 & 6 & -2 \end{bmatrix}$
e. $\begin{bmatrix} 2 & 2 & 5 & 2 \\ 1 & -2 & 4 & -1 \\ 0 & 3 & 1 & -2 \end{bmatrix}$
f. $\begin{bmatrix} -2 & 6 & 4 & 3 \\ 6 & 0 & -3 & 0 \\ 4 & 2 & -1 & 1 \end{bmatrix}$
g. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
h. $\begin{bmatrix} 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 5 \end{bmatrix}$

Answer

a. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 0 & \frac{77}{15} \\ 0 & 1 & 0 & -\frac{2}{15} \\ 0 & 0 & 1 & -\frac{8}{5} \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

? Exercise $11.7.A.3.10$

Compute the inverse of the given matrices

a. $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 4 & 0 \\ 2 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$

Answer

a. $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} \frac{5}{2} & 1 & -3 \\ -1 & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 1 \end{bmatrix}$

? Exercise $11.7.A.3.11$

Solve (find all solutions), or show no solution exists

a. $\begin{aligned} 4x_1 + 3x_2 &= -1 \\ 5x_1 + 6x_2 &= 4 \end{aligned}$

b. $\begin{aligned} 5x + 6y + 5z &= 7 \\ 6x + 8y + 6z &= -1 \\ 5x + 2y + 5z &= 2 \end{aligned}$

c. $$\begin{aligned} a+\ b+\ c\ &= -1 \\ a+5b+6c\ &= -1 \\ -2a+5b+6c\ &= 8 \end{aligned}$$

d. $$\begin{aligned} -2x_1+2x_2+8x_3 &= 6 \\ x_2+\ x_3 &= 2 \\ x_1+4x_2+\ x_3 &= 7 \end{aligned}$$

Answer

a. $x_1 = -2,\ x_2 = \frac{7}{3}$
b. no solution
c. $a = -3,\ b = 10,\ c = -8$
d. x_3 is free, $x_1 = -1 + 3x_3$, $x_2 = 2 - x_3$

? Exercise $11.7.A.3.12$

By computing the inverse, solve the following systems for $\vec{x}$.

a. $\begin{bmatrix} -1 & 1 \\ 3 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$
b. $\begin{bmatrix} 2 & 7 \\ 1 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Answer

a. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$
b. $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

? Exercise $11.7.A.3.13$

Compute the rank of the given matrices

a. $\begin{bmatrix} 7 & -1 & 6 \\ 7 & 7 & 7 \\ 7 & 6 & 2 \end{bmatrix}$
b. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$
c. $\begin{bmatrix} 0 & 3 & -1 \\ 6 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix}$

Answer

a. 3
b. 1
c. 2

? Exercise $11.7.A.3.14$

For the matrices in Exercise $11.7.A.3.13$, find a linearly independent set of row vectors that span the row space (they don't need to be rows of the matrix).

Answer

a. $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$
b. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$
c. $\begin{bmatrix} 1 & 0 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & 1 & -\frac{1}{3} \end{bmatrix}$

? Exercise $11.7.A.3.15$

For the matrices in Exercise $11.7.A.3.13$ find a linearly independent set of columns that span the column space. That is, find the pivot columns of the matrices.

Answer

a. $\begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 2 \end{bmatrix}$
b. $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$
c. $\begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$

? Exercise $11.7.A.3.16$

Find a linearly independent subset of the following vectors that has the same span.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$$

Answer

$\begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$

11.7.4: A.4: Subspaces, Dimension, and The Kernel

? Exercise $11.7.A.4.1$

For the following sets of vectors, find a basis for the subspace spanned by the vectors, and find the dimension of the subspace.

a. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$
b. $\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$
c. $\begin{bmatrix} -4 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$
d. $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

e. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

f. $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}$, $\begin{bmatrix} -5 \\ -5 \\ -2 \end{bmatrix}$

? Exercise 11.7.A.4.2

For the following matrices, find a basis for the kernel (nullspace).

a. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 5 \\ 1 & 1 & -4 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & -3 \\ 4 & 0 & -4 \\ -1 & 1 & 2 \end{bmatrix}$

c. $\begin{bmatrix} -4 & 4 & 4 \\ -1 & 1 & 1 \\ -5 & 5 & 5 \end{bmatrix}$

d. $\begin{bmatrix} -2 & 1 & 1 & 1 \\ -4 & 2 & 2 & 2 \\ 1 & 0 & 4 & 3 \end{bmatrix}$

? Exercise 11.7.A.4.3

Suppose a 5×5 matrix A has rank 3. What is the nullity?

? Exercise 11.7.A.4.4

Suppose that X is the set of all the vectors of $\mathbb{R}^3$ whose third component is zero. Is X a subspace? And if so, find a basis and the dimension.

? Exercise 11.7.A.4.5

Consider a square matrix A, and suppose that $\vec{x}$ is a nonzero vector such that $A\vec{x} = \vec{0}$. What does the Fredholm alternative say about invertibility of A.

? Exercise 11.7.A.4.6

Consider

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & ? & ? \\ -1 & ? & ? \end{bmatrix}.$$

If the nullity of this matrix is 2, fill in the question marks. Hint: What is the rank?

? Exercise 11.7.A.4.7

For the following sets of vectors, find a basis for the subspace spanned by the vectors, and find the dimension of the subspace.

a. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}2\\2\\2\end{bmatrix}$, $\begin{bmatrix}1\\1\\2\end{bmatrix}$

c. $\begin{bmatrix}5\\3\\1\end{bmatrix}$, $\begin{bmatrix}5\\-1\\5\end{bmatrix}$, $\begin{bmatrix}-1\\3\\-4\end{bmatrix}$

d. $\begin{bmatrix}2\\2\\4\end{bmatrix}$, $\begin{bmatrix}2\\2\\3\end{bmatrix}$, $\begin{bmatrix}4\\4\\-3\end{bmatrix}$

e. $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}2\\0\end{bmatrix}$, $\begin{bmatrix}3\\0\end{bmatrix}$

f. $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}2\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\2\end{bmatrix}$

Answer

a. $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ dimension 2,

b. $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}$ dimension 2,

c. $\begin{bmatrix}5\\3\\1\end{bmatrix}, \begin{bmatrix}5\\-1\\5\end{bmatrix}, \begin{bmatrix}-1\\3\\-4\end{bmatrix}$ dimension 3,

d. $\begin{bmatrix}2\\2\\4\end{bmatrix}, \begin{bmatrix}2\\2\\3\end{bmatrix}$ dimension 2,

e. $\begin{bmatrix}1\\1\end{bmatrix}$ dimension 1,

f. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}$ dimension 2

Exercise $11.7.A.4.8$

For the following matrices, find a basis for the kernel (nullspace).

a. $\begin{bmatrix}2&6&1&9\\1&3&2&9\\3&9&0&9\end{bmatrix}$

b. $\begin{bmatrix}2&-2&-5\\-1&1&5\\-5&5&-3\end{bmatrix}$

c. $\begin{bmatrix}1&-5&-4\\2&3&5\\-3&5&2\end{bmatrix}$

d. $\begin{bmatrix}0&4&4\\0&1&1\\0&5&5\end{bmatrix}$

Answer

a. $\begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -1 \end{bmatrix}$
b. $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$
c. $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
d. $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

? Exercise $11.7.A.4.9$

Suppose the column space of a 9×5 matrix A of dimension 3. Find

a. Rank of A.
b. Nullity of A.
c. Dimension of the row space of A.
d. Dimension of the nullspace of A.
e. Size of the maximum subset of linearly independent rows of A.

Answer

a. 3
b. 2
c. 3
d. 2
e. 3

11.7.5: A.5: Inner Product and Projections

? Exercise $11.7.A.5.1$

Find the s that makes the following vectors orthogonal: $(1, 2, 3)$, $(1, 1, s)$.

? Exercise $11.7.A.5.2$

Find the angle θ between $(1, 3, 1)$, $(2, 1, -1)$.

? Exercise $11.7.A.5.3$

Given that $\langle \vec{v}, \vec{w} \rangle = 3$ and $\langle \vec{v}, \vec{u} \rangle = -1$ compute

a. $\langle \vec{u}, 2\vec{v} \rangle$
b. $\langle \vec{v}, 2\vec{w} + 3\vec{u} \rangle$
c. $\langle \vec{w} + 3\vec{u}, \vec{v} \rangle$

? Exercise $11.7.A.5.4$

Suppose $\vec{v} = (1, 1, -1)$. Find

a. $\text{proj}_{\vec{v}}\big((1, 0, 0)\big)$

b. $\operatorname{proj}_{\vec{v}}((1,2,3))$
c. $\operatorname{proj}_{\vec{v}}((1,-1,0))$

Exercise 11.7.A.5.5

Consider the vectors $(1,2,3)$, $(-3,0,1)$, $(1,-5,3)$.

a. Check that the vectors are linearly independent and so form a basis.
b. Check that the vectors are mutually orthogonal, and are therefore an orthogonal basis.
c. Represent $(1,1,1)$ as a linear combination of this basis.
d. Make the basis orthonormal.

Exercise 11.7.A.5.6

Let S be the subspace spanned by $(1,3,-1)$, $(1,1,1)$. Find an orthogonal basis of S by the Gram-Schmidt process.

Exercise 11.7.A.5.7

Starting with $(1,2,3)$, $(1,1,1)$, $(2,2,0)$, follow the Gram-Schmidt process to find an orthogonal basis of $\mathbb{R}^3$.

Exercise 11.7.A.5.8

Find an orthogonal basis of $\mathbb{R}^3$ such that $(3,1,-2)$ is one of the vectors. Hint: First find two extra vectors to make a linearly independent set.

Exercise 11.7.A.5.9

Using cosines and sines of θ, find a unit vector $\vec{u}$ in $\mathbb{R}^2$ that makes angle θ with $\vec{\imath}=(1,0)$. What is $\langle\vec{\imath},\vec{u}\rangle$?

Exercise 11.7.A.5.10

Find the s that makes the following vectors orthogonal: $(1,1,1)$, $(1,s,1)$.

Answer

$s=-2$

Exercise 11.7.A.5.11

Find the angle θ between $(1,2,3)$, $(1,1,1)$.

Answer

$\theta \approx 0.3876$

Exercise 11.7.A.5.12

Given that $\langle\vec{v},\vec{w}\rangle=1$ and $\langle\vec{v},\vec{u}\rangle=-1$ and $\|\vec{v}\|=3$ and

a. $\langle 3\vec{u},5\vec{v}\rangle$
b. $\langle\vec{v},2\vec{w}+3\vec{u}\rangle$
c. $\langle\vec{w}+3\vec{v},\vec{v}\rangle$

Answer

a. -15
b. -1

c. 28

Exercise $11.7.A.5.13$

Suppose $\vec{v}=(1,0,-1)$. Find

a. $\operatorname{proj}_{\vec{v}}((0,2,1))$
b. $\operatorname{proj}_{\vec{v}}((1,0,1))$
c. $\operatorname{proj}_{\vec{v}}((4,-1,0))$

Answer

a. $\left(-\frac{1}{2},0,\frac{1}{2}\right)$
b. $(0,0,0)$
c. $(2,0,-2)$

Exercise $11.7.A.5.14$

The vectors $(1,1,-1)$, $(2,-1,1)$, $(1,-5,3)$ form an orthogonal basis. Represent the following vectors in terms of this basis:

a. $(1,-8,4)$
b. $(5,-7,5)$
c. $(0,-6,2)$

Answer

a. $(1,1,-1)-(2,-1,1)+2(1,-5,3)$
b. $2(2,-1,1)+(1,-5,3)$
c. $2(1,1,-1)-2(2,-1,1)+2(1,-5,3)$

Exercise $11.7.A.5.15$

Let S be the subspace spanned by $(2,-1,1)$, $(2,2,2)$. Find an orthogonal basis of S by the Gram-Schmidt process.

Answer

$(2,-1,1)$, $\left(\frac{2}{3},\frac{8}{3},\frac{4}{3}\right)$

Exercise $11.7.A.5.16$

Starting with $(1,1,-1)$, $(2,3,-1)$, $(1,-1,1)$, follow the Gram-Schmidt process to find an orthogonal basis of $\mathbb{R}^3$.

Answer

$(1,1,-1)$, $(0,1,1)$, $\left(\frac{4}{3},-\frac{2}{3},\frac{2}{3}\right)$

11.7.6: A.6: Determinant

Exercise $11.7.A.6.1$

Compute the determinant of the following matrices:

a. $\begin{bmatrix} 3 \end{bmatrix}$
b. $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$
c. $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 1 & 0 \\ -2 & 7 & -3 \\ 0 & 2 & 0 \end{bmatrix}$

f. $\begin{bmatrix} 2 & 1 & 3 \\ 8 & 6 & 3 \\ 7 & 9 & 7 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 2 & 5 & 7 \\ 0 & 0 & 2 & -3 \\ 3 & 4 & 5 & 7 \\ 0 & 0 & 2 & 4 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & 2 & 1 \\ 2 & -1 & -2 & 3 \end{bmatrix}$

? Exercise 11.7.A.6.2

For which x are the following matrices singular (not invertible).

a. $\begin{bmatrix} 2 & 3 \\ 2 & x \end{bmatrix}$

b. $\begin{bmatrix} 2 & x \\ 1 & 2 \end{bmatrix}$

c. $\begin{bmatrix} x & 1 \\ 4 & x \end{bmatrix}$

d. $\begin{bmatrix} x & 0 & 1 \\ 1 & 4 & 2 \\ 1 & 6 & 2 \end{bmatrix}$

? Exercise 11.7.A.6.3

Compute

$$\det\left(\begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & 8 & 6 & 5 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}\right)$$

without computing the inverse.

? Exercise 11.7.A.6.4

Suppose

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 7 & \pi & 1 & 0 \\ 2^8 & 5 & -99 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 5 & 9 & 1 & -\sin(1) \\ 0 & 1 & 88 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $A = LU$. Compute $\det(A)$ in a simple way, without computing what is A. Hint: First read off $\det(L)$ and $\det(U)$.

Exercise $11.7.A.6.5$

Consider the linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 1 & x \\ 2 & 1 \end{bmatrix}$ for some number x. You wish to make A such that it doubles the area of every geometric figure. What are the possibilities for x (there are two answers).

Exercise $11.7.A.6.6$

Suppose A and S are $n \times n$ matrices, and S is invertible. Suppose that $\det(A) = 3$. Compute $\det(S^{-1}AS)$ and $\det(SAS^{-1})$. Justify your answer using the theorems in this section.

Exercise $11.7.A.6.7$

Let A be an $n \times n$ matrix such that $\det(A) = 1$. Compute $\det(xA)$ given a number x. Hint: First try computing $\det(xI)$, then note that $xA = (xI)A$.

Exercise $11.7.A.6.8$

Compute the determinant of the following matrices:

a. $\begin{bmatrix} -2 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 9 & -11 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 1 & 0 \\ -2 & 7 & 3 \\ 1 & 1 & 0 \end{bmatrix}$

f. $\begin{bmatrix} 5 & 1 & 3 \\ 4 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}$

g. $\begin{bmatrix} 3 & 2 & 5 & 7 \\ 0 & 0 & 2 & 0 \\ 0 & 4 & 5 & 0 \\ 2 & 1 & 2 & 4 \end{bmatrix}$

h. $\begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 2 & -3 & 4 \\ 5 & 6 & -7 & 8 \\ 1 & 2 & 3 & -2 \end{bmatrix}$

Answer

a. -2
b. 8
c. 0
d. -6
e. -3
f. 28
g. 16
h. -24

Exercise $11.7.A.6.9$

For which x are the following matrices singular (not invertible).

a. $\begin{bmatrix} 1 & 3 \\ 1 & x \end{bmatrix}$
b. $\begin{bmatrix} 3 & x \\ 1 & 3 \end{bmatrix}$
c. $\begin{bmatrix} x & 3 \\ 3 & x \end{bmatrix}$
d. $\begin{bmatrix} x & 1 & 0 \\ 1 & 4 & 0 \\ 1 & 6 & 2 \end{bmatrix}$

Answer

a. 3
b. 9
c. 3
d. $\frac{1}{4}$

Exercise $11.7.A.6.10$

Compute

$$\det\left(\begin{bmatrix} 3 & 4 & 7 & 12 \\ 0 & -1 & 9 & -8 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-1}\right)$$

without computing the inverse.

Answer

12

Exercise $11.7.A.6.11$: (challenging)

Find all the x that make the matrix inverse

$$\begin{bmatrix} 1 & 2 \\ 1 & x \end{bmatrix}^{-1}$$

have only integer entries (no fractions). Note that there are two answers.

Answer

1 and 3

12: Appendix B- Table of Laplace Transforms

The function u is the Heaviside function, δ is the Dirac delta function, and

$$\Gamma(t)=\int_0^\infty e^{-\tau}\tau^{t-1}\,d\tau, \qquad \operatorname{erf}(t)=\frac{2}{\sqrt{\pi}}\int_0^t e^{-\tau^2}\,d\tau, \qquad \operatorname{erfc}(t)=1-\operatorname{erf}(t). \tag{12.1}$$

Table 12.1

$f(t)$	$F(s)=\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)dt$
C	$\frac{C}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
$t^p \quad (p>0)$	$\frac{\Gamma(p+1)}{s^{p+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$\operatorname{erf}\left(\frac{t}{2a}\right)$	$\frac{1}{s}e^{(as)^2}\operatorname{erfc}(as)$
$\frac{1}{\sqrt{\pi t}}\exp\left(\frac{-a^2}{4t}\right) \quad (a\ge 0)$	$\frac{e^{-as}}{\sqrt{s}}$
$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc}(a\sqrt{t}) \quad (a>0)$	$\frac{1}{\sqrt{s}+a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f(at) \quad (a>0)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$e^{-at}f(t)$	$F(s+a)$
$g'(t)$	$sG(s)-g(0)$
$g''(t)$	$s^2G(s)-sg(0)-g'(0)$
$g'''(t)$	$s^3G(s)-s^2g(0)-sg'(0)-g''(0)$
$g^{(n)}(t)$	$s^nG(s)-s^{n-1}g(0)-\cdots-g^{(n-1)}(0)$
$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$

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Glossary

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